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CONTRIBUTIONS FROM THE GRAY HERBARIUM OF HARVARD UNIVERSITY

New Series. — No. LX.

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CONTRIBUTIONS FROM THE GRAY HERBARIUM OF HARVARD UNIVERSITY,—NEW SERIES, NO. LX.

I. ON TROPICAL AMERICAN COMPOSITAE, CHIEFLY EUPATORIEAE.

BY B. L. ROBINSON.

Presented June 26. 1919. Received June 26. 1919.

DURING the past year the writer has continued his studies on the South American members of the genus *Eupatorium*, giving particular attention to those of Peru and Bolivia. Of the Peruvian it has proved possible to complete a recension, which, by bringing together widely scattered data, placing on record a considerable number of new species, and furnishing specific keys in each section, will, it is hoped, render a knowledge of these plants much more readily accessible and considerably facilitate their precise identification.

There have been several sources of new information regarding tropical American Eupatoriums and many facts have been observed which were not available during the preparation of the writer's revision of the Colombian, Venezuelan, and Ecuadorian species, a treatment completed about a year ago (Proc. Am. Acad. liv. 235–367, 1918). Thus, for instance, the later portions of Dr. F. W. Pennell's extensive collections from Colombia proved to contain interesting *Eupatorium* material to an unsuspected extent, including several additional novelties. A small Colombian collection, prepared by Bro. Ariste-Joseph and received at the Gray Herbarium in continuation of exchange from the United States National Herbarium, has yielded still another new species from the vicinity of Bogotá, a region already exceptionally rich in its representatives of the genus. A somewhat extended collection of the flora of Caracas and adjacent portions of northern Venezuela has been obtained from Prof. H. Pittier. This has given considerable supplementary information regarding the Eupatoriums of the regions explored. Finally, the expedition of Dr. J. N. Rose to Ecuador has brought in the most notable single collection as yet received from that country, including a suite of some twenty-eight different Eupatoriums, among which at least four merit recognition as new to science, while others show notable rediscoveries of plants not secured for many decades.

As has been the case in the preparation of several former papers relating to the *Eupatorium* tribe, much valued aid has been received from the New York Botanical Garden, the United States National Herbarium, and the Field Museum of Natural History, all of these establishments having lent for examination highly interesting material, which has formed the basis of many of the observations here recorded.

In elaborating the Peruvian species much aid has been derived from material borrowed from the Royal Gardens at Kew and from a considerable suite of specimens collected by Dr. A. Weberbauer and lent some years ago from the Royal Botanical Museum in Berlin for study and identification.

Prof. H. Pittier has kindly called to the writer's attention some geographical errors in the paper on the Venezuelan Eupatoriums. Among these are the employment of several obsolete and now supplanted place-names, and several orthographical slips which, while regretted, are happily not of a kind likely to cause serious error or misunderstanding. More annoying, however, in this respect was the writer's not unnatural but entirely erroneous identification of "Colonia Tovar"— a place frequently mentioned upon Fendler's labels — with the town of Tovar in the state of Mérida. In consequence, attention is here particularly directed to the fact that Colonia Tovar, the base of much of Fendler's Venezuelan work, is a small town not very distant from Caracas and situated in the mountains somewhat back from the coast in the northern part of the state of Aragua. This Tovar was unfortunately not recorded on the maps consulted during the preparation of the earlier paper.

A few notes, diagnoses, and transfers are here published concerning other *Compositae* which, mostly on account of habital similarity, have been submitted for identification in connection with these studies on the Eupatoriums.

The abbreviations employed in the present publication to indicate the different botanical establishments are the same as in former papers, those of most frequent occurrence being as follows: *Gr.* for the Gray Herbarium; *U. S.*, the U. S. National Herbarium; *N. Y.*, the New York Botanical Garden; *Field Mus.*, the Field Museum of Natural History, Chicago; *K.*, the herbarium of the Royal Botanical Gardens, Kew; *Brit. Mus.*, the British Museum of Natural History, South Kensington, London; *Par.*, the Muséum d'histoire naturelle, Jardin des plantes, Paris; *DC.*, the "Prodromus collection" in the

DeCandollean Herbarium, Geneva; *Berl.*, the Botanical Museum, Dahlem, Berlin. The few other similar abbreviations which occur are believed to be sufficiently clear without particular explanation.

Ophryosporus (§ **Euophryosporus**) **bipinnatifidus**, spec. nov., caulibus (vel ramis) curvato-adscendentibus vel erectis 2–3 mm. crassis ut videtur paullulo lignescentibus inferne delapsu foliorum nudatis nodulosis superne densissime foliosis; foliis alternis saepissime in axillis proliferis deltoideo-ovatis profunde bipinnatifidis utrinque dense griseo-pubescentibus 2–3 cm. longis et latis, lobis anguste oblongis vel oblanceolatis vix acutiusculis integris vel saepe lobatis; petiolo 5–12 mm. longo cuneatim alato; thyrso terminali solitario vel pluribus ovoideis densis ca. 3 cm. longis 2.5 cm. crassis; capitulis ca. 4-floris sessilibus vel breviter pedicellatis ca. 5.5 mm. longis; involucri squamis principibus plerisque 4 subaequalibus cum 1–2 multo minoribus calyculatis viridibus tenuibus sordide pubescentibus; corollis albis ca. 3.2 mm. longis, tubo proprio glandulari-puberulo; limbi dentibus 5 ovato-deltoideis patentibus; antheris apice omnino exappendiculatis basi subsagittatis; styli ramis filiformibus subtruncatis apice paullo incrassatis; achaeniis (valde immaturis) hispidulis; pappi setis ca. 18 inaequalibus ca. 2 mm. longis scabratis.— PERU: in very arid ground, Posco, between Mollendo and Arequipa, 6 Aug. 1914, *Dr. & Mrs. J. N. Rose*, no. 18,805 (U. S., phot. and fragm. Gr.). This beautifully distinct species adds a third to the alternate-leaved section of the genus. In habit and cutting of the leaves it recalls the Brazilian *Lomatozona artemisiaefolia* Bak. in Mart. Fl. Bras. vi. pt. 2, 199, t. 54, f. II (1876), but of course differs from it significantly in many technical characters, such as its exappendiculate anthers, longer pappus, alternate leaves, etc.

Ophryosporus (§ **Ophryochaeta**) **ovatus**, spec. nov., petiolis junioribus et inflorescentia puberulis exceptis glaberrimus; caule tereti purpurascente folioso; foliis oppositis graciliter petiolatis deltoideo-ovatis leviter crenato-serratis firmiusculis utrinque viridibus glaberrimis sublucidis a basi 3-nerviis minute prominulenterque reticulatis subacutis ca. 4 cm. longis 3 cm. latis; petiolo ca. 1 cm. longo; panicula ovoidea terminali ca. 1 dm. alta et crassa multicapitulata, ramulis pedicellisque crispe puberulis; capitulis 4.5 mm. altis 3 mm. diametro ca. 7-floris; involucri campanulati squamis ca. 6 tenuibus substramineis ellipticis subaequalibus apice rotundatis eroso-ciliolatis dorso glabris; corollis ca. 3.5 mm. longis a media parte modice ampliatis 5-dentatis extus ubique minutissime glanduloso-puberulis; antheris omnino exappendiculatis; styli ramis apice incrassatis et fuscescentibus; achaeniis (immaturis) 1.3 mm. longis praecipue in angulis paullo glandulari-puberulis; pappi setis ca. 27 inaequalibus barbellatis.— PERU: at Chachapoyas, *Mathews*, no. 1370 (K.). Apparently nearest *O. Sodiroi* Hieron. but differing in its firmer-textured broader leaves which are subtruncate at base and more coarsely crenate-serrate.

Eupatorium (§ **Subimbricata**) **anisodontum**, spec. nov., habitu indumento, aliis multis *E. Gascae* subtus descriptae simile sed differt conspicue foliis triangulari-hastatis caudato-acuminatis basi profundius cordatis irregulariter grosse dentatis (dentibus aliis parvis plerisque rotundatis aliis acutis majusculis usque ad 1 cm. longis); petiolo 3–5 cm. longo; inflorescentia corymbosa bene pedunculata densissima ca. 1 dm. diametro valde convexa; capitulis 1 cm. longis ca. 10-floris; involucri campanulato-subcylindrici squamis ca. 14 stramineo-viridibus tenuibus puberulis plerisque ovatis obtusis, intimis angustioribus acutis; corollis glabris 6 mm. longis tubulosis leviter sursum ampliatis; achaeniis 4 mm. longis glabris; pappi setis ca. 50 stramineo-albidis inaequalibus.— PERU: Province of Chachapoyas, *Mathews*, 1836. 87. H. (K., phot. and fragm. Gr.).

E. ARGUANS Robinson, Proc. Am. Acad. liv. 237, 288 (1918). The type of this recently described species, namely *Triana*, no. 1191 (K.), though believed to have come from Colombia, was unaccompanied by data as to the exact locality of collection. Happily the species has now been rediscovered by Bro. Ariste-Joseph, being represented by his no. A250 (Gr.) from Guadalupe, near Bogotá. This specimen recently received from the U. S. National Museum proves to agree in all features very closely with the original plant of Triana. The species may therefore be definitely located as occurring in the mountains near Bogotá in the Department of Cundinamarca.

E. (§ **Eximbricata**) **Aristei**, spec. nov., fruticosum patenter ramosum; caulibus teretibus griseo-brunnescentibus glabratis cum lenticellis instructis; ramis oppositis curvato-adscendentibus brevissime puberulis vel juventate sordide tomentellis vel griseo-tomentosis foliosis; foliis oppositis lanceolato-oblongis attenuatis acutis basi rotundatis crenulatis vel obtuse serrulatis vel subintegris coriaceis 7.5–9.5 cm. longis 2.5–3 cm. latis penniveniis supra (costa media paullo puberula excepta) glabris sublucidis sub lente valde reticulatis (areolis minutis, vix 0.3 mm. diametro), subtus paullo pallidioribus laxe pubescentibus obscuris; petiolo pubescente gracili 11–17 mm. longo; corymbis terminalibus planiusculis vel modice convexis multicapitulatis sordide pubescenti-tomentosis; bracteolis linearibus saepius patentibus curvatis stramineis firmiusculis; capitulis 10–12-floris; involucri squamis ca. 13 ovato-oblongis obtusis ciliatis dorso laxe pubescentibus; corollis albidis 4 mm. longis, tubo proprio 1 mm. longo glanduloso-atomifero, faucibus paullo ampliatis 3 mm. longis glabris; pappi setis ca. 17, sordide albis barbellatis; achaeniis (immaturis) ca. 2 mm. longis in angulis cum glandulis globosis minutis sessilibus instructis.— COLOMBIA: at Verjón, Sept., 1917, *Bro. Ariste-Joseph*, no. 150A (Gr.).

The leaves of this species vary from crenulate or serrulate to essentially entire and at different stages of development their reticulation, as seen from above, is of strikingly different size of mesh. These differences, however, are very clearly those of age or of individual development and have no classificatory worth. *E. Aristei* is clearly related to the Venezuelan *E. Jahnii* Robinson, Proc. Am. Acad. liv. 248 (1918), but that species has leaves of a more elliptic-oblong contour, scarcely more narrowed to the apex than to the base, their teeth somewhat larger, more salient, and mucronate, the petioles shorter, and the heads only about 6-flowered.

E. (§ **Subimbricata**) **Bridgesii**, spec. nov., ut videtur herbaceum (basi ignota) gracile erectum 3 dm. vel ultra altum; caule tereti 2–3 mm. diametro stramineo-viridi tenuiter puberulo (pilis brevissimis incurvatis); internodiis usque ad 6 cm. longis; foliis suboppositis vel superne alternis rhombeo- vel lanceolato-ovatis utroque attenuatis paullo supra basin 3–5-nervatis 5 cm. longis 1.5–1.8 cm. latis tenuiter membranaceis pallide viridibus supra praesertim ad marginem scabro-puberulis subtus vix pallidioribus in nervis et venis principibus albidis puberulis margine cum dentibus parvis mucronatis paucissimis (saepe utroque 1–2 solis) instructis; petiolo supra canaliculato anguste marginato 4–7 mm. longo; corymbo composito planiusculo 2–2.5 dm. diametro, ramis et pedicellis (3–6 mm. longis) stramineo-viridibus puberulis; capitulis numerosis ca. 25-floris 7 mm. altis 4 mm. diametro; involucri squamis ca. 22 valde inaequalibus gradatis pallide viridibus plerisque 2-costulatis, extimis brevibus ovato-lanceolatis acuminatis dorso paullo adpresse puberulis apice squarroso-recurvatis, intermediis gradatim longioribus lanceolatis attenuatis subglabris, intimis lineari-oblongis acutis glabris; corollis roseis 4 mm. longis glabris sursum sensim ampliatis limbo extus paullo granulatis; achaenis atro-brunneis in costis pallidioribus sursum hispidulis; pappi setis ca. 27 vix 2.5 mm. longis barbellatis.— BOLIVIA: without locality, *Bridges* (K., phot. and fragm. Gr.).

E. CELTIDIFOLIUM Lam., var. HIRTELLUM Robinson, Proc. Am. Acad.

liv. 238, 311 (1918). This continental variety of the West Indian *E. celtidifolium* was, at the time of its description, known to the writer only from Colombia, where it had been twice collected near Santa Marta. It has been rediscovered in northern Venezuela at Lower Cotiza (Agr. Exp. Sta.), near Caracas, on the sunny side of a gorge, alt. 925 m., *Pittier*, no. 7897 (Gr.). Prof. Pittier describes it as a shrub or small tree, 3–4 m. high, with small greenish flowers on drooping branchlets.

E. (§ **Eximbricata**) **choricephaloides,** spec. nov., ut videtur herbaceum (basi ignota) verisimiliter 1 m. vel ultra altitudine; caule ramisque gracilibus flexuosis teretibus dense patenterque glanduloso-puberulis; internodiis usque ad 1 dm. longis; foliis oppositis deltoideo-ovatis apice acuminatis basi subtruncatis vel leviter patenterque cordatis lateraliter serratis- vel crenato-dentatis (dentibus inaequalibus rotundatis vel subacutis utroque ca. 10) membranaceis supra dense puberulis et in nervis venisque maximis villosis ca. 6 cm. longis et 4.5 cm. latis a basi 3-nerviis; petiolo 1–3 cm. longo glanduloso-tomentello; panicula laxissima foliaceo-bracteata 4 dm. alta 3 dm. diametro dense glanduloso-puberula; pedicellis filiformibus 1–2.6 cm. longis; capitulis dissitis ca. 7 mm. altis 25–30-floris; involucri campanulati squamis subaequalibus anguste oblongis vel oblanceolatis acutis viridibus 2–3-costatis dorso puberulis ca. 4 mm. longis apicem versus scariosis et erosis; corollis albis glabris, tubo proprio fauces ampliatas subaequante; dentibus limbi brevissimis ca. 0.3 mm. longis; achaeniis pallide brunneis 1.8 mm. longis in angulis hispidulis in faciebus concavis laevibus; pappi setis ca. 12 albidis minute scabridulis.— PERU: Department Amazonas, Province Chachapoyas, *Mathews* (K., phot. and fragm. Gr.). In habit and very loose inflorescence recalling the Mexican *E. choricephalum* Robinson, but differing in many points.

E. (§ **Subimbricata**) **coelocaule,** spec. nov., fruticosum; ramis curvato-adscendentibus teretibus robustis griseo-brunneis maturitate glabris laevibus fistulosis; internodiis 3–4 cm. longis; foliis oppositis lanceolato-oblongis vel rhombeo-lanceolatis longe acuminatis serratis (basi cuneata integra excepta) penniveniis usque ad 17 cm. longis et 6 cm. latis firmiusculis glaberrimis (maturitate) supra viridibus subtus pallidioribus glaucis puncticulatis; petiolo 1–4 cm. longo rubescente glabro; foliis supremis minoribus lanceolatis integerrimis; corymbo sessili terminali composito leviter convexo 1–1.5 dm. lato dense crispeque fulvo-tomentello; capitulis numerosissimis inter se binis vel trinis aggregatis sessilibus vel brevissime pedicellatis ca. 5-floris 10–12 mm. longis 2–3 mm. crassis graciliter cylindricis; involucri squamis ca. 12 subtriseriatis stramineis extimis ovatis acutiusculis interioribus gradatim longioribus anguste ellipticis delicatule striatulis apice obtusiusculis glabris; corollis 7 mm. longis sensim sursum paullo ampliatis glabris; antheris fere liberis longiuscule appendiculatis; achaeniis 3 mm. longis nigris lucidis obsolete granulatis; pappi setis ca. 30 inaequalibus stramineo-albidis.—PERU: Province of Chachapoyas, 1835, *Mathews*, no. 1373 (K., phot. Gr.). In form the involucre of this species is nearly cylindrical yet the scales do not have the texture, nervation, or deciduous nature of § *Cylindrocephala* and it seems best placed in § *Subimbricata*.

E. (§ **Subimbricata**) **Cookii,** spec. nov., fruticosum dense villosum, pilis primo longis gracillimis sub lente pulcherrime purpureo-articulatis tardius plus minusve attritis; caule tereti purpureo, internodiis usque ad 13 cm. vel ultra longis; ramis patentibus vel divaricatis plerisque curvato-adscendentibus; foliis ovatis acuminatis crenato-serratis (basi rotundata vel subcordata integra excepta) ima a basi 3(–5)-nervatis supra obscure viridibus et crispe puberulis subtus paullo pallidioribus sordide pubescentibus et glandulari-atomiferis 3–6 cm. longis 1.8–3.6 mm. latis; petiolo 1–2 cm. longo; pedicellis 1–4 mm. longis; capitulis glomeratis ca. 22-floris 7 mm. altis 4 mm. diametro; involucri campanulati ca. triseriati substraminei squamis ca. 19 valde inaequalibus, interioribus anguste oblongis apice rotundatis sed apiculatis saepe 2-costulatis viridibus vel purpureo-tinctis saepissime dorso granulatis vel leviter puberulis, intermediis et extimis gradatim brevioribus acuminatis ciliatis distinctius puberulis; corollis albis 3.8 mm. longis, tubo proprio vix 0.8 mm. longo, faucibus sensim ampliatis cylindricis, limbo hispidulo; achaeniis fusco-brunneis 1.5 mm. longis glabris; pappi setis ca. 27 laete albis minute scabidis.—PERU: Dept. Cuzco: Lucumayo Valley, alt. 1800–3600 m., 19 June, 1915, *O. F. Cook & G. B. Gilbert*, no. 1352 (TYPE, U. S., phot. Gr.). An old specimen, *Mathews*, no. 1126, collected at "Andimarca" and now in the Kew Herbarium, is apparently of this species. It differs chiefly in its shorter pubescence, which on the mature branches is scarcely more than a somewhat glandular puberulence. However, the longer and articulated pubescence, so copious on the Cook & Gilbert specimen, is found to some extent on the younger petioles, etc., of the Mathews plant, and its absence from the older parts may well be due to some form of attrition, disarticulation, or absorption. At all events, species have been noted elsewhere in the genus which appear to pass from a stage with copious articulated and non-glandular pubescence to one in which glandular puberulence is prevalent.

It seems probable that the "Andimarca" mentioned is Andamarca in the Department of Junin, a locality about 300 km. distant from Lucumayo Valley. The leaves in the Mathews plant are somewhat more deeply cordate and a little more bluntly toothed than in the Cook & Gilbert plant, but in the presence of pretty close agreement in all essential features these differences do not appear of much classificatory moment. There being, however, a slight doubt as to the locality of the Mathews specimen, it has seemed better to select the other as the type, particularly as a species (described below) had already been dedicated to Alexander Mathews.

E. CUTERVENSE Hieron. in Engl. Bot. Jahrb. xl. 383 (1908). This species was originally found from near Cutervo and Tambillo in Northern Peru by von Jelski. It is described (Hieron. l. c.) as having the stem up to 2.5 cm. in thickness. This certainly must be a clerical error for 2.5 mm., since surely no plant with a stem 2.5 cm. thick would be characterized as "suffruticosa vel fruticulosa." The species is furthermore described as having the leaves sessile or short-petioled, the petioles scarcely 2 mm. long. The corolla is said to be glabrous externally, the pappus whitish, and the achenes roughish on the upper part of the angles. So far as known to the writer the species has not been subsequently reported. However, specimens have now been collected in the vicinity of Nabón, Ecuador, 25–26 Sept., *J. N. Rose, A. Pachano & G. Rose*, no. 23,014 (Gr., U. S., N. Y.), which correspond closely in nearly all described features to *E. cutervense*, having the same much-branched habit, small, roundish-ovate subcordate, acute to very shortly acuminate leaves, which are similarly crenate on the somewhat revolute margins. The articulated pubescence is the same, the inflorescence, and numbers of florets (35), scales (15–16) and pappus bristles (18) fairly approximate those given by Hieronymus. The chief differences observed are as follows: the leaves in the Ecuadorian plant are never really sessile and the petioles are sometimes as much as 4 mm. long; the corolla-teeth are dorsally hispid; the achenes are hispidulous on the angles throughout their length; and the pappus-bristles are distinctly roseate. These minor differences, however, do not appear sufficient to justify the separation of the Ecuadorian plant, at least until it is possible to have it carefully compared with the type of *E. cutervense*. It is therefore provisionally referred to that species.

E. (§ **Eximbricata**) **dasyneurum**, spec. nov., herbaceum erectum virgatum ut videtur annuum 6 dm. vel ultra altum; caule tereti densissime piloso, pilis patentibus attenuatis articulatis saepe curva-

tis; internodiis 1.5–5 cm. longis; foliis oppositis breviter petiolatis ovatis acuminatis obscure serratis basi rotundatis vel levissime cordatis 2–3 cm. longis 1.5–2 cm. latis supra subglabris solum sparse in nervis venisque pilosis bullato-rugosis post exsiccationem nigrescentibus subtus pallidioribus praecipue in nervis venisque sordido-hirsutis (pilis densis attenuatis curvatis maturitate firmiusculis) a basi 3-nervatis submembranaceis margine juventate forte revolutis; petiolo 2–4 mm. longo hirsuto; inflorescentia e corymbo unico composito terminali convexo 1 dm. diametro sistente vel e corymbis minoribus densiusculis pluribus ramos terminantibus paniculam plus minusve elongatam conjunctim formantibus; capitulis numerosis 7 mm. longis et crassis ca. 30-floris pedicellatis; involucri campanulati squamis ca. 17 plerisque aequalibus lanceolato- vel oblongo-linearibus acute mucronatis substramineis 2–3-costulatis ciliolatis dorso plus minusve sordido-pubescentibus, 2–3 extimis angustissimis subulatis paullo brevioribus; corollis albis ca. 3.8 mm. longis limbum versus paullo hispidulis; tubo proprio fauces distincte ampliatas subaequante; achaeniis 1.8 mm. longis maturitate nigris hispidulis; pappi setis ca. 18 albidis corollam subaequantibus apicem versus sensim incrassatis.— COLOMBIA: in field, Antizales, Dept. Bolívar, alt. 1500–1800 m., *Dr. F. W. Pennell*, no. 4460 (N. Y., Gr.).

This species is obviously close to *E. sotarense* Hieron., which, however, has the pubescence on the pedicels of wide-spreading gland-tipped hairs, while in *E. dasyneurum* it is of incurved non-capitate hairs. *E. sotarense* also has the leaves, if one may judge from its description, considerably less pubescent beneath, and the involucral scales less attenuate.

E. DENDROIDES Spreng. Syst. iii. 415 (1826). In treating this species in his recent study of the Ecuadorian Eupatoriums, Proc. Am. Acad. liv. 359 (1918), the writer by oversight omitted a reference to Bentham, Pl. Hartw. 135 (1844), where there is a record of its collection in the mountains of Loja, by Hartweg, no. 755. There is an unnumbered Hartweg specimen of the species from the same locality in the herbarium of the New York Botanical Garden. By error in the paper mentioned the species was given a parenthetical authority (HBK.) which should have been deleted.

E. DOMBEYANUM DC. Prod. v. 167 (1836). This species, described from material collected in South America by Dombey, but without indication of locality or even of the country, has never been satisfactorily identified. It belongs to § *Eximbricata* and is described as "fruticosum glaberrimum." The heads are in a loose panicle, many

of them nodding on curved or flexuous pedicels; the leaves are ovate and apparently membranaceous.

Entirely glabrous and at the same time thin-leaved members of § *Eximbricata* are not very numerous. Finding a plant of this general nature among the specimens collected by Mr. H. H. Smith near Santa Marta, Colombia, the writer (Proc. Am. Acad. liv. 315) ventured to place it doubtfully in *E. Dombeyanum* with which it appeared to have many points in common. However, further study of this and the related plants from Peru renders it decidedly unlikely that the Colombian plant can have anything to do with the original *E. Dombeyanum.* It is accordingly characterized below as a new species under the name *E. psilodorum.*

Similar efforts to identify with *E. Dombeyanum* DC. certain Peruvian plants, notably Weberbauer's nos. 860, 2766, and 3253, have likewise failed. No. 860 (described below as *E. stictophyllum*), while possessing rather closely the leaf-contour of *E. Dombeyanum* has considerably denser and corymbiform inflorescences, the young stems, branches, and pedicels are pulverulent-puberulent to a degree that DeCandolle would scarcely have described as "Glaberrimum;" the achenes are covered with sessile glands and the petioles are muriculate, which is clearly not the case in the type of *E. Dombeyanum* of which there is a photograph in the Gray Herbarium; finally, the leaves are rather conspicuously dark-punctate beneath — a feature, which had it been equally manifest in the type of *E. Dombeyanum,* would almost certainly have been mentioned by so careful a writer as DeCandolle.

Weberbauer's nearly related nos. 2766 and 3253 (below described as *E. simulans*) differ from *E. Dombeyanum* in having lanceolate (rather than ovate) leaves, which are pinnately veined rather than 3-nerved from above the base; the stems are much more leafy, with mostly short internodes; the branches, petioles, pedicels, and achenes are all perceptibly granular-puberulent, and the involucral scales are more pubescent than is indicated by DeCandolle in his character of *E. Dombeyanum.*

After prolonged effort to take into account all characters and make reasonable allowance for individual variation, it has seemed impossible to refer any of these specimens to *E. Dombeyanum.* On the other hand, they are so close as to give added strength to the view that the species will ultimately be found in Peru, where the greater part of Dombey's South American collecting was accomplished.

E. (**§ Subimbricata**) **drepanoides**, spec. nov., fruticosum usque

ad 4 m. altum glabrum; ramis curvatis subteretibus foliosis post exsiccationem paullo costulatis; internodiis 1–2 cm. longis; foliis oppositis lanceolatis falcatis acuminatis basi acutis serratis (dentibus 0.6 mm. altis 2–3 mm. latis) firmiusculis supra minute atomiferis planis (sine reticulatione prominulente venarum) subtus paullo pallidioribus supra basin 3-nerviis demum penniveniis 7–9 cm. longis 1.8–2.1 cm. latis; petiolo glabro ca. 2 cm. longo; corymbis terminalibus sessilibus compositis convexis 1 dm. vel ultra diametro, ramulis pedicellisque arachnoideo-puberulis; capitulis numerosis ca. 7-floris 6 mm. (valde immaturis) longis 3.5 mm. diametro; involucri anguste campanulati squamis ca. 13 ovato-oblongis acutiusculis striatulis purpurascenti-stramineis arachnoideis ciliatis; corollis graciliter tubulosis conspicue 5-nervatis granulatis; dentibus limbi 5 lanceolato-oblongis; achaeniis (valde immaturis) deorsum decrescentibus granulatis; pappi setis ca. 23.— Peru: open woods by a brook, Comin, Prov. Huari, Dept. Ancachs, alt. 3600–3700 m., 18 Apr., 1903, *Weberbauer*, no. 2918a (Berl., phot. and fragm. Gr.). This species, though pretty clearly undescribed, is represented only by fragmentary and immature material.

Near *E. coelocaule*, described above, but with much smaller and relatively narrower and duller leaves, arachnoid-ciliate involucral scales, etc.

E. (§ **Subimbricata**) **endytum**, spec. nov., fruticosum 1–2 m. altum breviter ferrugineo-velutinum; caulibus teretibus fistulosis usque in paniculam foliosis; internodiis 5–8 cm. longis; foliis oppositis oblongo-ovatis acutis serrulatis (dentibus ca. 0.7 mm. altis 2–3 mm. latis) basi rotundatis utrinque tomentellis subtus distincte pallidioribus penniveniis 8–12 cm. longis 3.6–6 cm. latis paullo firmiusculis vix membranaceis; petiolo subtereti 1.5–2 cm. longo; panicula ovoidea oppositiramea ca. 1.5–1.8 dm. alta et diametro apice rotundata; ramis inflorescentiae late patentibus infra nudis solum apicem versus capituliferis; capitulis ca. 37-floris 7 mm. longis 7 mm. diametro; involucri campanulati squamis ca. 21 acutiusculis 2–3-seriatim imbricatis sed vix gradatis exterioribus ovato-oblongis persistentibus dorso brunneo-tomentellis nervis obscurissimis, squamis interioribus angustioribus parce pubentibus mox deciduis; corollis sordide albis 3.5 mm. longis sursum gradatim ampliatis glabris; styli ramis filiformibus rectiusculis laevibus vix clavellatis; antheris apice cum appendice tenui ovata instructis; achaeniis immaturis 1.5 mm. longis glabris; pappi setis ca. 20 corollam subaequantibus paullo scabratis.— Peru: between Sandia and the tambo Azalaya, on the way from Sandia to Chunchusmayo, among bushes at an altitude of 1500–2000 m., 5 June 1902, *Dr. A. Weberbauer*, no. 1074 (Berl., phot. and fragm. Gr.).

E. (§ **Cylindrocephala**) **eripsimum**, spec. nov., suffruticosum 3–5 dm. altum ramosum decumbens; caule tereti et ramis saepissime curvato-adscendentibus foliosis teretibus puberulis; foliis parvulis oppositis lanceolatis vix petiolatis integriusculis vel utroque obscure 1–3 dentatis membranaceis 2.5–3.5 cm. longis (internodiis subaequantibus) 5–10 mm. latis a basi cuneata 3-nervatis supra glaberrimis sed saepe minute pustulatis subtus in nervis obscure appresseque pubentibus; corymbis terminalibus 3–8-capitulatis laxis; capitulis ca. 44-floris; involucri campanulato-cylindrici squamis ca. 3–4-seriatim gradatis, extimis ovato-oblongis, intimis oblongo-linearibus, omnibus caducissimis praeter marginem ciliolatum glaberrimis stramineis 1–4-costulatis lucidulis; receptaculo planiusculo sed apice pedicelli post delapsum squamarum denudato subcylindrico; corollis lilacinis (post exsiccationem nigrescentibus) graciliter tubulosis vix sursum ampliatis glaberrimis 8.5 mm. longis; dentibus limbi 5 angustis recurvatis; styli ramis longissimis leviter clavellatis rectiusculis patentibus ca. 6 cm. longis; achaeniis gracilibus deorsum paullo decrescentibus adpresse puberulis 4.5 mm. longis; pappi setis ca. 35 albis barbellatis subaequalibus corollam subaequantibus.— PERU: in open places among grasses, bromeliads, and cacti, alt. 2200–2500 m., Caraz, Dept. Ancachs, 19 May, 1903, *Weberbauer*, no. 3003 (Berl., phot. and fragm. Gr.).

A species somewhat resembling *E. serratuloides* HBK. but with narrower less toothed leaves somewhat hairy beneath, and considerably longer florets. Like *E. serratuloides* it has somewhat the habit of § *Praxelis*. The excessively caducous nature of the scales and florets, giving the specimens a dilapidated appearance, has suggested the specific name, which, happily, is not likely to have been hitherto employed.

E. (§ **Eximbricata**) **flexile**, spec. nov., suffruticosum gracile subscandens 2 m. altum; caulibus teretibus flexuosis molliter breviterque sordido-tomentellis; foliis ovatis caudato-acuminatis cordatis a basi 5–7-nerviis tenuibus supra puberulis subtus praecipue in nervis griseo-pubescentibus ca. 6 cm. longis 3 cm. latis plus minusve lateraliter leviter remoteque pauci-dentatis; petiolo gracili flexuoso ca. 1 cm. longo; panicula ampla pyramidata 2–4 dm. alta 1.5–2.5 dm. diametro laxa foliaceo-bracteatâ; pedicellis filiformibus flexuosis griseo-tomentellis; capitulis ca. 20-floris 7 mm. altis 6 mm. diametro; involucri campanulati squamis ca. 16, intimis lineari-oblongis obtusiusculis sed

acute mucronatis plerisque ca. 5 mm. longis; corollis viridescenti-flavidulis tubulosis gradatim paullulo sursum ampliatis 3.5 mm. longis; dentibus limbi 5 breviter deltoideis extus granulatis; antheris apice breviter crassiusculeque appendiculatis; styli ramis filiformibus paullo ad apicem incrassatis et nigrescentibus, achaeniis griseis 2.7 mm. longis deorsum decrescentibus angulis hispidulis; pappi setis ca. 32 corollam subaequantibus albis vix barbellatis.— PERU: woods near a river, Caraz, Dept. Ancachs, alt. 2200 m., May 21, 1903, *Weberbauer*, no. 3027 (Berl., phot. and fragm. Gr.).

In many respects similar to *E. solidaginoides* HBK. but not as yet connected by intermediates and rather too different in appearance to be regarded as a variety until intergradation has been demonstrated. In *E. solidaginoides* the branches of the inflorescence present a somewhat racemose appearance, the heads being rather evenly distributed along them. In *E. flexile* this is not the case, the branches being floriferous chiefly toward the tip. In *E. solidaginoides* the heads are 10–15-flowered and the involucre (when fresh or softened by boiling) is rather narrowly campanulate, the scales being thin. In *E. flexile* the heads are about 20-flowered, the involucre broadly campanulate, and the scales somewhat firmer in texture. The achenes of *E. solidaginoides* are from 1.8–2.1 mm. long, while in *E. flexile* they are about 2.7 mm. long.

E. FULIGINOSUM HBK. Nov. Gen et Spec. iv. 110 (1820); Robinson, Proc. Am. Acad. liv. 302 (1918). Of this little known Colombian species further material is now at hand extending its recorded distribution to the department of Huila, where these specimens were collected in forests on the Cordillera Oriental, east of Neiva, Aug. 1–8, 1917, by Drs. Rusby & Pennell, nos. 573 (N.Y.) and 977 (N.Y.). In all characters these correspond closely with a photograph (Gr.) of the type (Par.) and of their specific identity there can be no doubt. However, the receptacle bears a short, thin and very fugacious pubescence which commonly disappears so completely with the fall of the achenes as to leave the receptacle entirely glabrous and merely punctate, as it was described in the original diagnosis of the species. Technically, having a hairy receptacle, *E. fuliginosum* must be transferred to § *Hebeclinium*. In practice, however, it will be well to retain at least a cross-reference to it in § *Subimbricatum*, for the differential character from its obscurity and fugacious nature is here very likely to be overlooked. The labels of the material at hand record the plant as a tall shrub with greenish-yellow flowers.

E. (§ **Subimbricata**) **Gascae**, spec. nov., dense villoso-tomento-

sum, indumento brunneo vel fuscescenti patente glandulari longiusculo; caule (basi ignota) tereti 6 mm. crasso folioso medulloso; foliis oppositis petiolatis ovatis cordatis (sinu angusto) acutiusculis ca. 1 dm. longis 4–7 cm. latis supra atroviridibus conspicuiter bullato-rugosis puberulis subtus reticulatis sordido-tomentellis margine subregulariter dentatis, dentibus 1.5 mm. altis 3–4 mm. latis; petiolo 2–4 cm. longo dense piloso, pilis articulatis; corymbis terminalibus breviter pedunculatis densis 6–7 cm. diametro planiusculis vel modice convexis; capitulis ca. 22-floris 1.5 cm. longis 6 mm. crassis; involucri campanulato-subcylindrici squamis subtriseriatis ca. 21 ovatis acutis stramineo-viridibus costulato-striatulis dorso pubentibus, receptaculo plano; corollis graciliter tubulosis 5 mm. longis glabris paullulo sursum ampliatis; dentibus limbi 5 patentibus anguste deltoideis; achaeniis graciliter prismaticis 4 mm. longis; pappi setis ca. 45 stramineo-albidis firmiusculo-capillaribus vix scabratis.— PERU: Province of Chachapoyas, *Mathews* (TYPE, Gr.). The specimen at the Gray Herbarium bears no number, but an exact duplicate in the Kew Herbarium is marked "Mathews 1836. 91. H."

Though unusual to name plants for persons in no way connected with their discovery or investigation, the writer desires to perpetuate in the present interesting and attractive Peruvian species the memory of the extraordinary man, *Pedro de la Gasca* (born 1496, died 1569), styled by Charles V. the "President of the Royal Audience" and by the people of Peru "Father and Deliverer," concerning whom it may be said that he was the first Caucasian to reach Peru of whom the race has any reason to be proud.

E. (§ **Eximbricata**) **Gilbertii,** spec. nov., perenne vel fruticosum gracile glabriusculum; caule pallido-stramineo tereti ca. 2 mm. crasso flexuoso maturitate glaberrimo juventate sparse obscureque villosulo; foliis oppositis ovatis acutis vel leviter acuminatis mucronulato-serratis vel -crenatis (dentibus ca. 0.6 mm. altis inter se ca. 5 mm. distantibus) tenuibus membranaceis supra viridibus in nervis puberulis subtus distincte pallidioribus glabris 6–7 cm. longis 3–3.5 cm. latis basi rotundatis vel subcordatis integris; petiolo gracili ca. 1.7 cm. longo supra canaliculato puberulo; corymbis compositis planiusculis sublaxis obscure villosulis basi foliaceo-bracteatis; pedicellis filiformibus 4–10 mm. longis bracteolatis; capitulis parvis vix 5 mm. altis et diametro ca. 26-floris; involucri campanulati squamis ca. 17 subaequalibus oblongo-lanceolatis stramineo-viridibus saepius 2-costulatis et 3-veniis margine tenuibus subscariosis apice acutiusculis ciliatis dorso laxe pubescentibus (pilis tenuissimis fuscis non-glanduliferis); corollis albis ca. 3.3 mm. longis, tubo proprio glabro fauces campanulato-ampliatas subaequante, dentibus limbi hispidulis; achaeniis (valde immaturis) 1.2 mm. longis ut videtur glabris; pappi setis ca. 20 albis sursum hispidulis.— PERU: San Miguel, Urubamba Valley, Dept. Cuzco, alt. about 1800 m., 7 June, 1915, *O. F. Cook & G. B. Gilbert*, no. 1115 (U. S., phot. Gr.).

E. (§ **Eximbricata**) **gloeocladum,** spec. nov., fruticosum robustum vel arboreum; caulibus griseo-brunneis flexuosis teretibus juventate incurvo-puberulis glutinoso-viscosis tardius vix minute granulatis deinde omnino glabratis post exsiccationem longitudinaliter rugulosis usque ad 8 mm. diametro medullosis; medulla alba; internodiis 1–4 cm. longis; foliis ovato-oblongis argute acuminatis basi cuneatis 1–2 dm. longis 2.5–8 cm. latis integerrimis vel obsolete remoteque undulato-subdentatis penniveniis supra in nervo medio puberulis aliter glabris laevibus viridibus subtus pallidioribus reticulatis breviter denseque pubescentibus subcoriaceis; corymbis compositis valde convexis congestis ca. 1 dm. diametro a bracteis foliaceis plerumque superatis; pedicellis brevibus sordide pubescentibus; capitulis ca. 9-floris 8 mm. altis 4 mm. diametro; involucri campanulati squamis ca. 14 eroso-ciliatis dorso puberulis vel granulatis saepe aliquanto viscidulis vix imbricatis sed paucis extimis brevioribus et subgradatis; corollis verisimiliter albis vel roseis 5.5 mm. longis glabris paullo a basi ad limbum gradatim ampliatis; styli ramis vix clavellatis; achaeniis ca. 2.5 mm. longis juventate in angulis glanduloso-granulatis tardius glabratis laevibus brunneis vel fuscis; pappi setis ca. 26 stramineo-albidis vix scabratis.— *E. trichotomum* Sch. Bip. Bull. Soc. Bot. Fr. xii. 81 (1865), & Linnaea, xxxiv. 535 (1865–66), without char., not Sch. Bip. in sched. Riedel ex Bak. in Mart. Fl. Bras. vi. pt. 2, 305 (1876).— BOLIVIA: Department La Paz, Province Larecaja, in the neighborhood of Sorata, in temperate region, 2700–3000 m., on Mt. Chilieca, near Challapampa, July–Sept. 1858, *G. Mandon*, no. 258 (Gr., N. Y.).

In giving this marked species what is believed to be its first description the author has duly considered the possibility of retaining the name *E. trichotomum* long ago applied to the plant by Schultz-Bipontinus. However, Schultz working in his usual rapid manner not only failed to give any diagnosis of this *E. trichotomum* but soon employed in manuscript the same binomial for a wholly different plant. *E. trichotomum* Sch. Bip. in the latter sense has seen light and received a certain amount of definition in Mart. Fl. Bras. vi. pt. 2, 305 (1876) where established by Baker as a variety of *E. Vauthierianum*

DC. Under these circumstances it appears unwise to describe a *Eupatorium trichotomum* Sch. Bip. in another sense. Therefore, an entirely new name is here chosen.

E. (§ **Subimbricata**) **gracilentum**, spec. nov., herbaceum perenne gracile 3–4 dm. vel ultra altum; radice e fibris paucis gracilibus lignescentibus elongatis sistente; caulibus solitariis vel pluribus teretibus purpurascentibus sordide puberulis vel tomentellis; foliis oppositis detoideo-ovatis acutis vel acuminatis crenato-dentatis basi rotundatis vel truncatis vel subcordatis integris 1.8–3 cm. longis 1.1–2.3 cm. latis tenuibus membranaceis supra pubescentibus subtus griseo-tomentosis ima a basi 3-nervatis; petiolo tenui 4–8 mm. longo griseo-pubescente; capitulis ca. 25-floris 6 mm. longis 3.7 mm. diametro in cymos laxos 1–4-capitulatos ad apices ramorum patentium gestis in paniculam laxam foliaceo-bracteatam dispositis; involucri anguste campanulati squamis ca. 19 subtriseriatis stramineis, interioribus lanceolato-ellipticis obtusis laeviusculis 2–3-costulatis scarioso-marginatis, intermediis et extimis gradatim brevioribus ovato-lanceolatis acutis vel acuminatis brunneo-puberulis; corollis verisimiliter albis limbo excepto glabris; tubo proprio 0.7 mm. longo faucibus paullo ampliatis cylindricis 2.3 mm. longis; achaeniis 1.5 mm. longis fusco-brunneis, costis paullo pallidioribus hispidulis; pappi setis ca. 27 delicatule capillaribus sublaevibus albis.— PERU: without locality, *Mathews* (N. Y., phot. Gr.).

E. (§ **Subimbricata**) **hylophilum**, spec. nov., herbaceum ut videtur erectum obscure praecipue in novellis pulverulenti-puberulum; caule tereti flexuoso folioso pallide griseo-brunneo; foliis oppositis anguste lanceolatis utroque attenuatis peracutis subsessilibus remote denticulatis (dentibus 0.5–0.7 mm. altis inter sese ca. 1 cm. distantibus) supra glaberrimis viridibus subtus distincte pallidioribus nervo medio obsolete puberulis penniveniis (venis intra marginem anastomosantibus) ca. 1 dm. longis 1 cm. latis tenuiter membranaceis; panicula terminali ovoidea subthyrsoidea; ramulis pedicellisque fere filiformibus incurvato-puberulis; capitulis 5–6 mm. altis 2.6 mm. diametro ca. 21-floris; involucri squamis ca. 25, 3–4-seriatis gradatis stramineis tenuibus striatulis apicem versus griseo-tomentellis; corollis albis glabris graciliter tubulosis 3 mm. longis paullulo apicem versus ampliatis, dentibus limbi 5 deltoideis dorso granulatis; styli ramis cum appendice filiformi flexuosa papillato-puberula munitis; achaeniis 1.3 mm. longis nigris in angulis pallidioribus parce hispidulis; pappi setis capillaribus albis vix scabratis ca. 28 mm. longis.— COLOMBIA: forest, alt. 150–300 m., Boca Esmeralda, on Rio Sinu, Dept. Bolívar,

1 Mar. 1918, *Dr. F. W. Pennell*, no. 4561 (N. Y., phot. and fragm. Gr.).

This species with something of the habit of a *Solidago* is clearly of the affinity of *E. elatum* Steetz, *E. turbacense* Hieron., *E. Squiresii* Rusby, and *E. tovarense* Robinson. From all these it differs in its much narrower, much more gradually attenuate, and in texture more delicate leaves of somewhat different venation and essentially glabrous beneath. In style-branches and some other features it recalls *E. Trianae* Robinson but differs in its opposite leaves, slender-pedicelled heads, etc.

E. (§ **Subimbricata**) **hypargyrum**, spec. nov., fruticosum; ramis curvato-adscendentibus juventate subadpresso-lanatis tardius glabratis brunneo-griseis cum lenticellis munitis; foliis oppositis petiolatis rhomboideo-ovatis falcato-acuminatis basi cuneatis obscure remoteque cuspidato-serrulatis supra glabris viridibus subtus arcte lepidoto- vel pannoso-lanatis argyreis penniveniis (venarum jugis 1–2 paullo supra basin aliis multo longioribus, omnibus subglabris sursum incurvatis anastomosantibus) submembranaceis usque ad 1 dm. longis et 3.8 cm. latis; petiolo 1–1.6 cm. longo; corymbo terminali composito ca. 1 dm. diametro convexo multicapitulato; capitulis ca. 9-floris ca. 1 cm. longis 3.8 mm. diametro; involucri subcylindrico-campanulati squamis 3-seriatis ovato-lanceolatis vel anguste oblongis acutiusculis obscure 3-nervatis arachnoideo-ciliatis aliter subglabris viridibus saepius purpureo-tinctis; corollis laete purpureis glabris subcylindricis 6 mm. longis paullo sursum ampliatis sine faucibus distinctis; achaeniis (valde immaturis) 2.7 mm. longis granulatis; pappi setis ca. 28 laete stramineo-albis basin versus paullo incrassatis corollam subaequantibus vix scabratis.—ECUADOR: vicinity of Azogues, 16–17 Sept. 1918, *J. N. & G. Rose*, no. 22,774 (Gr., U. S., N. Y.). Not closely related to any hitherto described Ecuadorian species. The lower surface of the leaves is closely covered with a somewhat lepidote pubescence except the midnerve and chief veins which are essentially bare. In habit and foliage the species recalls several Eupatoriums of Peru and Bolivia, but these are of § *Eximbricata* rather than § *Subimbricata* to which the present species is clearly referable.

E. (§ **Eximbricata**) **iodotrichum**, spec. nov., herbaceum perenne ca. 8 dm. altum; caule tereti erecto vel paullo decumbente supra mediam partem oppositirameo atropurpureo villosulo, pilis aliis delicatulis attenuatis purpureo-articulatis aliis (plerumque in parte superiore) rectis divaricatim patentibus purpureis glanduloso-capitatis;

internodiis 4–11 cm. longis; foliis deltoideo- vel subreniformi-ovatis brevissime acuminatis cordatis grosse dentatis (dentibus utroque 5–7 vix acutis 1–1.8 mm. altis 3–4 mm. latis) a basi 3(–5)-nerviis utrinque sparse pubescentibus viridibus subtus paullo pallidioribus 2–2.5 cm. longis 2.5–2.8 cm. latis; petiolo ca. 1 cm. longo pubescente; cymis plerisque 3-capitulatis ramos graciles curvato-adscendentes folioso-bracteatos terminantibus laxis; pedicellis 1–1.5 cm. longis filiformibus dense cum glandulis stipitatis instructis; capitulis ca. 46-floris 1 cm. altis et crassis; involucri campanulati squamis ca. 20 ca. 6 mm. longis lanceolato-linearibus subaequalibus attenuatis glandulari-puberulis saepissime atropurpureis basin versus plerisque 2-costulatis; corollis 5–5.5 mm. longis ut videtur albis limbo hispidis aliter glabris; tubo proprio ca. 2.5 mm. longo, faucibus distincte ampliatis subcylindricis ca. 2.8 mm. longis; achaeniis (immaturis) ca. 2.5 mm. longis fuscescentibus in angulis sursum hispidulis; pappi setis ca. 17 caducis pulcherrime roseis corollam subaequantibus.— ECUADOR: vicinity of Cañar, 15–16 Sept., 1918, *J. N. & G. Rose*, no. 22,766 (Gr., U. S., N. Y.). Clearly of the affinity of *E. gracile* HBK., *E. prunellaefolium* HBK. etc., but characterized by leaves which are broader than long and rather deeply cordate.

E. IRESINOIDES HBK. Nov. Gen. et Spec. iv. 106, t. 340 (1820); Robinson, Proc. Am. Acad. liv. 285, 338 (1918), where vars. *villosum* Steetz, *glabrescens* Steetz, and *breviflorum* Hieron. are distinguished. Of this always characteristic yet somewhat polymorphous species it is now possible to add another variety as follows:

Var. **adenotrichum**, var. nov., habitu foliisque fere ut apud var. *a villosum* differt tamen ramulis pedicellisque inflorescentiae glandulari-puberulis; capitulis paullo minoribus ca. 5.3 mm. longis; involucri squamis numerosioribus ca. 22 omnibus peracutis vel attenuatis.— COLOMBIA: in a ravine, alt. 700–1500 m., Cordillera Oriental, east of Neiva, Dept. Huila, *Rusby & Pennell*, no. 480 (N. Y., Gr.).

It is, of course, impossible to say whether the glandular indumentum will prove to be regularly correlated with the slightly smaller heads and more numerous and sharper-pointed scales or whether these features will be found to have independent variability. However, the combination present in the material at hand seems to warrant varietal rather than formal distinction.

E. (§ **Eximbricata**) **isillumense**, spec. nov., fruticosum gracile subscandens exsiccatione nigrescens petiolis basique nervorum puberulis exceptis glaberrimum; caulibus teretibus 2.5 mm. diametro pallide brunneis, internodiis 3–8 cm. longis; foliis oppositis ovato-oblongis (supremis lanceolatis) acuminatis basi levissime cordatis mucronato-serratis (serraturis vix 1 mm. altis 6–8 mm. latis) tenuibus a basi 3–5-nerviis (nervis inter sese a venis transversis connexis) 10–12 cm. longis 3–4.5 cm. latis; petiolo brevi gracile visciduloto-mentello 4–6 mm. longo; panicula oppositiramea laxa ovoidea basi vel saepe ad mediam partem foliaceo-bracteata; pedicellis filiformibus glabris saepe cum bracteolis squamiformibus solitariis instructis 8–15 mm. longis; capitulis ca. 52-floris 7 mm. altis 9 mm. diametro; involucri late campanulati squamis ca. 30 (1–3 extimis brevioribus exceptis) subaequalibus linearibus acutissimis glabris obscure 1–3-nervatis; corollis 3 mm. longis albis vix sursum gradatim ampliatis glabris; antheris apice cum appendice ovata integra munitis; achaeniis (immaturis) ca. 2 mm. longis ut videtur glabris; pappi setis ca. 28 tenuibus albis sublaevibus.— PERU: in woods near the tambo Isilluma between Sandia and Chunchusnago, alt. 1000 m., 23 June, 1902, *Weberbauer*, no. 1206 (Berl., phot. and fragm. Gr.).

In habit and foliage recalling the Brazilian *E. laeve* DC. but readily distinguished by its much looser inflorescence, puberulent petioles, leaves nerved from the very base, more numerous florets, etc.

E. (§ **Eximbricata**) **lobatum**, spec. nov., herbaceum perenne pluricaule decumbens 1–4 dm. altum; caulibus teretibus viridibus vel fuscescentibus flexuosis simplicibus vel oppositirameis foliosis, internodiis 1–2 cm. longis; foliis oppositis graciliter petiolatis in axillis saepe proliferis, laminis ovatis 1–2 cm. longis fere aequilatis profunditer crenato-lobatis vel inferioribus etiam subpinnatim 3-foliolatis utrinque viridibus et pilosulis subtus vix pallidioribus, lobis 5–9 integris vel crenato-lobatis apice obtusissimis vel rotundatis, petiolo 5–15 mm. longo; capitulis mediocritibus solitariis et terminalibus vel in corymbis parvis paucicapituliferis erectis vel saepius nutantibus, pedicellis 3–10 mm.l ongis gracilibus tomentellis; involucro campanulato 8–11 mm. diametro multifloseculoso, squamis 1–2-seriatim imbricatis subaequalibus oblanceolati-oblongis subherbaceis acutis vel obtusis viridibus vel apice scarioso-purpurascentibus 6–8 mm. longis; corollis verisimiliter albis externe granulosis 4–5 mm. longis, tubo proprio gracili faucibus cylindricis breviori, dentibus limbi brevissimis suberectis; achaeniis gracilibus griseo-brunneis fere ab apice ad basin gradatim angustatis basi substipitatis in costis pulcherrime ciliolatis; pappi setis albis tenuissimis fragilibus 3–4 mm. longis.— *E. scopulorum* Sch. Bip. Bull. Soc. Bot. France, xii. 82 (1865) & Linnaea, xxxiv. 535 (1865–6); Rusby, Bull. N. Y. Bot. Gard. iv. 378 (1907); not Wedd.— BOLIVIA: on cliffs in the subalpine region, near

Yani, Prov. Larecaja, alt. 3500 m., *Mandon*, n. 263 (Gr., N. Y.); without exact locality, *Bang*, no. 1912 (Gr., N. Y., U. S.). Both specimens distributed as *E. scopulorum*. It is quite impossible to believe that this plant, with very characteristic, deeply lobed or even divided leaves, is conspecific with the plant which Weddell, Chlor. And. i. 216 (1857), described and illustrated as having serrate leaves.

E. (§ **Cylindrocephala**) **mallotum**, spec. nov., lignescens verisimiliter elatum et subscandens molliter tomentosum vel tomentellum, pilis densis brevibus incurvis; caule tereti oppositirameo; foliis oppositis deltoideo-ovatis acuminatis integris basi rotundatis vel subtruncatis vel etiam paullo cordatis 2–3.5 cm. longis 1.4–2.7 cm. latis ima a basi 3-nerviis supra griseo-tomentellis subtus canescenti-tomentosis; petiolo dense tomentoso 3–5 mm. longo; capitulis 13 mm. longis 8 mm. diametro ca. 30-floris numerosis corymbo composito planiusculo terminali dispositis; pedicellis 6–20 mm. longis; involucri maturitate cylindrici squamis 5–6-seriatis arcte imbricatis apice rotundatis ciliolatis, extimis ovatis dorso tomentosis, ceteris dorso glabriusculis saepe apicem versus fuscescentibus vel viridimaculatis; corollis verisimiliter violascentibus glabris, 6 mm. longis paullo gradatim sursum ampliatis; achaeniis gracilibus deorsum decrescentibus 5.2 mm. longis, faciebus nigris glabris, angulis pallidis sursum hispidulis; pappi setis sordide albis inaequalibus scabridis vix sursum incrassatis.— *E. Clematitis*, var. *tomentosum* Sch. Bip. Bull. Soc. Bot. Fr. xii. 81 (1865), without char., & Linnaea, xxxiv. 535 (1865–66), as *Clematidis* var. *tomentosa* and also without char. *E. conyzoides*, var. *incanum* Britton, Bull. Torr. Bot. Club, xviii. 333 (1891), and probably of Baker in Mart. Fl. Bras. vi. pt. 2, 278 (1876). — BOLIVIA: Dept. LaPaz, Prov. Larecaja: "viciniis Sorata; inter Munaypata et rivum Chalassayo, in schistosis. Reg. temp. 2600–2700 m.", *Mandon*, no. 249 (TYPE, Gr., N. Y.); Sorata, alt. 2440 m., *Rusby*, no. 1626 (N. Y.); Prov. Yungas, alt. 1220, *Rusby*, no. 1625 (N. Y.).

From the expression "*an spec.?*" appended to his identification Schultz seems to have surmised the probable distinctness of this plant, though provisionally treating it as a variety of *E. clematitis* DC. The latter, a rare and little known Peruvian species, founded upon Poeppig's no. 3108 (of which a small and poor fragment from the herbarium of the late F. W. Klatt is now in the Gray Herbarium), has narrowly triangular to lance-oblong thinnish and essentially glabrous leaves, much longer and more slender petioles, and a more elongated and looser inflorescence. Prof. Britton, l. c., is very likely

right in identifying the Bolivian plant with Baker's inadequately characterized *E. conyzoides* Vahl, var. *incanum* of Brazil. However, as has been shown elsewhere, *E. conyzoides* Vahl (as well as the much earlier *E. conyzoides* Mill.) drops definitely into the synonymy of *E. odoratum* L., a plant with smaller heads, paler involucral scales, more rhombic-ovate leaves of much greater size and less pubescence, longer petioles, etc. Of the varietal names, neither *tomentosum* nor *incanum* is satisfactorily available for use in the specific rank because of earlier hymonyms of possible validity. Accordingly it seems best to give this doubtful plant a new start as an independent species under a fresh name.

Var. ? **aporum**, var. nov., foliis lanceolato-ovatis gradatim acuminatis basi rotundatis supra densiuscule puberulis subtus pallidioribus griseo-tomentellis usque ad 5.2 cm. longis et 2.4 cm. latis; petiolo fere 1 cm. longo; aliter ut apud var. *typicum* supra descriptum.—BOLIVIA WITHOUT LOCALITY: *Bang*, no. 2875 (Gr., U. S.). This plant with its tomentellous round-based leaves and rather dense inflorescence cannot be satisfactorily placed in the Peruvian *E. clematitis* DC. (under which name it was distributed). Its fine and rather copious pubescence keeps it also from being placed in the nearly related *E. subscandens* Hieron. It is obviously close to the species just described as *E. mallotum*, but while the several collections of that plant already secured have consistently the small deltoid-ovate very short-petioled leaves which give the species a highly characteristic appearance, this possible variety differs considerably in the form and dimensions of the leaf-blade and in having decidedly longer petioles, in these regards showing an approach to *E. subscandens*. Until much more copious material is available to show the direction and extent of its variability, the plant must remain uncertain and for the present it seems best to give it the provisional classification above indicated.

E. (§ **Subimbricata**) **Mathewsii**, spec. nov., fulvo-tomentosum perenne verisimiliter fruticosum basi ignotum; caulibus ramisve curvato-adscendentibus teretibus foliosis 4 mm. diametro post exsiccationem paullulo costulatis; internodiis 3–6 cm. longis; foliis oppositis lanceolatis utroque attenuato-acuminatis caudatis remote cuspidato-serratis (dentibus acutis 1–2 mm. longis ca. 1.5 cm. distantibus) penniveniis supra lucidulis cum venulis depressis reticulatis in nervo medio pubescentibus aliter subglabris subtus rufescenti-pubescentibus leviter reticulatis ca. 1 cm. longis 2 cm. latis; petiolo 1–1.5 cm. longo; panicula oppositiramea pyramidata ca. 1.5 dm.

alta et diametro foliaceo-bracteata; pedicellis 3–7 mm. longis fulvo-tomentosis; capitulis ad apices ramorum subdense dispositis ca. 56-floris ca. 9 mm. altis et crassis; involucri campanulati squamis subtriseriatim imbricatis paullo gradatis ovato-lanceolatis costulatis purpureo-tinctis glanduloso-ciliolatis et dorso sparse tomentellis; receptaculo plano glabro; corollis 5.5 mm. longis sensim sursum ampliatis extus limbum versus paullulo granulatis; styli ramis clavellati-filiformibus; achaeniis pallide griseis 2.3 mm. longis plerisque basi curvata callosis; pappi setis ca. 25 laete albis 3.5 mm. longis corolla conspicue brevioribus.— PERU: Yambrasbamba, 1835, *Mathews*, no. 1386 (K., phot. Gr.).

E. (§ **Conoclinium**) **metense,** spec. nov., herbaceum erectum oppositirameum breviter tomentosum saltim 6 dm. altum (basi ignota); caule ramisque adscendentibus teretibus leviter multicostulatis; internodiis usque ad 7 cm. longis; indumento denso primo griseo tardius fusco-brunneo; foliis oppositis ovato-oblongis obtusis grosse crenatis saepius deflexis et conduplicatis 3–6 cm. longis 1.5–3 cm. latis basi abrupte angustatis supra griseo-viridibus tomentellis subtus pallidioribus griseo-tomentosis a basi 3-nerviis, venis laxe anastomosantibus; petiolo 2–6 mm. longo; corymbis ramos terminantibus irregulariter compositis densiusculis convexis; capitulis breviter pedicellatis 6.5 mm. altis 4.5 mm. diametro ca. 28-floris; involucri campanulati squamis ca. 20 subaequalibus leviter imbricatis oblongo-lanceolatis acutis pallide viridibus plerisque 2-costulatis et 3-nerviis dorso griseo-hirsutulis; corollis laete roseo-purpureis glabris 3.7 mm. longis supra mediam partem paullo in fauces indistinctas ampliatis; dentibus patenti-recurvatis; styli ramis longe exsertis vix incrassatis 3.5 mm. longis; achaeniis nigris glabris 1.7 mm. longis; pappi setis albis minute barbellatis 3 mm. longis basi in annulum paullo connatis.— COLOMBIA: grassy llano east of Villavicencio, Intendencia Meta, 1–2 Sept. 1917, *Dr. F. W. Pennell*, no. 1617 (N. Y., phot. and fragm. Gr.).

This species has so closely the habit and involucre of § *Conoclinium* that it seems best to place it there provisionally, although the receptacle is only moderately convex, considerably less so than is usual in the section. Among the Colombian Eupatoriums of § *Conoclinium* this new species approaches most nearly *E. ballotaefolium* HBK., but differs from it in its strictly opposite short-petioled leaves of oblong rather than deltoid-ovate contour and of considerably firmer texture.

E. (§ **Subimbricata**) **orgyaloides,** spec. nov., verisimiliter fruticosum (basi ignota) petiolis et ramis inflorescentiae obscure puberulum aliter glaberrimum: caulibus ut videtur erectis gracilibus paullo (post exsiccationem) costato-angulatis fusco-brunneis ad inflorescentiam foliosis; internodiis 4–5 cm. longis; foliis oppositis petiolatis lanceolato-oblongis utroque acuminatis paullo undulatis obscure remoteque cuspidato-denticulatis penniveniis (venis principibus utroque 4–6 prorsus curvatis) chartaceo-membranaceis supra atroviridibus sublucidis subtus paullo pallidioribus opacis utrinque glabris delicatule reticulato-venosis, maximis 1.6 dm. longis 6 cm. latis, petiolo supra canaliculato brevissime fusco-puberulo 4–7 mm. longo; panicula breviter lateque pyramidata 11 cm. alta 17 cm. diametro oppositiramea basi foliaceo-bracteata; ramis gracilibus divaricatim patentibus apicem versus solum capituliferis; capitulis ca. 20-floris 4.5 mm. altis et crassis brevissime pedicellatis subcongestis; involucri campanulati squamis ca. 21 fusco- vel viridistramineis ciliolatis aliter glabris modice gradatis, interioribus oblongis vel linearibus, extimis ovatis acuminatis 5-costulatis apice saepe recurvatis; corollis 2.5 mm. longis paullo sursum gradatim ampliatis glabris, dentibus limbi recurvantibus; achaeniis (immaturis) glabris 1.7 mm. longis; pappi setis ca. 30 vix scabratis 2 mm. longis.—PERU: Depart. Loreto, near Tarapoto, *Spruce*, no. 4546 (K., phot. Gr.). In habit recalling *E. orgyale* DC. of eastern Brazil, which however has pubescent conspicuously dentate leaves, 6–12-flowered heads, and other differences.

E. (§ **Subimbricata**) **Pachanoi,** spec. nov., suffruticosum 3 dm. vel ultra altum ramosissimum; caulibus teretibus plus minusve flexuosis basin versus delapsu foliorum nudatis nodulosis juventate cum indumento brevi lanuginoso griseo-fulvescente obtectis; ramis gracilibus saepissime oppositis curvato-adscendentibus foliosissimis, internodiis 3–12 mm. longis; foliis ovato-lanceolatis acutiusculis basi obtusis vel rotundatis supra mediam partem serratis (dentibus utroque 3–4) supra subaveniis obscure praecipue in nervo medio puberulis subtus vix pallidioribus laete viridibus punctatis glabris supra basin 3-nerviis 1.5–2.2 cm. longis 7–10 mm. latis; corymbis pluribus ca. 3 cm. diametro valde convexis subovoideis ramulos terminantibus a ramis foliiferis superatis; pedicellis 4–8 mm. longis fulvescenti-pubescentibus bracteolatis; capitulis ca. 10-floris 8 mm. altis 5 mm. diametro; involucri campanulati ca. 3-seriati laxe imbricatis squamis ca. 13 viridibus vel purpureo-tinctis ciliatis dorso pubescentibus et glanduliferis, extimis brevibus ovatis vel lanceolato-oblongis vix acutis, intermediis late spatulato-ovatis apice rotundatis, intimis lineari-oblongis; corollis ut videtur purpureis 4.3 mm. longis supra basin paullo constrictis aliter subcylindricis, usque ad mediam partem et in limbo extus atomiferis; achaeniis 2 mm. longis brunneo-fuscis deorsum decrescentibus in faciebus et in costis cum glandulis globosis sessilibus conspersis; pappi setis ca. 35 inaequalibus patenter barbellatis.— Ecuador: vicinity of Nabón, Prov. of Azuay, Sept. 25, 26, 1918, *Dr. J. N. Rose, A. Pachano, & George Rose*, no. 22,978 (Gr., U. S., N. Y.).

A well-marked species, in its habit and small ovate-lanceolate leaves recalling several species of *Vaccinium*. From *E. vaccineaefolium* Benth. of Colombia it differs in having longer internodes, membranaceous leaves, broader, blunter, and much more pubescent involucral scales, and more numerous florets. From *E. umbrosum* Benth. it differs in having considerably smaller ovate-lanceolate rather than elliptical leaves, which are fewer-toothed, and also in having a more imbricated involucre.

E. (§ **Hebeclinium**) **phoenicticum**, spec. nov., herbaceum ut videtur erectum oppositirameum; caulibus ramisque teretibus dense lanato-tomentosis; indumento primo pulchriter purpureo deinde brunneo vel griseo; internodiis 1–1.5 dm. longis; foliis majusculis oppositis ovatis vel (superioribus) lanceolatis acuminatis basi breviter cuneatis margine leviter obtuseque serratis penniveniis (venis lateralibus principibus utroque 3–4) supra scaberrimis dense papilloso-pubentibus subtus molliter fulvo-lanuginosis 7–14 cm. longis 3.5–7.5 cm. latis; petiolo 1–3.4 cm. longo tomentoso; corymbis terminalibus di- vel trichotomis densis; ramis curvato-adscendentibus; capitulis numerosis congestis sessilibus 5–6 mm. longis ca. 13-floris: involucri ovoideo-subcylindrici squamis ca. 25, 3–5-seriatis regulariter gradatis ovatis vel oblongo-lanceolatis obscure ciliolatis aliter glabris 1–3-nerviis purpureo-tinctis; receptaculo planiusculo vel leviter convexo longe piloso; corollis roseis graciliter tubulosis vix sursum ampliatis 4 mm. longis, glabris; styli ramis filiformibus flexuoso-recurvatis; achaeniis glabris 1.7 mm. longis (vix maturis); pappi setis ca. 36 laevibus vel vix barbellatis apicem versus paullulo incrassatis plerisque corollam subaequantibus albidis.— Colombia: herb in field, alt. 2200–2600 m., Murillo, Dept. Tolima, 18 Dec., 1917, *Dr. F. W. Pennell*, no. 3159 (N. Y., Gr.).

This species in habit and inflorescence rather closely resembles *E. fuliginosum* HBK., but differs from it in the serrate (not mucronate-denticulate) leaves, which have fewer lateral veins; also in having its involucral scales (which are purple rather than fuscous) glabrous except for an obscure ciliation. It is also related to *E. sericeum* HBK.,

which, however, has white woolly stems and petioles, and broader as well as more caudate-acuminate leaves, which are 3-nerved from above the base rather than feather-veined.

E. PSEUDO-DALEA (DC.) Gomez & Molt. Ann. Hist. Nat. Madrid. xix. 272 (1890). *Critonia pseudo-dalea* DC. Prod. v. 140 (1836). Recent material, agreeing with this species in all essential characters of habit, inflorescence, florets, and achenes, as well as in the toothing, venation, and punctation of the leaves, still exhibits such marked differences in the rounded leaf-base closely sessile upon the stem as to present a decidedly exceptional appearance when compared with typical specimens. There is also a tendency on the part of some material now available to have much more coarsely toothed leaves than the type. To provide appropriate classification for these variations the following varieties may be proposed.

Var. **typicum,** foliis breviter serrato-dentatis (dentibus 0.6–0.9 mm. altis 3–5 mm. latis) basi cuneatis; petiolo 6–8 mm. longo.— CUBA: *Ramon de la Sagra* (Gr.); in crevices of rocks at the Farallones, *Wright*, no. 1311 in part (Gr.).

Var. **apodophyllum,** var. nov., foliis breviter serrato-dentatis (dentibus ut apud var. *typicum*) basin versus angustatis sed ima basi rotundatis arcte sessilibus vel cum petiolo brevissimo 1–2 mm. longo munitis.— CUBA: base of coastal cliffs near Rio Yamuri, Oriente, 6, 8 Dec. 1910, *J. A. Shafer*, no. 783 (TYPE, Gr., N. Y.); margin of rivulets near Pinar del Rio, 3 Dec. 1860–1864, *Wright*, no. 1311 in part (Gr.).

The plant of Shafer has sessile leaves, that of Wright has leaves of closely similar form, but very shortly petioled (petioles 1–2 mm. long).

Var. **macrodontum,** var. nov., foliis argute serrato-dentatis, dentibus 2–4 mm. altis 4–6 mm. latis acuminatis; lamina basi cuneata; petiolo 1.4–2.5 cm. longo.— CUBA: Guajaibon, 4 Nov. 1860–1864, *Wright*, no. 2812 (TYPE, Gr.), limestone cliff, Baños San Vincente, Prov. Pinar del Rio, 12–16 Sept. 1910, *N. L. & E. G. Britton & C. S. Gager*, no. 7451 (Gr.).

E. (§ **Eximbricata**) **psilodorum,** spec. nov., aspectu primo glaberrimum solis in petiolis pedicellisque obscure puberulum herbaceum perenne vel fruticosum (basi ignota); caule tereti laevissimo flavido-viridi erecto folioso superne paniculatim ramoso; foliis oppositis lanceolato-ovatis longe acuminatis saepe falcatis argute serratis (dentibus ca. 1 mm. altis ca. 3–4 mm. latis incurvis) basi rotundatis aliquando obliquis utrinque glaberrimis flavido-viridibus a basi vel paulo supra basin 3–5-nerviis impunctatis 6.5–7.5 cm. longis 2–3 cm.

latis; petiolo gracili vix supra villosulo ca. 2 cm. longo; corymbis compositis terminalibus trifidis laxiusculis inferne foliaceo-bracteatis; pedicellis 8–15 mm. longis cum bracteolis 1–3 filiformibus instructis; capitulis ca. 22-floris 6–7 mm. altis 5 mm. diametro plerisque erectis; involucri campanulati squamis ca. 18 linearibus acutis glabriusculis substramineo-herbaceis apicem versus scariosis et fimbriato-ciliatis subaequalibus; corollis albis ca. 3.5 mm. longis glabris; tubo proprio ca. 1.5 mm. longo fauces campanulato-cylindricas aequante, limbi dentibus 5 ovato-deltoideis ca. 0.5 mm. longis; achaeniis 2 mm. longis maturitate nigris in angulis hispidulis; pappi setis 18–25 delicatulis albis scabridulis ca. 2.8 mm. longis fragilissimis.— *E. Dombeyanum* Robinson, Proc. Am. Acad. liv. 315 (1918), not DC.— COLOMBIA: in clearings, not common, Las Nubes near Santa Marta, Dept. Magdalena, alt. 1372 m., *Herbert H. Smith*, no. 621 (Gr., U. S., N. Y., Mo.).

The real *E. Dombeyanum* DC. (see p. 11) is probably a Peruvian plant, with roseate or purplish corollas. Its leaves, while of similar form, have decidedly more salient teeth and relatively shorter petioles. These in *E. Dombeyanum* are about one-eighth as long as the blade, while in *E. psilodorum* they are more than one-fourth its length. To *E. psilodorum* may be referred with scarcely a doubt also Dr. Pennell's no. 3190 from Murillo, Dept. Tolima.

E. (§ **Eximbricata**) **rhypodes**, spec. nov., herbaceum erectum 4–5 dm. altum virgatum vel superne mediocriter ramosum; caule tereti densissime glanduloso-tomentoso; pilis patentibus atropurpureis glandulari-capitulatis; internodiis 1.5–9 cm. longis, nodis paullo incrassatis; foliis oppositis vel ternatis membranaceis suborbiculari-ovatis breviter acuminatis serrato-crenatis (dentibus utroque 5–8 ca. 1 mm. altis 3 mm. latis) basi cordatis supra pubescentibus subtus dense villosis 2–4 cm. longis et latis, petiolo 7–10 mm. longo densissime glanduloso-tomentoso; panicula terminali plerumque trichotoma 5–9 cm. diametro planiuscula maturitate modice laxa; ramis et pedicellis (8–14 mm. longis) dense tomentosis; pilis atropurpureis aliis attenuatis articulatis aliis glandulari-capitulatis; capitulis ca. 45-floris 7–8 mm. altis et diametro; involucri campanulati squamis subaequalibus anguste lanceolati-oblongis acutis usque ad mediam partem plerisque 2-costulatis viridibus vel purpureo-tinctis cum pilis purpureis (aliis capitatis aliis attenuatis articulatis) sparsis; corollis albis 3.6–3.8 mm. longis; tubo proprio gracili 1.5 mm. longo; faucibus campanulato-cylindricis ca. 2.5 mm. longis, limbo patenter villoso 0.6 mm. longo; achaeniis nigris gracilibus 2.7 mm. longis ad apicem et ad basin paullo attenuatis praecipue in angulis

sursum hispidis basi callosis apice cum disco parvo flavescente pappifero coronatis; pappi setis albis roseo-tinctis sursum scabridulis apicem versus paullulo incrassatis.—ECUADOR: vicinity of Las Juntas, *J. N. Rose*, *A. Pachano*, & *G. Rose*, no. 23,179 (Gr., U. S., N. Y.).

This plant differs from *E. pichinchense* HBK. in its copiously gland-tipped pubescence, much smaller, fewer-toothed leaves, etc. From *E. glanduliferum* Hieron. it may be distinguished by having the leaves petiolate and their blades essentially as wide as long. In many respects it corresponds pretty closely with the Colombian *E. sotarense* Hieron., but that species, according to Hieronymus, has leaves glabrescent or subglabrescent (while they are here densely and permanently pubescent to villous on both surfaces), rounded at the base (while here distinctly cordate), and longer than wide.

E. RORULENTUM Robinson, Proc. Am. Acad. liv. 255 (1918). This species, described from young material with the heads merely in bud, has been collected near Laguna de Verjón, Colombia, by Bro. Ariste-Joseph, no. B20 (Gr.). From this material, which, agreeing as to habit and foliage in all details with the original, is decidedly more mature, it is possible to add the following characters to those recorded in the diagnosis: heads at length 8–9 mm. high; corolla whitish 5 mm. long, sparsely atomiferous, proper tube 1 mm. long, the throat about 3.4 mm. long, the limb about 0.6 mm. long, the teeth deltoid, acutish; mature achenes dark brown, about 3 mm. long.

- **E.** (§ **Cylindrocephala**) **Roseorum**, spec. nov., brevissime griseo-tomentellum; caule tereti usque ad 4 mm. vel ultra diametro pilis albis minutis curvatis eglandularibus tecto; internodiis ad 6 cm. longis; foliis oppositis saepe in axillis proliferis deltoideo-ovatis acute acuminatis basi subtruncatis lateraliter paullo crenato-serratis vel undulatis vel integris a basi 3-nervatis plerisque ca. 5 cm. longis 3.5 cm. latis membranaceis subconcoloribus vix subtus pallidioribus utrinque tomentellis; petiolo gracile flexuoso 10–18 mm. longo; corymbis modice convexis ramos oppositos vel supra alternos terminantibus multicapitulatis densiusculis conjunctim paniculam semiglobosam vel ovoideam foliaceo-bracteatam formantibus; pedicellis plerisque ca. 5 mm. longis; capitulis ca. 17-floris 1.5 cm. longis 4 mm. latis; involucri cylindrici squamis ca. 30 gradatim regulariterque imbricatis ciliolatis; extimis brevissime oblongis apice rotundatis dorso puberulis et sparsim atomiferis subherbaceis; intermediis et intimis gradatim longioribus ciliatis aliter glabriusculis obtusis vel (intimis) acutiusculis albis pulcherrime lilaceo-tinctis

3(–1)-viridi-nerviis; corollis glabris gracilibus ca. 5 mm. longis a basi ad limbum paullo gradatimque ampliatis; achaeniis graciliter subprismaticis basin versus paullo attenuatis griseo-fuscis adpresse in angulis et faciebus puberulis; pappi setis ca. 28 scabratis delicatulis albis ca. 4 mm. longis.— ECUADOR: vicinity of Guayaquil, 30 August to 2 Sept., 1918, *J. N. & G. Rose*, no. 22,464 (Gr., U. S., N. Y.). A handsome species with deltoid-ovate leaves recalling those of *E. iresinoides* HBK. but with involucre characteristic of § *Cylindrocephala*. The involucral scales (green, white, and lilac) recall those of *E. iridolepis* Robinson. An attractive and seemingly very distinct species which it is a pleasure to dedicate to Dr. J. N. Rose of the United States National Museum and to his son, who was his assistant on a recent collecting trip to Ecuador.

E. (§ **Subimbricata**) **roupalifolium**, spec. nov., robustum verisimiliter fruticosum; caule (vel ramo) ca. 5 mm. crasso medulloso purpureo-brunneo glaberrimo lucidulo; internodiis 1–2.5 cm. longis; foliis oppositis ovato-oblongis firme coriaceis crassiusculis penniveniis utroque angustatis apice vero obtusis basi acute cuneatis 8–11 cm. longis 3.5–5 cm. latis basi integra excepta grosse serratis (dentibus 1–2 mm. altis 4–7 mm. latis) glaberrimis lucidulis exsiccatione fuscescentibus; venis lateralibus numerosis parallelis aliis conspicuis aliis obscuris; petiolo 1.4–2 cm. longo; corymbo terminali composito trichotomo denso valde convexo multicapitulato fulvo-puberulo; bracteis inferioribus lanceolato-oblongis ca. 2 cm. longis 3–4 mm. latis obscure crenato-serratis, bracteolis subulatis minimis; capitulis ca. 5-floris subsessilibus (immaturis) ca. 6 mm. longis et 2.4 mm. crassis; involucri subcylindrici ca. 3-seriatim imbricati squamis crassis eveniis valde inaequalibus anguste oblongis glabris obtusis paullo carinatis apicem versus paullo brunnescentibus; corollis (valde immaturis) ca. 3 cm. longis brevissime 5-dentatis limbum versus granulatis vel hispidulis aliter glabris; antheris apice cum appendice membranacea ovato-oblonga instructis; pappi setis ca. 40 albis distinctis.— BRITISH GUIANA: Mt. Roraima, ledge, 18–12–'84, alt. 2135 m., *E. Jenman*, no. 311 (TYPE, U. S.); ledge, Roraima Exp., *E. F. Im Thurn* (K., phot. Gr.). These specimens appear absolutely identical, both being tips of branches carrying two or three of the upper leaves and a terminal corymb in bud. In each there has been some damage from insects, especially to the immature achenes. This is certainly the plant mentioned by Daniel Oliver, Trans. Linn. Soc. ser. 2, Bot., ii. 277 (1887), as "No label. EUPATORIUM, sp.? (not identified)." Although the material is poor, it discloses practically

all the features important for classification. It appears to be very different from any species known and may well have published record. It has somewhat the habit of a *Symphyopappus*, but the pappus-bristles are capillary and distinct to the base, showing no tendency to become firm or to be connate into a ring. It may here be mentioned that recent examination of *Symphyopappus* leads to the belief that it is an exceedingly weak genus of very doubtful distinctness and little taxonomic value. Its distinctions from *Eupatorium*, both technical and habital, break down completely.

E. SCIAPHILUM Robinson, Proc. Am. Acad. liv. 256 (1918). This interesting species, originally collected in Dept. Antioquia, Colombia, by *Kalbreyer*, appears to have been rediscovered by Dr. Pennell in the shrub zone, alt. 2800–3000 m., below Paramo de Chaquiro, Dept. Bolívar, Colombia, no. 4349 (N. Y.). Dr. Pennell's plant while agreeing with the original material in all the more essential points differs in having the leaves crowded, somewhat smaller (about 4 cm. long) and less rigidly coriaceous, the contour being elliptic-oblong rather than spatulate-oblong. In the presence of close agreement in the inflorescence, involucre, florets, achenes, pappus, pubescence, etc., it is probable that these differences are largely the result of individual environment or are at most only of a formal nature.

E. SCIATRAPHES Robinson, Proc. Am. Acad. liv. 257 (1918). Among the plants recently collected in Venezuela by Prof. H. Pittier is a *Eupatorium* in bud. It is labeled as follows: "Niquibao. Fls. pale lilac. Medicinal, pectoral. Around Caracas: altitude 800 to 1000 meters. Cultivated. June, 1918. *H. Pittier*, no. 7882." In habit, foliage, pubescence, as well as in the details of inflorescence and florets, so far as shown, this plant so closely coincides with the Santo Domingan *E. sciatraphes* that its specific identity seems practically assured. As in the original material the number of florets in the closely fastigiate heads shows considerable variability, ranging in the Venezuelan material from 4 to 8, while in the Santo Domingan specimens examined it ranged from 8 to 13. More mature material of the Venezuelan plant would be essential to prove with absolute certainty the identity, yet the highly characteristic habit, texture, lucidity, and nervation of the leaves, etc., give fairly conclusive evidence. It is to be observed that the Venezuelan material is from cultivated stock, so there is as yet no evidence that the plant is indigenous on the continent.

E. (§ **Eximbricata**) **simulans**, spec. nov., ut dicitur herbaceum vel fruticosum 1–2 m. altum; caule tereti griseo-brunneo juventate granulari-puberulo tardius subglabrato flexuoso oppositirameo folioso vel basin versus delapsu foliorum nudato et nodoso; foliis oppositis lanceolatis caudato-acuminatis argute serratis basi plerumque abrupte angustatis vel subcuneatis 5–7 cm. longis 1.3–2.5 cm. latis penniveniis (sed 2–3 venis inferioribus utroque latere aliis saepe distincte majoribus) membranaceis utrinque obscure viridibus; petiolis 8–11 mm. longis nullo modo muriculatis; corymbis caulem ramosque terminantibus valde convexis 8–10 cm. diametro laxiusculis; pedicellis filiformibus ca. 1 cm. longis bracteolatis glabris vel obscure granulato-puberulis; capitulis ca. 22-floris ca. 1 cm. longis et crassis; involucri campanulati squamis ca. 20 linearibus attenuatis dorso parce granulatis vix costulatis apicem versus textura flaccidis; corollis purpureis vel roseis glabris paullo sursum gradatim ampliatis 6.5 mm. longis; achaeniis valde immaturis primo aspectu glabris parce granulatis gracilibus obscure griseis cum costis paullo pallidioribus; pappi setis ca. 36 corollam subaequantibus scabridulis.— Peru: Dept. Ancachs: among bushes in the gorge of a brook on the slopes of the Cordillera Blanca, above Caraz, alt. 3200–3600 m., 9 June, 1903, *Dr. A. Weberbauer*, no. 3253 (type, Berl., phot. and fragm. Gr.); also among small bushes on a brook, above Ocros, Prov. Cajatambo, alt. 3300 m., 2 Apr. 1903, *Weberbauer*, no. 2766 (Berl., phot. and fragm. Gr.).

A species closely simulating *E. stictophyllum* described below, but having smaller impunctate feather-veined leaves and petioles without muriculate roughening.

E. solidaginoides HBK., var. **Armourii**, var. nov., foliis bracteisque triangulari-hastatis basi profunde cordatis; lobis basilaribus acutiusculis patenti-deflexis; dentibus marginis rotundatis.— Mexico: Palenque, Chiapas, Feb., 1895, *A. V. Armour*, no. 1 (Field Mus., phot. Gr.). This plant is exceedingly puzzling. The hastate lobing and deeply crenate instead of serrate-dentate margins of the leaves give it a very different appearance from the usual forms of *E. solidaginoides* HBK. However, the species exhibits considerable variability and specimens have long been known (e. g. *Pringle*, no. 3956, from limestone ledges, Tamasopo Cañon, San Luis Potosi) in which the leaves are in a similar manner cordate by a deep and narrow sinus and exhibit some tendency to become hastate through the production of an external basal angle at least on one side. It seems probable, therefore, with such approaches already evident, that complete intergradation as to foliage will ultimately be found between the typical form and the present variety, notwithstanding its striking character. In all traits of inflorescence, flowers, achenes, etc., the

correspondence is pretty close and in many details amounts to identity.

E. (§ **Subimbricata**) **Sprucei**, spec. nov., suffruticosum 2–3 dm. altum suberectum; radice e fibris paucis gracilibus duris elongatis sistente; caule curvato-adscendente, basin versus distincte lignescente 4 mm. crasso noduloso griseo-brunneo a delapsu foliorum nudato, in parte media folioso, internodiis brevibus, in parte superiore gracili erecto crispe fulvo-puberulo paniculatim florifero paullo foliaceo-bracteato, internodiis elongatis; foliis oppositis sessilibus oblanceolatis utroque attenuatis acutisque subremote denticulatis (dentibus 0.5 mm. altis 5 mm. distantibus) penniveniis (venis utroque 6–7 adscendentibus curvatis inter se anastomosantibus) membranaceis supra viridibus glaberrimis obscuris subtus pallidioribus praecipue in nervo venisque sordide pubescentibus 5–10 cm. longis 1–1.8 cm. latis; capitulis 6 mm. longis 5 mm. crassis ca. 18-floris; pedicellis filiformibus valde inaequalibus (1–6 mm. longis); involucri turbinati squamis ca. 20 lineari-oblongis valde inaequalibus obtusis stramineis plerisque medio 2-costulatis apicem versus dorso puberulis, extimis brevissimis ovalibus subherbaceis; corollis verisimiliter albis 3 mm. longis glabris, tubo proprio gracillimo 1.8 mm. longo, faucibus turbinatis distincte ampliatis 1.2 mm. altis; styli ramis cum appendicibus tenuibus attenuatis flexuosis papilloso-scabratis munitis; antheris apice cum appendice membranacea angusta instructis; achaeniis (submaturis) fuscis 1.5 mm. longis in angulis parce hispidulis; pappi setis ca. 25 delicatissimis albis 2.5 mm. longis vix scabratis.— PERU: along the River Huallaga, September, 1855, *Spruce*, no. 4167 (Gr.).

This low, upright undershrub is obviously a member of the peculiar group of § *Subimbricata* to which the following species may be referred: *E. elata* Steetz of the Panama region, *E. Squiresii* Rusby of the delta regions of the Orinoco in Venezuela and the Magdalena in Colombia, *E. turbacense* Hieron. also from Colombia, and *E. tovarense* Robinson from western Venezuela. In all these species the leaves are of the oblong-lanceolate type, opposite and feather-veined, the heads are small, in a loose, somewhat divaricately branched panicle; the involucre is stramineous, of delicate graduated scales, which are obtuse and usually a little tufted with a sparse puberulence on the back toward the tip; finally in all the style-branches have appendages which are slender-filiform, more delicate, more flexuous, and more distinctly hispidulous than is usual in *Eupatorium*. In regard to the style-branches these species recall conditions usual in *Vernonia* and in *Brickellia diffusa*. *E. Sprucei*, geographically remote from the others, differs from all of the others in its much lower stature, being only

2–3 dm. high as opposed to plants of a meter or so in height, considerably looser growth, and mostly herbaceous character. In *E. Squiresii* and *E. elatum* the upper surface of the leaves is distinctly though sparingly puberulent. In *E. tovarense* the leaves are lanceolate not oblanceolate and are strikingly glandular-atomiferous beneath. *E. turbacense* is said to attain 2 m. in height. It has lanceolate to ovate leaves 5–7 cm. wide, and its involucre is more campanulate and has the scales usually 3–4-costulate. The florets are also somewhat more numerous, 25–28.

It is clear that *E. Sprucei* cannot at present be united with any of these, yet they are all exceedingly close and may sometime be found to intergrade.

E. SQUALIDUM DC., var. **Rusbyanum**, var. nov., laxe ramosum; caule ramisque flexuosis griseo-tomentellis; pilis albidis minutis plerisque incurvis non evidenter articulatis; foliis membranaceis ovato-lanceolatis vel ovatis supra cum pilis minutis albis subappressis sparse conspersis, subtus sordide tomentellis et glandulari-punctatis, indumento densiusculo glandulas obscurante; capitulis 18–20-floris; involucri squamis laevissimis saepissime brunnescentibus apice rotundatis vel obtusis arcte appressis ciliolatis; achaeniis 2.5 cm. longis.— *E. scabrum* Britton, Bull. Torr. Bot. Club, xviii. 333 (1891), not L. f. *E. Martiusii* Ktze. Rev. Gen. iii. pt. 2, 148 (1893), not DC. — BOLIVIA: at Guanai in Prov. Larecaja, Dept. La Paz, alt. 610 m., May, 1886, *Rusby*, no. 1623 (TYPE, N. Y., phot. Gr.); in Prov. (East) Velasco, Dept. Santa Cruz, alt. 200 m., July, 1892, *Kuntze* (N. Y.). PERU: without locality, *Mathews* (N. Y.).

This plant differs from *E. scabrum* L. f. considerably in the form and texture of the leaves as well as in the much shorter not obviously jointed pubescence. This variety may be distinguished from both var. *Martiusii* (DC.) Bak. and var. *subvelutinum* (DC.) Bak. by its denser pubescence on the lower surface of the leaves, the hairs being so close as to obscure almost wholly the glandular punctation which is clearly evident, indeed rather conspicuous, in the varieties just mentioned. From var. *tomentosum* (Sch. Bip.) Bak. the present plant differs in its distinctly smaller heads, fewer florets, more cylindrical and less ovoid involucre, etc.

E. SQUIRESII Rusby in Robinson, Proc. Am. Acad. liv. 258, 337 (1918). Soon after the publication of this species two specimens were received at the Gray Herbarium of much interest in connection with it. They were Dr. Pennell's nos. 3928 and 3929, collected in an alluvial thicket, alt. 80–90 m., at Badillo, Rio Magdalena, Dept.

Santander, Colombia, 16 Jan. 1918. These correspond closely in all essential and most minor features with the original material of *E. Squiresii* from the delta of the Orinoco and certainly appear to be conspecific with it. The only differences found during a rather detailed examination were that the leaves of the Colombian plant were slightly firmer in texture and even more shortly petioled or subsessile. These differences are precisely of the kind and degree that occur very frequently between exposed and shade forms of the same plant. The range of the species is thus extended some 1500 km. and over the watershed from the Orinoco Valley into that of the Magdalena. However, there is little difference in the latitude, and the habitat, low alluvial thickets, is similar. Renewed examination of the plants of this group brings out what had not been previously noticed, namely, the affinity between this species and *E. turbacense* Hieron. Of the latter species the writer has not seen the type, Stübel's no. 51, collected at Turbaco, Dept. Bolívar, in the delta region of the Magdalena. However, Hieronymus when publishing upon the plants of Lehmann (Engl. Bot. Jahrb. xxviii. 573) identifies with his *E. turbacense* Lehmann's no. 5971, and of this a leaf and a bit of the inflorescence were received at the Gray Herbarium some years ago in an exchange from the Royal Botanical Garden at Berlin. This plant of Lehmann's was collected on the Rio Ortega in the Dept. Cauca, that is to say, some 900 km. to the south of the original station. If Hieronymus has been right in referring it to his *E. turbacense,* the following differences may be pointed out between that species and the later *E. Squiresii.* In *E. turbacense* the leaves are entirely glabrous above, while in *E. Squiresii* they are puberulent at least on the midnerve and sometimes perceptibly so on the surface as well; in *E. turbacense* the lowest two pairs of lateral veins leave the midnerve at an angle of about 40° and in length considerably exceed those arising at a greater distance from the base, while in *E. Squiresii* the lower pairs of veins are no longer, indeed are usually shorter than some of the others, and all of them leave the midnerve at a considerable angle, usually at about 70°. In *E. turbacense* the involucre is campanulate, in *E. Squiresii* it is somewhat longer and campanulate-subcylindric (in the fresh state) or campanulate-subturbinate (in the dried state). In *E. turbacense* the pubescence of the pedicels is distinctly longer and more sordid-tawny than in *E. Squiresii.* The two species are certainly very close. In both the lower leaves have a peculiar form, the lance-oblong blade being narrowed at the base into a more or less elongated portion like a broadly winged petiole. It

must be emphasized that the validity of the distinctions made above depends upon the accuracy with which Hieronymus identified Lehmann's no. 5971 from the region of Popayan with Stübel's no. 51, the original material of *E. turbacense* from near the mud-volcano of Turbaco. The writer has had no opportunity to have this identification controlled by a re-examination.

E. (§ **Subimbricata**) **Steetzii**, spec. nov., fruticosum ramosum robustum 2–3 m. altum; ramis teretibus dense cum tomento purpureo-brunneo obtectis, capillis articulatis patentibus; foliis oppositis ovato-oblongis acutis vel acuminatis, crenato-serratis basi saepius rotundatis rarius paullo cordatis 7–20 cm. longis 5–12 cm. latis supra cum basibus incrassatis capillorum dense scabratis et cum glandulis subsessilibus interspersis subtus griseo-tomentosis supra basin 3-nervatis; petiolo 1.2–3.8 cm. longo crasso dense tomentoso; corymbis densis trichotomis pluricapitulatis; capitulis ca. 15–18-floris breviter pedicellatis; involucri campanulati squamis ca. 15 subtriseriatis stramineo-purpureis oblongo-lanceolatis acutiusculis vel subattenuatis dorso glabris; corollis 5 mm. longis gradatim sursum ampliatis glabris albis marginem versus purpurascentibus (Fendler); achaeniis 2.8 mm. longis brunneis cum glandulis sessilibus scabratis; pappi setis ca. 40 stramineo-albidis barbellatis 5 mm. longis saepius purpureo-tinctis.—*E. Vargasianum* Robinson, Proc. Am. Acad. liv. 289 (1918), as to character, not DC., also Robinson, l. c. 339 excl. pl. of Vargas.—VENEZUELA: in the State of ARAGUA: near Colonia Tovar, *Fendler*, no. 647 (TYPE, Gr.); without exact locality, *Cruger* (K.). In the FEDERAL DISTRICT: Caracas, *Linden*, no. 137 (K.); in forest, Boca de Tigre, Altos de Galipán, Cerros del Avila, alt. 1600 m., *Pittier*, no. 8302 (Gr.); La Ciénega, alt. 2280 m., Silla de Caracas, Coastal Range, alt. 2000–2640 m., *Pittier*, no. 8314 (Gr.). Dedicated to the memory of *Joachim Steetz* (1804–1862), a physician of Hamburg and a discriminating investigator of the *Compositae*, who by some unpublished notes on a drawing of this plant now in the Gray Herbarium appears to have been the first to recognize its probable novelty.

E. (§ **Eximbricata**) **stictophyllum**, spec. nov., fruticosum usque ad 1 m. altum; caule tereti purpureo-brunneo oppositirameo juventate obscure puberulo; foliis oppositis ovato-lanceolatis caudato-acuminatis basin versus rotundatis deinde abrupte subacuminatis serratis firmiuscule membranaceis glabris supra obscure viridibus subtus pallidius viridibus nigro-puncticulatis delicatule reticulatos venulosis (venulis non prominulentibus) 6–8 cm. longis 3–4 cm. latis a puncto ca. 5 mm. supra basin 3-nervatis, nervis lateralibus brevi spatio furcatis vel aliquando pro nervo uno latere nervis duabus proximis substitutis; petiolo puberulo parce muriculato; corymbis terminalibus compositis planiusculis foliaceo-bracteatis, particularibus densiusculis fastigiatim ramosis convexis; capitulis ca. 24-floris 1 cm. longis; pedicello filiformi 6–10 mm. longo bracteolato; involucri campanulati squamis ca. 24 anguste lanceolato-linearibus inaequalibus sed vix imbricatis acutis purpureo-viridibus extus granulari-pulverulentis; corollis 7.8 mm. longis roseis glabris vix sursum ampliatis; achaeniis (immaturis) 3 mm. longis cum glandulis subsessilibus instructis; pappi setis ca. 27 albis scabridis corollam subaequantibus. — PERU: in bushy places about Cuyocuyo, Prov. Sandia, Dept. Puno, alt. 3100 m., *Dr. A. Weberbauer*, no. 860 (Berl., phot. and fragm. Gr.).

E. (§ **Subimbricata**) **tarapotense**, spec. nov., verisimiliter fruticosum oppositirameum; caulibus subteretibus purpureis crispe fulvo-puberulis (pilis articulatis), internodiis 4–6 cm. longis; foliis oppositis ovato-ellipticis petiolatis acuminatis basi rotundatis remote serratis (dentibus ca. 1 mm. altis ca. 1 cm. inter se distantibus) penniveniis (venis principibus ca. 5-jugis) coriaceis supra atroviridibus nitidis in nervo venisque depressis puberulis subtus pallidioribus tomentellis et atomiferis 4–8 cm. longis 1.5–4 cm. latis; petiolo ca. 1 cm. longo fulvo-puberulo; cymis paucicapitulatis ramos ramulosque terminantibus conjunctim paniculam laxiusculam foliaceo-bracteatam formantibus; pedicellis 1–3 mm. longis; capitulis ca. 37-floris ca. 1 cm. altis; involucri campanulati squamis ca. 18 gradatis acutiusculis, extimis late ovatis 5–7-costulatis fusco-brunneis dorso pubescentibus, intermediis ovatis 3–5-costulatis substramineis, intimis lineari-oblongis acutis stramineis ciliolatis dorso plus minusve puberulis; corollis glabris ca. 6 mm. longis, tubo proprio gracili ca. 4 mm. longo, faucibus paullo sed distincte ampliatis 1.5 mm. altis, dentibus 5 detoideis 0.5 mm. longis extus granulatis; achaeniis graciliter prismaticis fusco-brunneis 2.3 mm. longis; pappi setis inaequalibus corolla distincte brevioribus.— PERU: in the mountains along the river Mayo, near Tarapoto, Dept. Loreto, July–Aug., 1850, *Spruce*, no. 4014 (Gr.). In habit similar to *E. Lobbii* Klatt, but readily distinguished by its much thicker, pinnately veined leaves, which are shining above.

E. (§ **Eximbricata**) **uber**, spec. nov., fruticosum 4 m. altum; ramis oppositis robustis griseo-brunneis foliosis granuloso-puberulis medullosis; foliis oppositis magnis petiolatis deltoideo-ovatis acuminatis integris vel obsolete undulato-denticulatis basi abrupte subcuneatis supra glabris viridibus (venulis reticulatis depressis) subtus sordide puberulo-tomentellis 16–19 cm. longis 6–9 cm. latis; nervis lateralibus principibus (2-jugis) ca. 1 cm. supra basin laminae nervo medio orientibus; petiolo 3–5 cm. longo granulari-puberulo; panicula terminali corymbiformi planiuscula usque ad 2 dm. diametro breviter crispeque pubescenti; capitulis numerosissimis breviter pedicellatis ca. 13-floris ca. 9 mm. longis; involucri anguste campanulati squamis ca. 16 lineari-oblongis acutiusculis (obscure 2-costatis) subaequalibus (1–3 extimis brevioribus) dorso granularibus margine eroso-ciliolatis; corollis ca. 5 mm. longis albis sursum gradatim mediocriter ampliatis; limbi dentibus 5 deltoideis; achaeniis griseo-brunneis deorsum decrescentibus 2.5 mm. longis in angulis hispidulis basi conspicue callosis; pappi setis ca. 31 barbellatis corollam fere aequantibus sordide albidis.— PERU: woods near a brook, below Pampa Romas, between Samanco and Caraz, Dept. Ancachs, alt. 2100 m., 29 May, 1903, *Weberbauer*, no. 3184 (Berl., phot. and fragm. Gr.). A species of exceptional luxuriance both as to flowers and foliage, somewhat resembling the Bolivian *E. longipetiolatum* Sch. Bip. ex Rusby, Mem. Torr. Bot. Club, iii. no. 3. 52 (1893), which, however, has a more open inflorescence, shorter, relatively broader, and dorsally more pubescent involucral scales, purple corollas, more or less clearly crenate leaves, etc. *E. uber* is also close to the Bolivian *E. gloeocladum* Robinson described above, which, however, has smaller lance-ovate, regularly feather-veined leaves, viscid stems, etc.

E. (§ **Subimbricata**) **urubambense**, spec. nov., herbaceum et perenne vel fortasse fruticosum (basi ignota); caulibus (vel ramis) erectis teretibus purpureis crispe albo-puberulis; foliis oppositis ovato-lanceolatis fere a basi ad apicem gradatim angustatis sed apice vero saepissime obtusiusculis basi obtusis vel rotundatis crenatis a puncto supra basin 3-nervatis crassiusculo-membranaceis supra obscure viridibus puberulis leviter rugulosis subtus griseo-tomentosis 5–6.6 cm. longis 1.8–2.7 cm. latis; petiolo 5–8 mm. longo; corymbis trichotomis foliaceo-bracteatis; inflorescentiis particularibus densis rotundatis 3–6 cm. diametro; capitulis ca. 10-floris ca. 8 mm. altis subsessilibus; involucri subcylindrico-campanulati squamis ca. 3-seriatis apice rotundatis, exterioribus brevissimis, intermediis late ellipticis brunneo-stramineis saepissime 3-nervatis et 4-costulatis aliquanto marginatis, intimis lineari-oblongis apice paullo eroso-ciliatis; corollis 4.8 mm. longis graciliter tubulosis supra basin sensim constrictis glabris; achaeniis fuscis 2.6 mm. longis deorsum attenuatis sursum in costis villosis; pappi setis ca. 28 lucidulis flavido-albidis 4.3 mm. longis sublaevibus.— PERU: Urubamba in the Valley of Ymay [?], *Pentland* (K., phot. and fragm. Gr.).

E. VALLINCOLA DC. Prod. v. 168 (1836). Of this species two varieties

with markedly different pubescence but otherwise of close similarity can be distinguished, namely:

Var. α. **typicum**, caulibus et ramis et petiolis dense patenterque villosis; pilis longis tenuibus flexuosis articulatis.—PERU: without locality, *Haenke* (DC., phot. Gr.); on calcareous rock, near Lima, alt. 300–600 m., *Weberbauer*, no. 1650 (Berl., fragm. Gr.).

Var. β. **brevipilum**, var. nov., puberulum vel tomentellum; pilis plerisque brevibus inarticulatis paucis passim longioribus.—PERU: on the Lima-Oroya Railroad, between Matucana and Tambo de Viso, on rocks, alt. 2370–2650 m., 26 Dec. 1901, *Weberbauer*, no. 103 (TYPE, Berl., fragm. Gr.); on slopes of eruptive rock, Matucana, alt. 2370 m., 24 Dec. 1901, *Weberbauer*, no. 66 (Berl.).

E. VARGASIANUM DC. Prod. v. 155 (1836). Further study of this species leads to the belief that it has been misinterpreted and that to it should be referred *E. macrophylloides* Robinson, Proc. Am. Acad. liv. 249, 340 (1918). The only two collections of the species known to the writer are as follows: at Caracas, *Vargas* (DC., phot. Gr.), and at Sanchorquiz, *Eggers*, no. 13,413 (U. S.). The localities, both in the Federal District of Venezuela, are not far apart.

The species has been reported twice from Colombia, namely by Klatt in Engl. Bot. Jahrb. viii. 36 (1887), on the basis of Lehmann's no. 938 from near Popayan, and by Heering, Mém. Soc. Neuchât Sci. Nat. v. 420 (1913), on Mayor's no. 392, from near Medellin. The writer has seen neither of these specimens and cannot confirm the accuracy of their determination, indeed is inclined to question it.

A related plant, confused with *E. Vargasianum* by Klatt in herb. and by the writer, ll. cc., appears to be a new species and is described above as *E. Steetzii*. The confusion of these two plants and the accidental omission of one species from the author's key to § *Subimbricata* of the Venezuelan Eupatoriums, Proc. Am. Acad. liv. 337 (1918), necessitates a revision of the last few lines of the key, thus:

i. Shrubs; leaves (5–) 6–13 cm. wide *j*.
  j. Heads in thyrsoid panicles; scales rounded or obtuse; stem glabrate . . . . 19. *E. morifolium*.
  j. Heads in corymbosely branched flattish-topped panicles; scales acute; stem tomentose *k*.
    k. Leaves about two-thirds as wide as long; petioles 1.2–3.8 cm. long; heads 15–18-flowered; achenes covered with sessile globules . . . . 20. *E. Steetzii*
    k. Leaves about four-fifths as wide as long; petioles 4–6 cm. long; heads 10–11-flowered; achenes hispid-puberulent . . . . 21. *E. Vargasianum*.

i. Perennial herbs; leaves 2–4.5 (–5) cm. wide; heads crowded *l.*
  l. Heads about 7-flowered, sessile in dense subglobose glomerules; involucre subcylindric . . . . .22. *E. tenuifolium.*
  l. Heads 20–25-flowered; pedicels 1–9 mm. long; involucre turbinate-campanulate . . . . .23. *E. pycnocephalum.*
i. Delicate annual; leaves thin, 2–5.5 cm. wide; inflorescence diffuse . . . . .23. *E. microstemon.*

Brickellia ? Arsenei, spec. nov., herbacea perenne virgata 7 dm. vel ultra alta; caule tereti gracili usque ad 3 mm. crasso folioso paullo flexuoso atropurpureo crispe griseo-puberulo; internodiis 1.5–4 cm. longis; foliis suboppositis vel saepe ternatim verticillatis sessilibus lanceolatis argute acuminatis basi rotundatis serrulatis chartaceis 3–4.5 cm. longis 1–1.6 cm. latis penniveniis supra reticulatis et minute granulatis subtus paullo pallidioribus reticulatis et in costa media et in venis majoribus puberulis; inflorescentia terminali multicapitata corymbosa convexa; capitulis graciliter pedicellatis ca. 12-floris ca. 1 cm. longis; involucri squamis ca. 20, 3–4-seriatim imbricatis regulariter gradatis, extimis brevissimis puberulis ceteris lanceolato-oblongis obtusiusculis glabris 3-nerviis in parte exposita atro-vinaceis; flosculis valde immaturis; corollae dentibus anguste oblongis; antheris angustis ad apicem longiuscule appendiculatis; pappi setis albidis vix scabridis.— Mexico: vicinity of Morelia, State of Michoacan, 26 Oct., 1911, alt. 2500 m., *Bro. G. Arsène*, no. 5608 (Gr., U. S.), distributed as *Eupatorium pulchellum* HBK. It is unfortunate that this beautifully distinct plant, obviously of the *Eupatorium* tribe, is available only in bud, so that it is impossible to determine the form of the mature achenes. However, as it is unlikely that the locality will be visited by a botanist for some time to come, and as the habit and such details of floral structure as are shown pretty clearly indicate that this is a new species of *Brickellia* § *Steviastrum*, the writer ventures to give it provisional record on this theory. It is a pleasure to dedicate it to Bro. Arsène, whose careful collections are among the most extended and valuable which have come from Mexico in recent years.

Calea caracasana (HBK.) Ktze., var. pilosior Ktze. Rev. Gen. i. 324 (1891). Of this more hairy variety of this common and in northern South America somewhat widely distributed and variable species, there is, besides the more common radiate state, a discoid form which may be recorded as follows:

Forma **discoidea**, forma nova, pubescens ut apud var. *pilosiorem*, capitulis homogamis, radiis nullis.— Colombia: forest, alt. 1200–1500 m., "La Virginia," Libano, Dept. Tolima, 22 Dec., 1917, *Pennell*, no. 3264 (N. Y.). Florets greenish-yellow.

SCHIZOTRICHIA EUPATORIOIDES Benth. in Benth. & Hook. f. Gen. ii. 410 (1873). A specimen of this rare and highly interesting generic monotype has been found among some Eupatoriums kindly lent to the writer from the New York Botanical Garden. Although unidentified, this specimen, collected in Peru by Mathews and corresponding in all described details with Bentham's clear diagnosis, is with scarcely a doubt a part of the original material secured by Mathews at Chachapoyas about 1836. Bentham in his treatment (l. c.) places the genus among the *Helenieae*, but adds at the end the remark that the plant might perhaps rather belong to *Eupatorium*. However, its short, blunt, and strongly recurved style-branches, its involucral scales marked with the immersed elongated glands characteristic of certain *Helenieae*, the subterete achenes, and the peculiar pappus of fimbriate scales all are foreign to or very unusual in the *Eupatorium* tribe, with which it certainly can have nothing to do.

Although Bentham gives a somewhat detailed generic character and mentions a specific name, he gives no specific diagnosis. To place the plant upon a somewhat more regular footing in this regard the following brief character may be put on record:

S. EUPATORIOIDES Benth., l. c., lignescens paullo pubescens; ramis dichotomis subteretibus foliosis; foliis elliptico-ovatis crenulatis tenuibus apice rotundatis et mucronatis basi rotundatis penniveniis 2–2.5 cm. longis 1–1.5 cm. latis breviter petiolatis; cymis sessilibus laxis compositis; bracteis minutis squamiformibus, pedicellis filiformibus glabris usque ad 1.5 cm. longis; capitulis ca. 8 mm. altis et crassis; involucri squamis principibus ca. 8 elliptico-oblongis tenuibus glabris subaequalibus dorso cum glandulis flavidis elongatis munitis; flosculis ut a Benthamio descriptis.—PERU: Chachapoyas, *Mathews*. In habit somewhat suggesting *Porophyllum*.

II. A RECENSION OF THE EUPATORIUMS OF PERU.

The literature of the Peruvian Eupatoriums is not extensive, the more important records on the subject being as follows: In 1786, Lamarck (Encycl. ii.) described as from Peru four species of *Eupatorium*, but of these one has since proved a *Stevia* and the others, collected by Joseph de Jussieu, came presumably from what is now Ecuador. Kunth in 1820, having worked over the collection of Humboldt & Bonpland, published (HBK. Nov. Gen. et Spec. iv.) descriptions of six species of *Eupatorium* from Peru, all being new to science. In 1836 the eldest DeCandolle (Prod. v.) indicated the Peruvian occurrence of twenty-one species of the genus, but several of these have subsequently dropped into synonymy or passed to other genera such as *Ophryosporus* and *Helogyne*. Poeppig in 1845 (Nov. Gen. ac Spec. iii.) added to the group five Peruvian species. Of these, however, at least two have since fallen into synonymy. In 1857, Weddell (Chlor. And. ii.) ascribed six species of *Eupatorium* to Peru, of which four were described as new, two of them being his own and the other two being species proposed but never characterized by Schultz-Bipontinus. In 1876, Baker (in treating the genus *Eupatorium* for von Martius's Flora Brasiliensis vi. pt. 2) incidentally mentions eleven species as extending into Peru. In 1883, Oliver (in Hook. Ic. xv. t. 1462) added an interesting species secured by John Ball. Klatt (Abh. Nat. Gesellsch. Halle, xv. and Ann. Nat. Hofmus., Vienna, ix.) described two new Eupatoriums from Peruvian material. Finally Hieronymus (in Engl. Bot. Jahrb. xxix., xxxvi., xl., and Verh. Bot. Ver. Brandenburg, xlviii.), after working over collections chiefly of von Jelski and of Weberbauer, mentions or discusses some twenty-eight species and varieties of Peruvian Eupatoriums, nineteen of these being characterized as new.

At no time has any effort been made to catalogue the Peruvian members of this group as such or to synopsize or key them. The present paper is put forth to assemble data previously scattered and to place on published record several species found in material recently worked. It is hoped that the treatment, while representing merely a stage in the course of work still in progress, will furnish what has certainly never before existed, namely, a means of speedy and accurate identification of the Peruvian Eupatoriums, besides adding some new elements to the group.

To save space, references are made where possible to the sectional, specific, and varietal descriptions in the author's recent paper on the Eupatoriums of Colombia, Venezuela, and Ecuador. In the case of Peruvian species and varieties not occurring in any of these countries and therefore not described in the paper just mentioned, diagnoses are here given. Thus, by the use of the two papers together, anyone desiring to identify a Peruvian *Eupatorium* will be able to consult a fairly detailed diagnosis of each species and variety of the genus thus far known from the country.

Sect. I. CYLINDROCEPHALA DC. (See Robinson, Proc. Am. Acad. liv. 270.)

KEY TO SPECIES.

a. Involucral scales squarrose, considerably altered in texture and somewhat spreading at the subtruncate or very bluntly pointed tip; leaves linear or narrowly oblong, nearly or quite sessile .... 1. *E. ivaefolium.*
a. Involucral scales closely appressed at the often darkened but not greatly modified tip; leaves lanceolate to deltoid- or rhombic-ovate, petiolate *b*.
  b. Heads solitary, long-pedicelled; leaves scabrous above, glabrous beneath; habit of § *Praxelis* .... 2. *E. serratuloides.*
  b. Heads corymbose or cymose-paniculate; leaves never pubescent above if glabrous beneath *c*.
    c. Heads 40–50-flowered; leaves small, lanceolate, 2.5–3.5 cm. long .... 3. *E. eripsimum.*
    c. Heads 7–35-flowered; leaves (except in *E. squalidum*) rarely less than 4 cm. long *d*.
      d. Pedicels glabrous or at most minutely granulated, furrowed; leaves glabrous on both surfaces, elliptical to rhombic, acute at both ends, thickish .... 4. *E. laevigatum.*
      d. Pedicels (when developed) pubescent *e*.
        e. Heads very slender, 7–10-flowered, acute in bud *f*.
          f. Involucre essentially glabrous; leaves attenuate-cuneate at base, mostly thin and nigrescent in drying .... 5. *E. leptocephalum.*
          f. Involucral scales (at least when young) definitely pubescent, especially toward the dark and mucronulate tip; leaves subrotund or only shortly cuneate at base, thickish-membranaceous, drying green .... 6. *E. tenuicapitulatum.*
        e. Heads thicker, mostly 15–35-flowered, obtuse in bud *g*.
          g. Heads about 8 mm. long, borne in a leafy panicle; leaves 1.5–4 cm. long; petioles 1–4 mm. long........................ 7. *E. squalidum*, v. *Rusbyanum.*
          g. Heads 1–1.2 cm. long; leaves 6–10 cm. long on petioles (except in *E. scabrum*) 1–2 cm. long *h*.

h. Heads in dense terminal and lateral corymbs *i*.
i. Stems when young furrowed and angulate, covered with long, spreading conspicuously dark-jointed hairs; leaves olive-green beneath, prominently and loosely reticulate-veiny on lower surface, in age strongly bullate-rugose above . . . . 8. *E. scabrum.*
i. Stems rather finely ribbed, nearly terete, softly and often sparsely pubescent, the hairs short, light-colored, not conspicuously jointed; leaves not prominently netted beneath nor bullate above . . . . . 9. *E. odoratum.*
h. Heads in (at maturity) open panicles *j*.
j. Leaves deltoid-lanceolate, truncate at the base . . . . 10. *E. clematitis.*
j. Leaves elliptic-lanceolate, cuneate at the base . . . . 11. *E. Jelskii.*

1. E. IVAEFOLIUM L. Syst. ed. 10, 1205 (1759) [as *iuaefolium*]; Robinson, Proc. Am. Acad. liv. 275 (1918). *E. fasciculare* Poepp. in Poepp. & Endl. Nov. Gen. ac Spec. iii. 54 (1845).—HUANUCO: Cassapi, Jan. 1830, *Poeppig*, no. 1660 (Vienna Hofmus., phot. Gr.). WITHOUT LOCALITY: *Mathews*, in 1862 (N. Y.). The plant of Poeppig is rather slender and short-leaved in the manner of var. EXTRORSUM (Sch. Bip.) Bak. in Mart. Fl. Bras. vi. pt. 2, 290 (1876), as *extrorsa*, which, however, appears to be merely a reduced state, the result perhaps of local drought. [Braz. to southern U. S.]

2. E. SERRATULOIDES HBK. Upright opposite-branched smoothish herb 4–6 dm. high; stem round, striate but nearly smooth; internodes rather long, much exceeding the leaves; these opposite, ovate, short-petioled, crenately 4–6-toothed on each side, 3-nerved, narrowed to an obtuse or rounded apex (not actually acute as stated in original diagnosis), roughish above, ciliate on the margin, glabrous beneath, 1.5–3 cm. long, half as wide, entire at the cuneate base, membranaceous; petiole 3.5–4.5 mm. long; heads subsolitary on the short upper branches, pedicellate, 8 mm. long; involucre campanulate-cylindric, the scales numerous, closely appressed in several series, rounded at the dark-purple tip, ciliate but otherwise glabrous and shining, excessively caducous; florets 7–8 mm. long, scarcely exceeding the involucre; corolla slenderly tubular, scarcely ampliated above, violet and smooth; pappus whitish, barbellate, nearly as long as the corolla.—Nov. Gen. et. Spec. iv. 117 (1820); DC. Prod. v. 143 (1836). *Osmia serratuloides* (HBK.) Sch. Bip. Pollichia, xxii.–xxiv. 252 (1866).—PIURA: steep slopes in cool region, Huancabamba, *Humboldt & Bonpland*, no. 3525 (Par., phot. Gr.).

3. E. ERIPSIMUM Robinson (p. 14). Decumbent undershrub; stems

(3–5 dm. high) and branches curved-ascending to erect, terete, incurved-puberulent, the hairs minute, white, non-glandular; leaves opposite, lanceolate, about equalling the internodes, 2.5–3.5 cm. long, 5–10 mm. wide, subentire or remotely and obscurely 1–3-toothed on each side, 3-nerved from the cuneate short-petioled or subsessile base, attenuate to a narrow but ultimately rounded tip, glabrous but often minutely pustulate above, finely appressed-pubescent on the nerves beneath; heads erect, terminal on the branches of 3–several-headed at length rather loose corymbs; pedicels 5–45 mm. long; involucre cylindric, the scales several-ranked and apparently graduated, the outer ovate to oblong, so excessively caducous as to have entirely fallen away in the material at hand; the inner linear-oblong to linear, all pale, stramineous, 1–4-costulate, especially toward the narrowed but rounded tip, ciliolate, otherwise glabrous; florets about 44; receptacle flattish; corolla slenderly tubular, lilac when fresh but drying dark-purple toward the scarcely dilated throat, 8.5 mm. long, glabrous; teeth 5, narrow, recurved; stamens with an oblong appendage; style-branches lilac, drying dark-purple, 6 mm. long; clavate, widely spreading; achenes prismatic, slightly tapering toward the often curved base, slender, 4.5 mm. long, finely appressed-pubescent, the faces grayish-brown, the ribs stramineous; pappus-bristles about 45, white, barbellate, nearly equalling the corolla.— ANCACHS: in an open formation consisting both of shrubs and herbs, especially bromeliads and cacti, alt. 2200–2500 m., at the town of Caraz, 19 May, 1903, *Weberbauer*, no. 3003 (Berl., phot. and fragm. Gr.).

In habit somewhat suggesting § *Praxelis*, but referred better to § *Cylindrocephala* on account of its flattish receptacle and the nature of its involucre. Most nearly related to the preceding, but certainly distinct.

4. E. LAEVIGATUM Lam. Encyc. ii. 408 (1786); Bak. in Mart. Fl. Bras. vi. pt. 2, 286 (1876); Hieron. in Engl. Bot. Jahrb. xxxvi. 465 (1905); Robinson, Proc. Am. Acad. liv. 273, 333, 346 (1918). *E. resinosum* Poepp. in Poepp. & Endl. Nov. Gen. ac Spec. iii. 54 (1845).— CAJAMARCA: between Chota and Cutervo, *von Jelski*, no. 793, acc. to Hieron. l. c. LORETO: Moyobamba, *Mathews* (Gr.). CUZCO: Santa Ana, *Cook & Gilbert*, no. 1493 (U. S.); Machu Picchu, alt. 2100 m., *Cook & Gilbert*, no. 1025 (U. S.). HUANUCO: on open savannahs, subandean region, at Cassapi, *Poeppig* no. 1217 (Hofmus. Vienna, phot. Gr.); without locality, *Mathews*, nos. 1361, 1362, and 1363 in part (all N. Y.). [Mex. to Argent.]

5. E. LEPTOCEPHALUM DC. Prod. v. 148 (1836); Robinson, Proc.

Am. Acad. liv. 278, 346 (1918). PERU: on the Andes, but without record of exact locality, *Haenke* acc. to DC. l. c. [Ecuador, Colombia.] No Peruvian material or more precise record of this species has been seen by the writer, though it appears to be tolerably frequent in Ecuador.

6. E. TENUICAPITULATUM Hieron. Opposite-branched, erect, 5 dm. high, herbaceous; stems and branches terete, dull purple, covered at first with a scanty delicate appressed and jointed pubescence; leaves opposite, ovate-lanceolate, caudate-acuminate, finely serrate, firmly membranaceous, appressed-puberulent on the nerves and chief veins, green and glabrous above, lighter green and punctate beneath, about 1 dm. long and 3 cm. wide, base entire, rounded but at the insertion subcuneate; petiole 5–12 mm. long; corymbs terminal, compound, crowded, flattish; heads about 10-flowered, subsessile by 2's and 3's, about 8 mm. long and 2 mm. thick (acc. to Hieron.); involucral scales about 20, stramineous, with dark purplish mucronate pubescent at length slightly squarrose tips.— Hieron. in Engl. Bot. Jahrb. xxxvi. 465 (1905).— CAJAMARCA: near Tambillo, *von Jelski*, no. 598 (Berl., fragm. Gr.).

7. E. SQUALIDUM DC. Prod. v. 142 (1836); Bak. in Mart. Fl. Bras. vi. pt. 2, 281, t. 77 (1876); Robinson, Proc. Am. Acad. liv. 334 (1918).

[Var. TYPICUM Robinson, l. c. Branches villous-hirsute, the hairs mostly straight, widely spreading, attenuate, with dark articulations; leaves roundish-ovate, of firm texture, very shortly petioled.— Eastern Brazil, e. g. at Marianna, Minas Geraës, *Vauthier*, no. 279 (DC., Gr.), etc.].

Var. RUSBYANUM Robinson (p. 34). Stem and branches covered with short whitish mostly incurved and unjointed hairs; leaves ovate-lanceolate to rhombic-ovate, above sparingly appressed-puberulent, beneath sordid-tomentellous and glandular-punctate, the indumentum sufficiently dense to obscure the punctation.— PERU WITHOUT PRECISE LOCALITY: *Mathews* (N. Y.). [Boliv.]

8. E. SCABRUM L. f. Suppl. 354 (1781); J. E. Sm. Ic. iii. t. 67 (1791); Bak. in Mart. Fl. Bras. vi. pt. 2, 299 (1876); Robinson, Proc. Am. Acad. liv. 277 (1918).— PERU WITHOUT LOCALITY: Baker, l. c., extends the distribution of this Colombian species to Peru but does not mention the collector [Mathews?]. The species has been so variously interpreted in the past that this single and undetailed record of its occurrence in Peru must until verification be subject to considerable doubt. [Colombia.]

9. E. ODORATUM L. Syst. ed. 10, 1205 (1759); Robinson, Proc. Am. Acad. liv. 280, 346 (1918). *E. conyzoides* Mill. Dict. ed. 8, no. 14 (1768);

Vahl, Symb. iii. 96 (1794); Bak. in Mart. Fl. Bras. vi. pt. 2, 277 (1876). *E. floribundum* HBK. Nov. Gen. et Spec. iv. 118, t. 344 (1820). *E. conyzoides*, var. *floribundum* (HBK.) Hieron. in Engl. Bot. Jahrb. xxxvi. 463 (1905); as *floribunda*. *E. conyzoides*, var. *tambillense* Hieron. l. c. 464.— CAJAMARCA: near Tambillo, *von Jelski*, nos. 783 (Berl., fragm. Gr.), 785 (Berl., fragm. Gr.), and acc. to Hieron. l. c. (under his varieties founded on the highly inconstant number of florets and degree of pubescence) also *von Jelski*, nos. 780–782, 784, 786. [Widely distributed in the warmer parts of America; common and variable.]

10. E. CLEMATITIS DC. Shrub with round flexuous smoothish green or brownish stems and opposite divaricate curved-ascending branches; leaves opposite, slender-petioled, rather thin, ovate- or deltoid-lanceolate, acuminate, subentire or shallowly and bluntly 2–3-toothed on each side, subtruncate, rounded or obtusely pointed at the base, 3-nerved, green on both sides, sparingly hirtellous chiefly on the nerves and glandular-punctate or resinous-atomiferous (the resin globules at first golden brown, at length turning whitish), 3–5 cm. long, 1.3–3 cm. wide; petiole about 1 cm. long; corymbs open, mostly few-headed; heads cylindric, 1 cm. long, about 25-flowered; pedicels mostly 6–15 mm. in length; involucral scales smooth, closely appressed, rounded or obtuse at the greenish tips; achenes slender, fuscous, smooth, 4 mm. long.— Prod. v. 144 (1836).— PERU? without exact locality, *Poeppig*, no. 3108 (DC., phot. and fragm. Gr.).

This species has been studied from a clear photograph of the type in the Prodromus Herbarium at Geneva and a fragment of the type number in the Klatt collection purchased by the Gray Herbarium.

11. E. JELSKII Hieron. Hirsute-villous undershrub 1 m. high; branches somewhat virgate, terete, purplish, leafy up into the terminal opposite-branched panicle; hairs articulated, rusty, spreading or somewhat tangled; leaves thin, lanceolate, coarsely few(2–8)-toothed, acute, cuneate at base, 6–9 cm. long, 1.4–3 cm. wide, above sparingly sordid-villous or at length only scabrid, below densely villous-hirsute chiefly on the nerves and larger veins, 3-nerved from near the base; petiole mostly 5–7 mm. long, densely tawny-villous; heads slender, cylindrical, about 20-flowered, about 1 cm. in length, on filiform pedicels at maturity nearly as long; involucral scales about 30, reddish-purple, green-nerved toward the rounded or obtuse slightly puberulent and ciliolate tip; corollas apparently purple, slightly enlarged toward the summit, granulate on the outside; achenes dark-colored.— Hieron. in Engl. Bot. Jahrb. xxxvi. 464 (1905).— CAJAMARCA: near Tambillo, *von Jelski*, no. 665 (Berl., fragm. Gr.). WITHOUT LOCALITY: *Mathews* (N. Y.).

Sect. II. Subimbricata (DC.) Hoffm. (See Proc. Am. Acad. liv. 281.)

Key to Species.

a. Leaves pinnate-veined, the lower veins not conspicuously longer or larger.
b. Leaves on wingless petioles; blade cordate or hastate at the broad base.
Petioles very short, rarely over 1.5 mm. long........54. *E. Weberbaueri.*
Petioles 1 cm. or more in length.
Leaves cordate, with an open triangular sinus.....24. *E. glomeratum.*
Leaves with a deep narrow sinus at the base.
Leaves cordate, acutish, almost regularly crenate-dentate.......................................25. *E. Gascae.*
Leaves hastate, caudate-attenuate, with coarse and very unequal rounded teeth...............26. *E. anisodontum.*
b. Leaves on wingless petioles; blade rounded at the base (in *E. persicifolium* sometimes shortly acuminate at the point of insertion from a generally rounded base).
Leaves ovate or elliptical, 1.5–2 times as long as wide, acute or barely acuminate.
Leaves membranaceous, dull and puberulent to tomentellous above, the veins not depressed...........27. *E. endytum.*
Leaves coriaceous, shining and glabrous above, the veins reticulated and depressed..................28. *E. tarapotense.*
Leaves ovate- or lance-oblong, 2.5–4 times as long as wide, gradually acuminate or attenuate.
Heads about 25-flowered; involucre campanulate-subcylindric, scales 30–40, at length of firm texture, in 4–6 rows; achenes 5.5 mm. long; leaves permanently canescent-tomentellous beneath.............12. *E. persicifolium.*
Heads 16–19-flowered; involucre campanulate, scales 16–20, thin, in 3–4 rows; achenes 4–4.5 mm. long; leaves at length green, subglabrate and prominently reticulated beneath............................15. *E. Salvia.*
b. Leaves lanceolate to ovate or broadly elliptical, cuneate or at least obtusely pointed at base; petioles wingless, rarely less than 1 cm. long *c.*
c. Leaves puberulent to villous beneath *d.*
d. Leaves lanceolate, caudate-attenuate, remotely few-toothed; pubescence conspicuously ferruginous...29. *E. Mathewsii.*
d. Leaves lance-oblong, denticulate, crenulate, or subentire; pubescence white or merely dull, not ferruginous.
Heads 13–15-flowered.
Leaves denticulate, canescent-tomentellous beneath, often 15–18 cm. long.................13. *E. buddleaefolium.*
Leaves serrulate, green and finely puberulent on the veins beneath, rarely over 1 dm. long....17. *E. pseudarboreum.*
Heads about 25-flowered; leaves somewhat canescent-tomentellous beneath.
Leaves subentire, scarcely crenulate, cuneate at the base.......................................14. *E. discolor.*
Leaves crenate-denticulate, the base rounded but sometimes shortly acuminate at the point of insertion.............................12. *E. persicifolium.*

d. Leaves ovate or elliptical.
Heads about 16-flowered; involucral scales obtuse; leaves rhombic-ovate, 3–4.5 cm. wide.....31. *E. helianthifolium.*
Heads 20–24-flowered; involucral scales acute; leaves elliptical, at maturity about 1 dm. wide.......32. *E. vestitum.*
c. Leaves glabrous or merely atomiferous beneath.
Heads in a divaricately branched pyramidal panicle; leaves undulate-margined and with a delicate prominulent reticulation above.......................33. *E. orgyaloides.*
Heads in broad terminal rounded corymbs; veinlets not prominulent on the upper surface of the serrate leaves.
Involucral scales conspicuously arachnoid-ciliate; leaves dull above, 1.5–2.3 cm. wide..........34. *E. drepanoides.*
Involucral scales minutely and obscurely ciliolate; leaves lucid above, 3–6 cm. wide............35. *E. coelocaule.*
b. Leaves narrowly lance-oblong to linear, sessile or on very short petioles (1–4 mm. long) *e*.
e. Heads large, 1.4–2.5 cm. long.
Involucral scales regularly graduated in 3–4 series; heads 1.4–1.8 cm. long.
Leaves linear, about 3 mm. wide................18. *E. Gayanum.*
Leaves lance-oblong, 8–15 mm. wide..............19. *E. Ballii.*
Involucral scales much imbricated, but the outer scarcely shorter; heads 2.5 cm. long.................20. *E. Carsonii.*
e. Heads 7–10(–12) mm. high *f*.
f. Heads mostly 30-40-flowered, in loosely fastigiate-branched panicle..........................36. *E. amygdalinum.*
f. Heads 7–15-flowered, mostly in dense thyrsoid inflorescence *g*.
g. Leaves linear or nearly so, 3–6 mm. wide.
Leaves strongly deflexed, tardily glabrate and smoothish (not conspicuously rugose-bullate) above, entire (the margins strongly revolute) 21. *E. lavandulaefolium.*
Leaves spreading, promptly glabrate and strongly rugose-bullate above, appearing crenulate from the depressed veins in the very revolute margins..................................22. *E. chotense*
g. Leaves narrowly lance-oblong, 1–2 cm. wide in the middle.
Shrubs 1–2 m. high; leaves thickish, of firm texture, chartaceous or coriaceous.................23. *E. Volkensii.*
Herbaceous or nearly so, 2–3 dm. high; leaves thin, membranaceous, of soft texture............37. *E. Sprucei.*
b. Leaf-blade contracted below into a winged petiolar portion, either cuneate or cordate-auriculate at its insertion on the stem.
Heads 14–17-flowered; leaves entire, tomentulose beneath. 38. *E. pilluanense.*
Heads about 20-flowered; leaves coarsely toothed or lobed, coarsely pubescent on the veins beneath.........39. *E. Commersonii.*
Heads 200–300-flowered. Here may be sought.......81. *E. nemorosum.*
a. Leaves palmately 3–7-nerved from or somewhat above the base, or subpinnately veined, but with one of the lower pairs of veins considerably longer or thicker than the others *h*.
h. Heads 4–7-flowered *i*.

i. Leaves velvety-puberulent to tomentose beneath.
Leaves entire, rounded at base............40. *E. trachyphyllum.*
Leaves crenate-dentate, cordate at base.......24. *E. glomeratum.*
i. Leaves glabrous beneath *j.*
j. Leaves 4–10 cm. long or more.
Leaves broadly ovate, thin, rounded or cordate at the wide base..........................41. *E. acuminatum.*
Leaves ovate-lanceolate to elliptical, thickish, acute at the base.
Heads subsessile on the flexuous ascending branches of an ample racemose panicle; leaves elliptical, reticulated on both surfaces............42. *E. crenulatum.*
Heads in dense round-topped corymbs.
Involucral scales conspicuously arachnoid-ciliolate; leaves dull above, 1.5–2.3 cm. wide..34. *E. drepanoides.*
Involucral scales minutely and obscurely ciliolate; leaves lucid above, 3–6 cm. wide.......35. *E. coelocaule.*
j. Leaves small, less than 1 cm. in diameter, broadly ovate, obovate, or suborbicular..............53. *E. incarum.*
h. Heads 8–12-flowered *k.*
k. Leaves lanceolate, ovate, or suborbicular, obtuse, rounded or cordate at base (in some cases very shortly acuminate at attachment of petiole) *l.*
l. Leaves small (about 1.6 cm. long), obtuse. 55. *E. chamaedrifolium.*
l. Leaves larger (4–12 cm. long), acute or acuminate, or conspicuously narrowed to a blunt apex *m.*
m. Leaves rather deeply cordate with a triangular sinus; heads nearly sessile in globose glomerules. 24. *E. glomeratum.*
m. Leaves obtuse, rounded, truncate at base, or shallowly and broadly cordate; heads often crowded in compound corymbs but not in dense globose glomerules.
Petioles (of the mature cauline leaves) 17–32 mm. long.
Leaves at length rugose-bullate above, conspicuously reticulated beneath, the nerves connected by prominulent and subregular cross-veins, these densely villous....................43. *E. Stuebelii.*
Leaves membranaceous, neither bullate above nor prominulently reticulated beneath......44. *E. callacatense.*
Petioles 8–10 mm. long.
Involucral scales broad, the intermediate and inner elliptical, stramineous, nearly smooth, 3–5-ribbed, the outer suborbicular, loosely pubescent.........................45. *E. urubambense.*
Involucral scales narrowly oblong, densely white-tomentose, obscurely 1-nerved......46. *E. leucophyllum.*
k. Leaves rhombic-ovate or -lanceolate, acutish to acuminate or attenuate at base.
Leaves merely acutish at base and definitely petioled, soft-pubescent above, glandular-punctate and puberulent beneath............................31. *E. helianthifolium.*
Leaves attenuate at base to an often short and obscure petiole, somewhat scabrous to tomentose above, densely canescent-tomentose beneath........47. *E inulaefolium.*

h. Heads 14–30-flowered *l*.
l. Perennial herbs or shrubs *m*.
m. Leaves rhombic-ovate, attenuate to a subsessile base . . . . .47. *E. inulaefolium.*
m. Leaves ovate-lanceolate to suborbicular, obtuse, rounded, or cordate at base.
Leaves pubescent or scabrous above.
Middle and outer involucral scales acute.
Heads rather few and loosely disposed in open panicles; pubescence of short light-colored hairs neither gland-tipped nor conspicuously jointed. 48. *E. gracilentum.*
Heads many, glomerate at ends of branches of an open panicle, pubescence dense, dark, purplish-brown, the hairs perceptibly dark-jointed. 49. *E. Cookii.*
Middle and outer involucral scales obtuse or rounded at the tip . . . . .56. *E. marrubiifolium.*
Leaves glabrous above, and merely puberulent on the nerves and veins beneath . . . . .50. *E. marginatum.*
l. Delicate annual; leaves slender-petioled . . . . .51. *E. microstemon.*
h. Heads 45–100-flowered.
Heads numerous, glomerate or at least crowded at the ends of the branches in a leafy-bracted panicle; pedicels mostly 0–3 mm. long . . . . .30. *E. Lobbii.*
Heads numerous, in an open loose panicle; pedicels mostly 1–2 cm. long; involucral scales multistriate . . . . .52. *E. vitalbae.*
Heads in few (mostly 1–5)-headed corymbs; pedicels 5–20 mm. long; involucral scales 3-ribbed . . . . .16. *E. tahonense.*

12. E. PERSICIFOLIUM HBK. Nov. Gen. et Spec. iv. 130 (1820); Ball, Jour. Linn. Soc. xxii. 43 (1885); Robinson, Proc. Am. Acad. liv. 350 (1918). *E. arboreum* HBK. l. c. 131; Robinson l. c. 351. *E. compactum* Benth. Bot. Sulph. 112 (1844).—LIMA: Huamantango, *Barclay* (K., phot. Gr.); Obrajillo, *Wilkes Exp.* (Gr.); Chicla, *Ball*, acc. to Ball, l. c. ANCACHS: Prov. Cajatambo: on grassy plains with numerous scattered bushes, alt. 3400–3700 m., Ocros, 28 March, 1903, *Weberbauer*, no. 2682 (Berl., fragm. Gr.). CUZCO: Ollantaytambo, alt. 3000 m., *Cook & Gilbert*, nos. 330 (U. S.), 335 (U. S.); Piñasniocj, Panticalla Pass, alt. 3600 m., *Cook & Gilbert*, no. 1809 (U. S.).

In a recent paper the writer, l. c., endeavored to maintain *E. arboreum* HBK. as distinct from *E. persicifolium*, but the characters are so slight that it now seems impossible to keep these species satisfactorily apart or key them from each other with any clearness whatever. [Ecuad.]

13. E. BUDDLEAEFOLIUM Benth. Pl. Hartw. 135 (1844); Robinson, Proc. Am. Acad. liv. 350 (1918).—AMAZONAS: Prov. Chachapoyas, *Mathews* (Gr.). [Ecuad.]

14. E. DISCOLOR DC. Shrubby; branches flexuous, slender, hexagonal, nearly smooth; leaves opposite, oblong-lanceolate, acuminate,

acute at base, slightly revolute and obscurely crenulate (at first sight entire), above glabrous but finely bullate-rugulose, beneath closely cinereous-tomentose, feather-veined (the veins 22–28 on each side, somewhat darker than the rest of the surface), about 9 cm. long, 1.5–2.5 cm. wide; petiole about 8 mm. long; corymbs terminal, few-headed, rather dense, 3–5 cm. in diameter, convex; heads about 25-flowered, short-pedicelled; involucre several-seried, subturbinate-campanulate; scales acute, ciliate; achenes roughened on the angles.— Prod. v. 161 (1836).— HUANUCO: near the city of the same name, *Haenke* (DC., phot. Gr.). The locality was cited by DeCandolle under the unrecognized name "Huanaceria," but it seems probable that this was a mere misreading of Huanuco, formerly spelled Huanacco, where Haenke did much of his work. The species is closely related to *E. persicifolium* HBK., but the leaves are cuneate at base, more decidedly discolorous, and nearly entire.

15. E. SALVIA Colla. Shrub 8–10 dm. high, somewhat sticky on the younger parts; branches subterete, somewhat ribbed (when dried), purplish-brown to buff, at first granular-puberulent, soon glabrate, ascending, usually very leafy, terminating in compound round-topped corymbs; leaves (often proliferous in the axils) lanceolate, gradually narrowed from below the middle to a sharp point, irregularly dentate (teeth small, obtuse or merely acutish), of firmish texture but rather thin, feather-veined, rounded or obtusish at base, above glabrous and bullate, beneath finely reticulated, paler, more or less arachnoid-puberulent but often nearly or quite glabrate and pale-green, 5–8 cm. long, 1.5–2.4 cm. wide; petiole 2–8 mm. long, slender; corymbs 3–8 cm. in diameter, somewhat peduncled (i. e. uppermost leaves reduced and bractlike); heads 16–19-flowered, 1.3 cm. high, 8 mm. in diameter, pedicelled; involucral scales 16–20, graduated, 2–3-seriate, partly purplish-tinged, the outer ovate, somewhat viscid-pubescent, the inner lance-oblong to linear, nearly glabrous, finely nerved; corollas purple, 6 mm. long, glabrous; achenes 4.3 mm. long, slender, obscurely puberulent, with dark concave faces and lighter-colored ribs, 4.3 mm. long; pappus-bristles about 46, often purple-tinged.— Mem. Tor. xxxviii. 8 (1835); Gay, Fl. Chil. iii. 473 (1847); Reiche, Fl. Chil. iii. 264 (1901). *E. reticulatum* Hook. & Arn. Bot. Beech. 29 (1830), not Desv.— PERU WITHOUT LOCALITY: *Hayne* (Berl.). [Chile.] VERNACULAR NAME: *Salvia Macho*, acc. to Gay, l. c.

16. E. TAHONENSE Hieron. Suffruticose, attaining 1 m. in height; branches round, at first sticky-glandular; lower internodes 1–2.5 cm. long, the upper as much as 7 cm. in length; leaves opposite, ovate,

acutish or obtusish, cordate at base but with a short acumination at the insertion, subcrenate-dentate (teeth 15–20 on each side, the largest 1.5 mm. high, 2.5 mm. broad, with revolute mucro), above slightly roughened in age and somewhat bullate, beneath loosely glandular-tomentulose, 3-nerved from the very base, about 4 cm. long, 2.25 cm. wide; corymbs about 5-headed from the axils of the upper leaves; heads 70–100-flowered, on sticky-glandular pedicels (5–20 mm. in length); involucre campanulate, 3–4-seriate; scales 40–55, the inner densely glandular, lance-linear, acute, the outer gradually shorter and broader, acuminate, ovate-oblong, 5–7-nerved at the base, the nerves anastomosing; corollas about 5.5 mm. long, scarcely ampliated toward the summit, glabrous; style-branches moderately thickened; achenes brown, 4 mm. long, roughened throughout; pappus-bristles 30–40, dirty-white.— Hieron. in Engl. Bot. Jahrb. xl. 372 (1908).— CAJAMARCA: below the Hacienda La Tahona near Hualgayoc, alt. 2600 m., *Weberbauer*, no. 4037. Not seen by the writer, the description here compiled with condensation from the original character.

17. E. **PSEUDARBOREUM** Hieron. Shrub; branches somewhat hexagonal, glabrous, grayish-brown, when young viscid and slightly vernicose; internodes 1–5 cm. long; leaves opposite, lanceolate to lance-oblong, acuminate, serrulate except near the cuneate base (teeth about 0.7 mm. high, 1–1.5 mm. wide), feather-veined (chief veins about 15 on each side), deep green, smooth, lucid, and nearly glabrous above, beneath olive-green, puberulent on the prominulent and reticulated veinlets, chartaceo-membranaceous, 7–10 cm. long, 2.4–3.5 cm. wide; petiole slender, about 1 cm. long; corymbs terminal on the branches, together forming a leafy panicle; heads about 14-flowered, on puberulent pedicels (2–5 mm. long); involucre 3–4-seriate, graded, ovate to lanceolate, acute, ciliate, green, with 3–5 pale ribs; corollas glabrous, slender, slightly and gradually enlarged upwards, 5 mm. long; achenes yellowish-brown, ciliolate on the angles, 3.3 mm. long (scarcely mature); pappus-bristles about 40, whitish with a slight yellow tinge, nearly smooth.— Hieron. in Engl. Bot. Jahrb. xxxvi. 469 (1905).— CAJAMARCA: Callacate, May, 1879, *von Jelski*, no. 766 (Berl., fragm. Gr.).

18. E. **GAYANUM** Wedd. Shrubby, 1 m. high; branches slender, ascending, nodulose after the fall of the leaves, at first closely enveloped in a yellowish-white wool, at length nearly glabrate, brownish, terete; internodes 5–35 mm. long; leaves opposite, subsessile crowded on the branchlets, linear, 3–5 cm. long, about 2 mm. wide, obtusish, grayish-pubescent and somewhat bullate above, densely covered beneath with yellowish-white wool; margins revolute, apparently entire; cymes loose,

terminal or somewhat lateral, 3–5-headed; pedicels woolly, 5–15 mm. long, filiform, mostly recurved; heads rather large, 1.3 cm. high, about 1 cm. in diameter; involucre narrowly campanulate, the scales lanceolate, attenuate, the inner long, stramineous, smoothish, the outer progressively shorter, rather densely woolly; corollas glabrous, greenish (Weberbauer), 8–10 mm. long, the style-branches clavate, purple; achenes (immature) pale, 4 mm. long, slightly hispidulous on the angles. — Chlor. And. i. 216, t. 40A (1857).— CUZCO: on mountains, *Gay*, acc. to Wedd. l. c. JUNIN: calcareous rocks, alt. 3700–4000 m., between Tarma and La Oroya, Feb., 1903, *Weberbauer*, nos. 2524 (Berl.) and 2583 (Berl., fragm. Gr.).

19. E. BALLII Oliver. Similar to the preceding in habit and inflorescence; leaves narrowly lance-oblong, acute, cuneate at the subsessile base, 5–8 cm. long, 7–13 mm. wide, thinnish, sparingly arachnoid and finely bullate above, paler and sparingly sordid-woolly beneath; cymes loose, few-headed; heads large, 1.5–1.8 cm. long, mostly nodding on slender (at maturity) dark glandular-puberulent pedicels; involucral scales ovate-elliptic, acuminate, thin, finely many-striate, 3–5-seriate, graduated, the outer distinctly shorter and more woolly; florets as in the preceding.— Oliv. in Hook. Ic. xv. 49, t. 1462 (1884).— LIMA: on ledges of the Andes, near Chicla, alt. 3660–3965 m., *Ball* (Gr.).

20. E. CURSONII Robinson. Probably shrubby and with the habit of the two preceding species; branchlets round, slender, fusco-tomentellous, leafy toward the summit; internodes about 8 mm. long; leaves opposite, linear-lanceolate, attenuate to both ends, above slightly puberulent and bullate-rugulose; beneath paler, reticulate-venose, tomentose, 9 cm. long, 8 mm. wide, thickish, the margin strongly revolute, slightly crenulate; petiole about 1 mm. long; heads in terminal 3-headed cymes, about 80-flowered, very large for the genus, 2.5 cm. long about equally thick; pedicels slender (short in fragmentary type but with greater maturity presumably elongating); involucral scales much imbricated in 3–5 series but not graduated as to length, lanceolate, acute, the outer thickish, 2 cm. long, 4 mm. wide, finely striate, granular-puberulent on the back, subherbaceous, scarious-margined; the inner narrower, smoother and more stramineous; receptacle apparently somewhat paleiferous toward the edge; pales very narrow, filiform-attenuate; corollas 1 cm. long, slenderly tubular, glabrous; style-branches, clavate, somewhat flattened, dark purple; achenes 7 mm. long, slightly roughened on the angles; pappus-bristles about 50, some of the outer shorter.— Proc. Am. Acad. xlii. 38 (1906).— AREQUIPA: Arequipa, *Curson* (Brit. Mus., phot. Gr.).— Known only from a single

branch bearing a few leaves and a very large, fairly mature head flanked by two heads still in bud.

21. E. LAVANDULAEFOLIUM DC. Slender branched shrub; branches suberect, when young densely clothed with yellowish-white wool; internodes 2–4 cm. long; leaves opposite, subsessile, linear, entire, slightly narrowed to an obtusish point, strongly revolute on the margins, 2–3 cm. long, 1.3–3.4 mm. wide, above glabrate, bullate-rugulose, beneath white-woolly; heads glomerate at the ends of the branches of small terminal panicles, subsessile, 9–10-flowered, about 8 mm. high; involucre narrowly campanulate, 3–4-seriate; scales graduated, thin, scarioso-stramineous, obtusish, the outermost somewhat woolly; corollas conspicuously granulate toward the limb.— Prod. v. 154 (1836).— PERU WITHOUT LOCALITY: in valleys of the Andes, *Haenke* (Gr.), from South America, presumably Peru, *Dombey* (fragm. and sk. Gr.).

22. E. CHOTENSE Hieron. Erect branching shrub attaining 5 dm. in height; branches round, when young cinereous-tomentose, soon only sordid-arachnoid, internodes 2–7 (the uppermost as much as 12–15) mm. long; leaves opposite, shortly but distinctly petiolate, linear-oblong, obtuse, 3.4–4.5 cm. long, 4–6 mm. wide, distinctly feather-veined, the lateral veins 20 or more on each side, leaving the midnerve at a large angle (about 80°–85°), upper surface glabrous, lucidulous, bullate-reticulated, the lower gray-tomentose, margin strongly revolute and from the depressed veins appearing crenulate; corymbs compound, terminal, strongly convex subhemispherical, 3 cm. in diameter; heads 10–11-flowered; involucre subcylindric, multiseriate in the manner of § *Cylindrocephala* but of looser nature, the scales about 24, scarioso-stramineous, the inner purple-tinged toward the somewhat narrowed but at the apex rounded tips, the short outer scales webby-woolly; corollas glabrous, slightly enlarged upward, about 4 mm. long; achenes dark, nearly or quite smooth, about 2 mm. long.— Hieron. in Engl. Bot. Jahrb. xxxvi. 466 (1905).— CAJAMARCA: between Chota and Cutervo, *von Jelski*, no. 794 (Berl., fragm. Gr.); near Cutervo, *von Jelski*, no. 631 (Berl., fragm. Gr.).

23. E. VOLKENSII Hieron. Branching shrub, attaining a height of 1 m. (Weberbauer) or 1.2–1.5 m. (Cook & Gilbert); branches at first slightly pubescent, at length glabrate; leaves opposite, linear-lanceolate, narrowed to an acutish or mucronulate tip, cuneate at base, crenate-serrate (teeth scarcely 1 mm. high, as much as 3.5 mm. wide), above scabrid-puberulent, beneath white-pubescent, feather-veined (the more prominent veins 8–10 on each side), chartaceous, 6–12 cm. long and 9–19 mm. wide, the reticulated veinlets depressed above, prominulent

beneath; petiole 1–4 mm. long; heads in terminal corymbs, 10–12-flowered, about 1 cm. high; involucre narrowly campanulate; scales about 20, dusky-stramineous, scarious, obtuse, slightly lacerate-ciliate toward the tip, the outer gradually shorter, about 5-nerved; corollas gradually enlarged toward the summit, scarcely 5 mm. long, glabrous, pale-blue or pale-purple; achenes dark, hispidulous on the angles and slightly so on the upper part of the faces; pappus-bristles 30–35, yellowish-white.— Hieron. in Engl. Bot. Jahrb. xl. 370 (1908).— CUZCO: on the hill Sacsahuamán, near Cuzco, alt. 3500–3600 m., *Weberbauer*, no. 4850; Ollantaytambo, alt. about 3000 m., *Cook & Gilbert*, nos. 369 (U. S.) and 589 (U. S.).

This species in habit and many essential characters is exceedingly close to *E. salicinum* Lam., of which it may ultimately prove a variety. However, the leaves are less thick and not so firm nor so deeply wrinkled; their margins are less strongly revolute, and the upper surface permanently hirtellous — the hairs (not readily visible except with a lens) being white, subappressed, and slightly tuberculate-thickened at the base. In *E. salicinum*, on the other hand, the leaves, even when young, are quite glabrous on the upper surface, which is strongly bullate-rugose and somewhat lucid. Furthermore, the achenes (at least in the form occurring in Colombia) are sparsely covered with short-stiped capitate glands. The flower-color in *E. Volkensii* was noted by Dr. Weberbauer as bluish, a statement questioned by Hieronymus. Cook & Gilbert give the flower-color of their no. 369 as "pale wistaria violet" and of their no. 589 as "pale blue." In the dried material of both of these numbers the limb and upper part of the throat of the corollas still show in the dried material a pale purple coloration.

In regard to the recorded Peruvian occurrence of *E. salicinum*, see below (p. 87).

24. E. GLOMERATUM DC. Suffruticose, shortly but rather coarsely spreading-pubescent; branches terete, leafy to the inflorescence; internodes 6 cm. or more long; leaves opposite, ovate-lanceolate, acuminate to scarcely acute, crenate-dentate except at base, where cordate by an open sinus, slightly pubescent above, softly but not very densely sordid-tomentose and somewhat veiny beneath, 4–6 cm. long, 2–3 cm. wide; petiole slender, 8–14 mm. long; globose glomerules 1–3 cm. in diameter, terminal on the divaricate branches of a leafy-bracted open ovoid or subpyramidal panicle; heads sessile, 7–8-flowered; involucre subcylindric-campanulate, the scales brownish-stramineous, striate, the inner oblong, obtuse, essentially glabrous, the outer progressively shorter, ovate, somewhat hairy dorsally near the blunt or rounded tip.— Prod.

v. 154 (1836).—LIMA: Obrajillo, *Wilkes Exp.* (U. S., phot. Gr.). Originally described from material from the herbarium of *Haenke* (DC., phot. Gr.), said by DeCandolle to have been found "inter Chilenses."

25. E. GASCAE Robinson (p. 15). Robust, densely covered with spreading and jointed dark-brownish glandular hairs; leaves opposite, ovate, cordate, subacute, rather coarsely and evenly dentate, deep green, strongly bullate-rugose and puberulent above, somewhat paler, sordid-tomentose and conspicuously netted beneath, about 1 dm. long, 4–7 cm. wide; petiole densely dark-hairy, 2–4 cm. long; corymbs terminal, dense, sessile; heads about 22-flowered, 1.5 cm. high, 6 mm. thick; involucral scales about 21, greenish-straw-colored, ovate, acute, finely striate-ribbed; receptacle flat; corollas slender, gradually and very slightly enlarged toward the top; achenes 4 mm. long.—AMAZONAS: Prov. Chachapoyas, *Mathews* (Gr.). A very marked species with foliage suggesting some of the coarser Salvias. In habit and inflorescence recalling § *Conoclinium*, but with a flat receptacle.

26. E. ANISODONTUM Robinson (p. 6). In habit, foliage, and pubescence similar to the preceding, but leaves triangular-hastate, caudate-acuminate, with unequal teeth (some short and rounded, others 2–3 times as long and acutish), basal sinus deep and narrow; corymb pedunculate, strongly convex, congested; heads about 10-flowered, 1 cm. long; involucre cylindric-campanulate; corollas 6 mm. long, smooth; achenes 4 mm. in length, glabrous.—AMAZONAS: Prov. Chachapoyas, 1836, *Mathews*, no. 87 H (K., phot. and fragm. Gr.).

27. E. ENDYTUM Robinson (p. 13). A stoutish shrub, 1–2 m. high, covered with a short, dense and somewhat matted reddish-brown velvety pubescence; stems round, hollow, leafy well into the large opposite-branched ovoid panicle; leaves opposite, oblong-ovate, acute, serrulate, feather-veined, thickish-membranaceous, tomentellous on both surfaces, paler beneath, 8–12 cm. long, 3.6–6 cm. wide; petiole 1.5–2 cm. long, densely rusty-velvety; heads crowded at the ends of the widely spreading branches of the panicle, about 39-flowered, 7 mm. high and thick; involucre campanulate, 2–3-seriate; scales scarcely graded, acutish, obscurely nerved, the outer ovate, persistent, dorsally tomentellous, the inner narrower, subglabrous, promptly deciduous; corollas 3.5 mm. long, smooth; achenes glabrous; pappus-bristles about 20, somewhat unequal, nearly as long as the corolla.—PUNO: between Sandia and the tambo Azalaya, on the road from Sandia to Chunchusmayo, in bushy places alt. 1500–2000 m., *Weberbauer*, no. 1074 (Berl., phot. and fragm. Gr.).

28. E. TARAPOTENSE Robinson (p. 37). Rusty-tomentellous shrub; leaves opposite, elliptic-ovate, acuminate, remotely serrate (teeth about

1 mm. high and 1 cm. apart), rounded at base, coriaceous feather-veined, above sparingly pubescent, lucid, deeply rugose-reticulated, beneath paler, pubescent and minutely atomiferous, 4–8 cm. long, 1.5–4 cm. wide; petiole about 1 cm. long; heads about 37-flowered, about 1 cm. high and thick, densely cymose at the ends of the leafy-bracted branches in an open panicle; involucre campanulate; scales about 18, clearly graduated, acutish, distinctly striate-costulate; corollas about 6 mm. long, with slightly but clearly enlarged throat.— LORETO: in mountains along the river Mayo, near Tarapoto, *Spruce*, no. 4014 (Gr.).

29. E. MATHEWSII Robinson (p. 23). Tawny-velvety, probably shrubby (the base unknown); stems and branches terete, leafy to well within the inflorescence; internodes 3–6 cm. long; leaves opposite, lanceolate, caudate-acuminate, attenuate to a relatively short-petioled base, remotely and rather sharply 4–6-toothed on each side (teeth 1–2 mm. long, about 1.5 cm. apart), feather-veined, 1 dm. long, 2 cm. wide, above sparingly tawny-pubescent on the midrib, the surface otherwise subglabrous, somewhat shining, the reticulated veinlets depressed, beneath rusty- or tawny-pubescent or subtomentose, paler, the veinlets slightly prominent; panicle leafy; the heads crowded at the ends of the branches, about 56-flowered, nearly 1 cm. high and thick; involucre campanulate, about 3-ranked; scales ovate to (the inner) linear, acutish, striate-costate, glandular-ciliolate, the outer dorsally a little tawny-hairy; corollas 5.5 mm. long, slightly and gradually enlarged toward the summit, the upper part externally a little granulated; achenes dull-grayish, 2.3 mm. long; pappus-bristles about 25, white, scarcely roughened, about a third shorter than the corolla.— DEPARTMENT NOT ASCERTAINED: at Yambrasbamba (an unidentified name perhaps not correctly read from a somewhat obscure label), 1835, *Mathews* no. 1386 (K., phot. Gr.).

30. E. LOBBII Klatt. Shrub; branches round, striate-costulate (after drying), fistulose, densely purplish- or later tawny-tomentose (at last subglabrate), the hairs slender, spreading, attenuate, purple-jointed; internodes 3–7 cm. long; leaves rhombic-ovate, attenuate to a slightly obtusish apex, abruptly cuneate at the base, quite entire or undulate-dentate (with 3–6 low broad irregular rounded teeth on each side), 7–9 cm. long, 2.8–3.5 cm. wide, membranaceous, dark-green and finely tawny-pubescent above, paler, tawny-tomentellous beneath, 3-nerved from a point about 1 cm. above the base; petiole about 1.2 cm. long, panicle ovoid, opposite-branched, leafy-bracted, 1.3–2 dm. in height and diameter; heads clustered at the ends of the branches, 1 cm. high and thick; involucre campanulate, 2–3-seriate; scales moderately

graduated, 2-ribbed and 3-nerved, the inner stramineous, acutish, tawny-tomentellous toward the tip, the outer oblong, round-tipped, subherbaceous, dorsally tawny-tomentellous; corollas slenderly tubular, without evident throat, glabrous, 5.5 mm. long; achenes 2.8 mm. long, slender, tapering downward, fuscous, upwardly setulose toward the summit; pappus-bristles about 30, dull-white, nearly smooth, almost equalling the corolla.— Ann. Naturh. Hofmus. Vienna, ix. 356 (1894). *E. sordescens* Bak. in Mart. Fl. Bras. vi. pt. 2, 306 (1876), in part, not DC.— LORETO: in mountains along the Mayo River, near Tarapoto (incorrectly given by Klatt as Jarapoto), *Spruce*, no. 4804 (Gr.). WITHOUT EXACT LOCALITY: *Lobb* (Hofmus. Vienna, sk. and fragm. Gr.).

This is one of several widely different plants included by Baker under *E. sordescens* DC., which in its typical form is a pretty well marked species of Atlantic Brazil with much smaller (about 25-flowered) heads, linear and acute involucral scales, and petioles (as stated by DeCandolle) 1.8–2 cm. long. The real *E. sordescens* seems to be well represented by Riedel's no. 1348 (Gr.) determined by Schultz-Bipontinus.

31. E. HELIANTHIFOLIUM HBK. Shrubby; branches opposite, round, dark-hispid; internodes 4–6 cm. long; leaves opposite, ovate, narrowed from below the middle to an acute point, serrate except toward the abruptly pointed base (teeth 1–1.7 mm. high, 4–6 mm. wide, mostly acute), feather-veined, membranaceous, above scabrid, beneath hispid with minute hairs and closely beset with orange-colored glands, the largest leaves about 12 cm. long, 5 cm. wide; petiole subterete, densely hispid, canaliculate above, 7–10 mm. long; inflorescence a trifid very leafy flattish panicle; heads pedicellate, about 10-flowered, 8 mm. high; involucre campanulate-cylindric, the scales about 16, imbricated, but not very strongly or regularly graduated, the inner oblong-linear, smoothish, obtuse, the outer somewhat shorter, oblong, acutish, hispidulous.— Nov. Gen. et Spec. iv. 127 (1820).— PIURA: near Ayavaca, *Humboldt & Bonpland* (Par., phot. Gr.).

From the photograph in the Gray Herbarium this species appears to resemble in habit the variable *E. inulaefolium* HBK. The leaves, however, appear to be more regularly feather-veined, the petiole more sharply defined, and the inflorescence more leafy. From the original character it is to be inferred that the pubescence is more hispid and of darker color. The type-material is immature and until rediscovery the status of this species must probably remain doubtful.

32. E. VESTITUM Poepp. Erect shrub with terete somewhat flexuous leafy branches; pubescence on younger axes, as well as on the midrib and chief veins of the leaves short, dense, rusty-velvety; leaves petiolate,

opposite, ovate-elliptic, acuminate, serrate, except toward the obtuse or rounded base (teeth 0.4–0.7 mm. high, 4.5–7 mm. wide), feather-veined (chief veins about 5 on each side, leaving the midnerve at an angle of 60°–65° and curving forward), reticulate-veiny, above scabrid, dark-green, beneath pubescent, paler, membranaceous, 12–15 cm. long, half as wide; petiole densely rusty-velvety, 8–25 mm. long; corymbs pedunculate, raised well above the leaves, compound, rounded, the branches wide-spreading; 1.5–4.5 cm. in diameter; heads short-pedicelled, about 7.5 mm. high; involucre campanulate; scales 2–3-seriate, graduated, the outer apparently ovate-oblong, narrowed to an obtusish apex; corollas white, scentless.— Poepp. in Poepp. & Endl. Nov. Gen. ac Spec. iii. 55 (1845).— ANCACHS (or possibly HUANUCO) in mountain woods at Pampayaco (Cuchero), *Poeppig* (Hofmus. Vienna, phot. Gr.).

33. E. ORGYALOIDES Robinson (p. 24). Shrub or tall herb with slender dark-brown branches, nearly smooth and leafy to the inflorescence: internodes 4–6 cm. long; leaves opposite, lance-oblong, acuminate at each end, undulate-margined and remotely cuspidate-denticulate, chartaceo-membranaceous, feather-veined, glabrous and delicately net-veined on both surfaces, a little paler beneath, the largest 1.6 dm. long, 6 cm. wide; petiole minutely dusky-puberulent 4–7 mm. long; panicle broadly pyramidal, leafy-bracted below, its slender opposite branches widely spreading and flowering only toward the tip; heads about 20-flowered, 4.5 mm. high and thick; involucre bell-shaped, dusky- or greenish-stramineous, the scales imbricated but not very clearly graduated, ciliolate but otherwise glabrous, the inner oblong to linear, the outermost ovate, acuminate, often recurved at the tip; corollas glabrous, only slightly and gradually enlarged upward, 2.5 mm. long; achenes (immature) 1.7 mm. long, glabrous; pappus dull-white, 2 mm. long, the bristles about 30.— LORETO: near Tarapoto, 1855–56, *Spruce*, no. 4546 (K., phot. Gr.).

In many respects near the preceding, but with essentially glabrous stems and obscurely toothed, or subentire leaves acuminate at the base.

34. E. DREPANOIDES Robinson (p. 12). Glabrous shrub reaching a height of 4 m.; branches nearly round, curved-ascending, leafy; internodes 1–2 cm. long; leaves opposite, lanceolate, broadly scythe-shaped, acuminate, acute at base, serrate (teeth 0.6 mm. high, 2–3 mm. wide), firmish, above flat, dull, sprinkled with fine granules, beneath slightly paler, 3-nerved above the base, then feather-veined, 7–9 cm. long, 1.8–2.1 cm. wide; petiole glabrous, about 2 cm. long; corymbs terminal, sessile, compound, rounded, 1 dm. or more in

diameter; branchlets and pedicels webby-puberulent; heads many, about 7-flowered, 6 mm. (very immature) high, 3.5 mm. in diameter; involucral scales about 13, ovate-oblong, acutish, finely striate, stramineous with purplish tinge, graduated, webby-ciliate; corollas slenderly tubular, 5-nerved, granulated; achenes tapering downward, finely granulated; pappus-bristles about 23.— Ancachs: open woods by a brook at Comin, Prov. Huari, alt. 3600–3700 m., *Weberbauer*, no. 2918a (Berl., phot. and fragm. Gr.).

35. E. COELOCAULE Robinson. Shrubby; branches stout, round, grayish-brown, at length glabrous, smooth, hollow; internodes 3–4 cm. long; leaves opposite, lance-oblong or rhombic-lanceolate, long-acuminate, serrate (except at the cuneate base), feather-veined, of firmish texture, at maturity green, glabrous, and somewhat shining above, somewhat glaucous and puncticulate beneath; petiole 1–4 cm. long, smooth, reddish; upper leaves smaller, lanceolate, entire; corymb terminal, sessile, compound, slightly convex, 1–1.5 cm. broad, densely covered with a short and curly fulvous tomentum; heads very numerous, shortly pedicelled or sessile by 2's or 3's, about 5-flowered, 10–12 mm. high, 2–3 mm. thick, slenderly cylindrical; scales about 12, stramineous, about 3-seriate, the outer ovate, acutish, the inner gradually longer, narrowly elliptical, delicately striatulate, obtusish, glabrous; corollas smooth, slightly enlarged upward, 7 mm. long; achenes dark, shining, obsoletely granulate; pappus-bristles about 30, unequal, yellowish-white.— Amazonas: Prov. Chachapoyas, ***Mathews***, no. 1373 (K., phot. and fragm. Gr.).

36. E. AMYGDALINUM Lam. Encyc. ii. 408 (1786); Hieron. in Engl. Bot. Jahrb. xxxvi. 470 (1905); Robinson, Proc. Am. Acad. liv. 301, 339 (1918). *E. dodoneaefolium* DC. Prod. v. 161 (1836). *E. fraternum* DC. l. c. 163.— Loreto: near Tarapoto, *Spruce*, no. 4116 (Gr.). San Martin: open grassy plains at Yurimaguas, Prov. Maynas, *Poeppig*, no. 2075 (DC., phot. Gr.). Cajamarca: Tambillo, *von Jelski*, no. 728, acc. to Hieron. l. c. Cuzco: Santa Ana, alt. 900 m., *Cook & Gilbert*, no. 1613 (U. S.). Ancachs: on open hills at Cuchero, *Poeppig*, no. 18 (DC., phot. Gr.). Without locality: *Mathews* (N. Y.). The original specimen of *E. amygdalinum* said to have been collected in Peru by Joseph de Jussieu is still extant (Par., phot. Gr.), but probably came from what is now Ecuador. [Nicaragua to Bolivia and Brazil.]

37. E. Sprucei Robinson (p. 33). Low undershrub, 2–3 dm. high; stem curved-ascending and distinctly woody toward the soon defoliated base; internodes very short below; leaves opposite, oblan-

ceolate, sessile, attenuate to each end, rather remotely denticulate, green and glabrous above, paler and on the midnerve sordid-pubescent beneath, 5–10 cm. long, 1–1.8 cm. wide, thin, membranaceous, feather-veined, veins about 6 or 7 on each side, curved forward and anastomosing with each other; panicle terminal, its slender branches and filiform pedicels clothed with short dirty-brownish curly hairs; heads about 18-flowered, about 6 mm. high; involucre turbinate, several-seried, its scales stramineous, mostly linear-oblong, obtuse or rounded at the tip and often slightly tufted on the back near the summit, mostly 2-costulate; corollas 3 mm. long, glabrous; proper tube slender 1.8 mm. long, the throat 1.2 mm. high, turbinate; style-branches with flexuous very delicate attenuate and papillose appendages; achenes blackish, 1.5 mm. long, sparingly hispidulous on the angles; pappus-bristles about 25, very delicate, nearly smooth, white.— LORETO: along the Huallaga River, *Spruce,* no. 4167 (Gr.).

38. E. PILLUANENSE Hieron. Suffruticose, 5 dm. high; stems puberulent, striate; leaves opposite, elliptic- to rhombic-ovate, acute or obtuse, entire, cuneately narrowed below but cordate and half-clasping at the sessile base, chartaceous, puberulent above, densely tomentulose beneath, feather-veined (lateral veins 6–8 on each side, leaving the midrib at about 45° and then curved-ascending, connected by reticulation), the largest 12 cm. long, 4–4.5 cm. wide; panicle terminal, compound, leafy-bracted below; heads corymbose or subracemose toward the tips of the branches, some also borne directly from the axils of the bracts, pedicellate or more rarely sessile, 14–17-flowered; involucre about 4 mm. long, the scales about 20, obtuse, the inner linear-oblong, stramineous, 1–3-nerved and striate, the outer gradually shorter and broader, 4-nerved, the outermost ovate, about 3 mm. long; achenes about 1.5 mm. long, dark, smooth.— Verh. Bot. Verein Brand. xlviii. 201 (1907).— LORETO: Salinas de Pilluana, *Ule,* no. 6780.

This species has not been seen by the writer. The description is condensed from the original, in which the plant is said to be nearly related to *E. iresinoides* HBK. but to differ in its firmer more pinnately veined leaves, larger heads, and more numerous florets.

39. E. COMMERSONII (Cass.) Hieron. Nearly herbaceous, decumbent, rather densely grayish-pubescent, 3–5 dm. high; stem round, stoutish, curved-ascending, leafy below, nearly naked above the middle to the terminal, rather dense, often nodding inflorescence; leaves chiefly opposite, oblong, obtuse, 1 dm. long, 3 cm. wide, irregularly and coarsely serrate-crenate or lobed except on the long gradually cuneate

petiole-like entire base; heads short-pedicelled, in dense rounded corymb or at length loose irregular corymbose panicle, about 20-flowered; involucre campanulate, about 2-seriate, the inner scales oblong 2–3-ribbed and smoothish to the middle, ending in tomentellous and erubescent rounded slightly dilated tips of softer texture, a few of the outer scales progressively shorter and narrower, the outermost linear, acutish; corollas said to be red, glabrous, slightly and gradually enlarged upward; achenes dark, 2.5 mm. long, puberulent, tapering to a sharp villous base; pappus-bristles about 30, yellowish-white, barbellate.— Hieron. in Engl. Bot. Jahrb. xxii. 771 (1897). *Gyptis Commersonii* Cass. Dict. xx. 178 (1821). *Eupatorium Baeleanum* DC. Prod. v. 157 (1836). *Gyptis Commersoni* [Cass.] Bak. in Mart. Fl. Bras. vi. pt. 2, 314 (1876), in syn. — Peru without locality (as "inter Peruanas"): *Haenke*, acc. to DeCandolle, l. c. [Southern Brazil, Uruguay, and Argentine.]

40. E. trachyphyllum Hieron. Woody climber; branches round, closely clothed with a short and stiffish reflexed tawny pubescence, the hairs pointed, non-glandular; leaves opposite, ovate-oblong, caudate-acuminate, rounded at base, entire, concolorous, firmly membranaceous, 3-nerved from somewhat above the base, above dull green, scabrous-puberulent, rugulose-reticulated, beneath tomentellous especially on the prominent reticulation, mostly 7–12 cm. long, about a third as wide, petiole 1–2 cm. long, retrorsely velvety-puberulent; panicle branched at right angles, the branches naked except at the densely floriferous tip; heads 6–7-flowered, subsessile, numerous in close subglobose glomerules; involucres narrowly campanulate, scarious, stramineous, the scales very unequal, mostly linear-oblong, slightly ciliate toward the obtuse or rounded tip, 3–5-costulate, the inner promptly deciduous; corollas pale, probably flesh-colored, glabrous, slenderly tubular, slightly and gradually enlarged upward, 6 mm. long; achenes purplish-black 2.3 mm. long tapering toward the callose base, at the summit slightly constricted into a short neck bearing the pappiferous disk; pappus-bristles about 30, white with a light yellow tinge, barbellate.— Hieron. in Engl. Bot. Jahrb. xxxvi. 467 (1905).— Cajamarca: near Tambillo, 29 Aug., 1878, *von Jelski*, no. 697 (Berl., fragm. Gr.).

41. E. acuminatum HBK. Nov. Gen. et Spec. iv. 107 (1820); Hieron. in Engl. Bot. Jahrb. xxxvi. 469 (1905); Robinson, Proc. Am. Acad. liv. 290 (1918).— Cajamarca: near Tambillo, 13 Aug., 1878, *von Jelski*, no. 790, acc. to Hieron. l. c. [Colombia.]

The writer has had no opportunity to verify the Peruvian record of this species, indeed has seen no specimens of it except from central and southwestern Colombia where it appears to be abundant. If Hieronymus is correct in his identification, the plant of von Jelski extends the known range southward toward 1000 km.

42. E. CRENULATUM Spreng. Glabrous shrub, with ascending slender more or less flexuous purple branches leafy up to the loose terminal racemose panicle; leaves opposite, lance-oblong to rather broadly elliptic, acute or obtuse, cuneate at base, finely serrate, 3-nerved from the base, of firmly coriaceous texture, punctate, scarcely paler beneath, 5–7.5 cm. long, 1–3.5 cm. wide, 5–15 mm. long; heads very numerous, about 5-flowered, about 6 mm. high, shortly pedicelled, racemosely disposed on long flexuous branches of a more or less pyramidal panicle; involucre subcylindric, of few purple-tinged stramineous very unequal elliptic-lanceolate firmish round-tipped scales, the inner paler, ciliolate, the outer darker and somewhat puberulous; corollas 4.5 mm. long, the proper tube slender, granulated, 2.5 mm. long, the throat definitely though not greatly enlarged, smoothish, limb short densely granulated on the outside; achenes at maturity black and shining, almost prismatic; pappus-bristles 25–30, firmish, stramineous, tapering from near the base, obscurely barbellate.—Spreng. ex Hieron. in Engl. Bot. Jahrb. xxii. 776 (1897). *Baccharis crenulata* Spreng. Syst. iii. 465 (1826). *Eupatorium dendroides* Bak. in Mart. Fl. Bras. vi. pt. 2, 321 (1876), excl. syn. *Mikania arborea* Kunth. For further synon. see Bak. l. c.—CUZCO: San Miguel, Urubamba Valley, alt. 1800 m., *Cook & Gilbert*, no. 1113 (U. S.). PERU WITHOUT LOCALITY: 1902, *Weberbauer* without number (Berl.).

This species is not closely related to any other in Peru. Its leaves recall those in some species of *Baccharis*, the genus to which it was first referred. [Brazil, Argentina, Bolivia.]

43. E. STUEBELII Hieron. in Engl. Bot. Jahrb. xxi. 329 (1895); Robinson, Proc. Am. Acad. liv. 288, 354 (1918).—AMAZONAS: Prov. Chachapoyas, *Mathews* (Gr.). This Peruvian specimen was identified with the type at Berlin by Dr. J. M. Greenman. [Ecuador, Colombia.]

44. E. CALLACATENSE Hieron. Suffruticose; branches slender, terete, spreading, flexuous, covered with short dense somewhat deflexed brownish-gray scarcely or not at all glandular pubescence; leaves opposite, ovate, acute, crenate-serrate except at the nearly truncate base (teeth 0.8–1.4 mm. high, 3–4 mm. wide), 3-nerved from a slightly acuminate attachment (the lateral nerves soon branched

on the outer side), dull-green above, slightly paler beneath, sordid-pubescent on both surfaces, 5 cm. long, 2.8–3.5 cm. wide; petiole slender, 2–2.5 cm. long; corymbs trifid, leafy-bracted at base, the bracts lanceolate, entire, petiolate; partial inflorescences rounded, rather dense, 2–4 cm. in diameter, the branchlets and filiform pedicels (1–3 mm. long) densely covered with a short brownish-gray pubescence; heads 5–6 mm. high, about 10-flowered; involucral scales about 15, obtuse, brownish-stramineous, somewhat scarious, the outer short, ovate, loosely-pubescent, the intermediate elliptic-oblong, 3–5-ribbed, slightly puberulent toward the tip; corollas about 4 mm. long, with slender proper tube 2 mm. long, slightly enlarged into a cylindric throat 1.7 mm. long; style-branches filiform, not clavate; achenes (very immature) 2 mm. long, grayish-pubescent; pappus-bristles about 30, delicate, white, distinctly shorter than the corolla.— Hieron. in Engl. Bot. Jahrb. xxxvi. 468 (1905).— CAJAMARCA: near Callacate, May, 1879, *von Jelski*, no. 670 (Berl., fragm. Gr.).

45. E. URUBAMBENSE Robinson (p. 38). Perennial herb or perhaps shrub (the base unknown); stems (or branches) erect, terete, purple, covered with fine white crisped puberulence; internodes 1.5–(the upper)8–11 cm. long; leaves opposite, ovate-lanceolate, gradually narrowed from much below the middle to a mostly obtusish tip, crenate except toward the obtusish or rounded base (the teeth 0.7–1.5 mm high, 3–4 mm. wide), 3-nerved from somewhat above the base (the lateral nerves quickly branching), thickish-membranaceous, dull-green, slightly rugulose and puberulent above, gray-tomentose beneath, 5–6.6 cm. long, 1.8–2.7 cm. wide; petiole 5–8 mm. long, slender except at the somewhat expanded base; corymbs trichotomous, leafy-bracted at the base (the bracts like the leaves only smaller, similarly crenate and petioled); partial inflorescences dense, rounded, 3–6 cm. in diameter; heads about 10-flowered, about 8 mm. high, essentially sessile; involucre subcylindric-campanulate; the scales about 3-seriate, rounded at the summit, the outer very short, the intermediate broadly elliptical, brownish-stramineous, mostly 3-nerved and 4-costulate, somewhat margined, the innermost linear-oblong, slightly erose-ciliate at the blunt summit; corrollas 4.8 mm. long, slenderly tubular, perceptibly constricted just above the base and sometimes slightly so just beneath the limb, glabrous; teeth lanceolate; style-branches filiform, scarcely clavellate; anthers with long narrow apical appendage; achenes fuscous, 2.6 mm. long, tapering downward, upwardly villous on the nerves; pappus-bristles about 28, lucid, yellowish-white, tapering, 4.3 mm. long, nearly smooth.—

Cuzco: Urubamba in the Valley of Ymay [?], *Pentland* (K., phot. and fragm. Gr.); Ollantaytambo, alt. about 3000 m., *Cook & Gilbert*, no. 336 (U. S.).

46. E. LEUCOPHYLLUM HBK. Distinctly shrubby, the stems, spreading-ascending terete branches, inflorescence, and under-surface of leaves closely white-woolly; leaves opposite, narrowly ovate, attenuate, acutish, crenulate, 3-nerved essentially from the rounded to subtruncate entire base, pale-green and very finely puberulent above, white-tomentose but with perceptibly darker nerves and reticulated veins beneath, 3–5 cm. long, half as wide; petiole about 1 cm. long; heads about 10-flowered, 5 mm. long, very numerous in rounded intricately branched corymbiform terminal panicles; involucre subcylindric-campanulate; the scales about 15, narrowly oblong, obtuse, very unequal, dorsally white-woolly, often with a single perceptible mid-nerve; corollas glabrate, 2 mm. long; proper tube shorter than the gradually enlarged throat; achenes glabrous, shining, 1.7 mm. long; pappus-bristles, whitish, essentially smooth.— Nov. Gen. et Spec. iv. 115 (1820); Benth. Pl. Hartw. 135 (1844); Jameson, Pl. Aeq. ii. 82 (1865).— Although generally attributed to Peru and with scarcely a doubt extending into the northern part of the country, this plant seems never to have been collected south of the present boundary of Ecuador, the only collections known to the writer being in favored spots of the temperate region of the Andes near the villages Cajanuma and Gonzanama, alt. 1976 m., *Humboldt & Bonpland* (Par., phot. Gr.), and in mountains of Loja, *Hartweg*, no. 756. Of the latter collection (though unnumbered) there is a specimen in the herbarium of the New York Botanical Garden.

This species was by oversight omitted from the writer's recension of the Eupatoriums of Ecuador. It is closely related to *E. niveum* but may be distinguished by its more attenuate leaves which are dull and finely pubescent above instead of being green, glabrous, and lucid as in *E. niveum*. In *E. leucophyllum* the involucral scales are thicker, narrower, more densely tomentose, and less scarious than in *E. niveum*. *E. leucophyllum* furthermore gives the impression of being rather the more xerophytic of the two.

47. E. INULAEFOLIUM HBK. Nov. Gen. et Spec. iv. 109 (1820); Hieron. in Engl. Bot. Jahrb. xxii. 765 (1897), which see for synon.; Robinson, Proc. Am. Acad. liv. 291 (1918). *E. decemflorum* DC. Prod. v. 154 (1836); Poepp. in Poepp. & Endl. Nov. Gen. ac Spec. iii. 54 (1845); Klatt in Engl. Bot. Jahrb. viii. 34 (1887).— Cuzco: Santa Ana, alt. about 900 m., *Cook & Gilbert*, no. 1633 (U. S.). Huan-

UCO: in bushy places at Cassapi, *Poeppig*, no. 30 (DC., phot. Gr.). WITHOUT INDICATION OF DEPARTMENT: *Haenke*, acc. to DC., l. c.; in Andes of Peru, *Wasner* [doubtfully legible], no. 1349 [apparently of the Mathews series] (N. Y.); *Mathews*, without number (Gr.).

Forma SUAVEOLENS (HBK.) Hieron. in Engl. Bot. Jahrb. xxix. 11 (1900), xxviii. 572 (1901), xxxvi. 470 (1905); Robinson, Proc. Am. Acad. liv. 292 (1918). *E. suaveolens* HBK. Nov. Gen. et Spec. iv. 109 (1820).— CAJAMARCA: near Tambillo, 7 August, 1878, *von Jelski*, nos. 692, 742, acc. to Hieron. l. c. xxxvi. 470. [Ecuad., Colomb., Venez.]

48. E. GRACILENTUM Robinson (p. 18). Slender perennial herb 3–4 dm. or more high; root of a few strong slender lignescent elongated fibres; stems 1–several from the base, erect, or at least decumbent, terete, purplish, 1–2 mm. in diameter, sordid-puberulent or -tomentellous; internodes 2–11 cm. long; leaves opposite, deltoid-ovate, acute to acuminate, crenate-dentate except at the rounded, truncate or subcordate base, 1.8–3 cm. long, 1.1–2.3 cm. wide, thin, membranaceous, softly pubescent above, grayish-tomentose beneath, 3-nerved from the insertion; petiole slender, subterete, gray-pubescent, 4–8 mm. long; heads about 25-flowered, 6 mm. long, 3.7 mm. in diameter, borne in loose irregular 1–5-headed cymes at the ends of the spreading branches of an open leafy-bracted panicle; involucre narrowly campanulate, scales about 19, about 3-seriate, stramineous, the inner narrowly lance-elliptic, obtuse, smoothish, 2–3-costulate, scarious-margined; the intermediate and outer progressively shorter, ovate-lanceolate, acute to acuminate, brownish-puberulent; corollas probably white, glabrous except at the short limb; proper tube 0.7 mm. long, throat slightly enlarged, cylindrical, 2.3 mm. long; style-branches filiform, scarcely at all clavellate, achenes 1.5 mm. long, fuscous-brown, with lighter-colored obscurely hispidulous ribs; pappus-bristles about 27, delicately capillary, white, essentially smooth.— PERU WITHOUT INDICATION OF LOCALITY: *Mathews* (N. Y., phot. Gr.). Like several other specimens of Mathews's Peruvian plants from the herbarium of Meisner, two sheets of this plant, now in the herbarium of the New York Botanical Garden, bear a yellow label in the hand of Meisner, reading merely "Peruvia Matthews, 1862." Alexander Mathews, the well-known collector in Peru (who spelled his name with one *t*) died in 1841. It has been impossible to get information of any subsequent collector in Peru of this name and it is accordingly inferred that errors have here arisen in the copying of labels, and that these plants were in reality collected by Alexander Mathews about 1835–1840.

49. E. Cookii Robinson (p. 9). Shrubby, densely spreading-villous, the hairs at first long, soft, and under a lens beautifully purple-jointed; stems round, purple, the internodes elongated, sometimes 13 cm. or more in length; branches spreading, usually curved upward, bearing about 2 pairs of leaves and terminating in a mostly dense trifid corymb; leaves ovate, acuminate, crenate-serrate except at the rounded, or subcordate base, dull-green and sparingly pubescent above, slightly paler, sordid-villous and somewhat glandular-atomiferous beneath, about 5 cm. long and 3–3.4 cm. wide; petioles 1.2–2 cm. long; corymbs trifid, dense; pedicels short, woolly, with long soft brown and jointed hairs; heads crowded, about 20-flowered, about 7 mm. high, 4 mm. in diameter; involucres campanulate, about 3-seriate, substramineous, the scales about 19, lance-oblong, acute, mostly 2-ribbed and 3-veined, dorsally pubescent, the innermost narrower, less pubescent, obtusish or rounded at the more or less scarious summit; corollas apparently white, hispidulous on the limb, otherwise essentially glabrous, about 3.8 mm. long, the proper tube only 0.7–0.8 mm. long, much exceeded by the cylindrical throat; achenes dark brown, nearly black, 1.5 mm. long, glabrous; pappus-bristles about 40, clear white, scarcely roughened.— Cuzco: in the Lucumayo Valley, alt. 1800–3600 m., *Cook & Gilbert*, no. 1352 (U. S., phot. Gr.). Junin: Andamarca, *Mathews*, no. 1126 (K., phot. and fragm. Gr.), a doubtful form with shorter pubescence and more distinctly cordate leaves, with crenate rather than serrate margins.

50. E. **marginatum** Poepp. Climbing shrub; branches round, slender, divaricate as they leave the stem, but curved-ascending, very finely puberulent; internodes 4–7 cm. long; leaves opposite, lance-ovate, acuminate, serrate, except at the rounded base, 3–4.5 cm. long, 1–2.4 cm. wide, membranaceous, 3-nerved essentially from the base, glabrous except for a delicate puberulence on the nerves beneath; the margin narrowly revolute and slightly indurated; petiole slender, 5–10 mm. long; panicle opposite-branched, leafy-bracted, ovoid, open, about 4 dm. long; heads small, about 15-flowered, slender-pedicelled, in convex fairly dense terminal corymbs on the branchlets of the finely puberulent composite inflorescence; involucral scales about 3-seriate, narrowly oblong, stramineous, the inner glabrous, obtuse, the outer pubescent: corollas white; achenes glabrous.— Poepp. in Poepp. & Endl. Nov. Gen. ac Spec. iii. 54 (1845).— Huanuco: in bushy places at Cassapi, August, *Poeppig*, no. 1254 (Hofmus. Vienna, phot. Gr.). This species has somewhat the habit of an *Ophryosporus*, into which genus it would fall should the anther-tips

prove to be unappendaged, a matter not to be determined from a photograph, the only source of information at hand.

51. E. MICROSTEMON Cass. Dict. xxv. 432 (1822); Robinson, Proc. Am. Acad. liv. 295, 340, 356 (1918).— CUZCO: in cultivated ground, Santa Ana, alt. about 900 m., *Cook & Gilbert*, no. 1544 (U. S.). A weed widely distributed in the warmer parts of America. Not present in the earlier collections from Peru and perhaps a recent introduction in the country.

52. E. VITALBAE DC. Prod. v. 163 (1836); Bak. in Mart. Fl. Bras. vi. pt. 2, 305 (1876), excl. syn. *E. remotifolium* DC. which is exceedingly different.— LORETO: near Tarapoto, *Spruce*, no. 4106 (Gr.).

53. E. INCARUM Robinson. Smooth, but sticky, much branched shrub; leaves small, suborbicular, obtuse, shortly subcuneate at base, denticulate, 3-nerved from the base, punctate; heads about 5-flowered, short-pedicelled, racemosely disposed on the branchlets of the panicle, about 6 mm. long; involucre subcylindric; scales about 3-seriate, oblong, rounded at the tip, obscurely 2–3-ribbed, thin, subscarious brownish straw-colored, slightly puberulent; corollas 3.7 mm. long, glabrous; the proper tube slender, 1.7 mm. long, the throat distinctly enlarged, cylindrical, 2 mm. long; anthers with an ovate round-tipped membranaceous apical appendage; style-branches filiform, recurved, rounded and perceptibly enlarged at the tip; achenes dark-brown, 1.5 mm. long, tapering downward, somewhat atomiferous on the faces; pappus-bristles about 30, fulvous, attenuate, 3 mm. long, nearly smooth.— Mem. Gray Herb. i. 122 (1917). *Baccharis microphylla* DC. Prod. v. 406 (1836), not HBK. *B. Candolleana* Steud. Nom. ed. 2, i. 177 (1840). *Brickellia microphylla* (DC.) Hieron. in Engl. Bot. Jahrb. xxviii. 583 (1901), not Gray.— PERU WITHOUT LOCALITY, *Haenke* (DC., detail sk. Gr.).

54. E. WEBERBAUERI Hieron. Round-branched shrub 1 m. high; branches dark-velvety, leafy; internodes rarely exceeding 6 mm. in length; leaves opposite, broadly ovate-oblong, obtusish or acutish, cordate, nearly entire or slightly crenulate, feather-veined and reticulated, above a little puberulent on the nerves, but soon glabrate and somewhat shining, below shortly white-velvety, the largest 1.5 cm. long, 1 cm. wide, on thickish petioles rarely more than 1.5 mm. long; heads densely and cymosely paniculate, 7–8-flowered, on pubescent pedicels (2–5 mm. long); scales of the 4–5-seriate involucre about 15, obtusish or (the outer) acutish, 3–5-striate, scarious, ochraceous, puberulent, deciduous; corollas dull-yellowish, turbinate-subcylindric, gradually a little enlarged toward the summit, sprinkled with short-

stalked glands; achenes brown, covered throughout with subglandular hairs; pappus-bristles 50–60, yellowish-white.— Hieron. in Engl. Bot. Jahrb. xl. 369 (1908).— AMAZONAS: near Molinopampa east of Chachapoyas, alt. 2000–2300 m., *Dr. A. Weberbauer*, no. 4359.

Not seen, the description here condensed from the original diagnosis by Hieronymus.

55. E. CHAMAEDRIFOLIUM HBK. Nov. Gen et Spec. iv. 113 (1820); Benth. Pl. Hartw. 135 (1844); Robinson, Proc. Am. Acad. liv. 353 (1918).— PIURA? Originally collected "on the whole range of mountains" between Gonsanama, Ecuador, and Ayavaca, Peru, obviously close to the boundary between the two countries, *Humboldt & Bonpland* no. 3485 (Par., phot. Gr.). A specimen collected in the mountains of Loja, Ecuador, by Hartweg, and although unnumbered doubtless being a portion of his no. 759 cited by Bentham, l. c., is in the herbarium of the New York Botanical Garden. [Ecuador.]

56. E. MARRUBIIFOLIUM Hieron. Grayish villous-hirsute herb; stems terete, purplish, covered with a loose, at first spreading, at length mostly deflexed pubescence of slender white hairs; leaves opposite, broadly ovate, obtuse, crenulate except at the subcordate base, 1–1.5 cm. long 8–13 mm. wide, thickish, above finely bullate-reticulate with depressed veins, gray-pubescent, beneath shaggy gray-velvety, 3(–5)-nerved nearly from the base; petiole 3–5 mm. long, shaggy-villous; corymbs trifid, dense, the parts rounded; heads about 20-flowered, subsessile or shortly pedicelled; involucre campanulate, 3–4-seriate, brownish-stramineous; scales mostly ovate-elliptical, thin, rounded at the summit; corollas 3.5 mm. long, glabrous, scarcely enlarged upward; achenes (young) 2.2 mm. long, stramineous, at length turning dark-brown, slightly hispid toward the summit; pappus-bristles about 30, bright white, nearly equalling the corolla.— Hieron. in Engl. Bot. Jahrb. xxxvi. 466 (1905).— CAJAMARCA: between Chota and Cutervo, June, 1879, *von Jelski*, no. 799 (Berl., fragm. Gr.).

Sect. III. EXIMBRICATA (DC.) Hoffm. (See Robinson, Proc. Am. Acad. liv. 303.)

KEY TO SPECIES.

a. Heads 3–10-flowered *b*.

b. Leaves pinnately veined . . . . 58. *E. exserto-venosum.*

b. Leaves 3–5-nerved from or somewhat above the base.

Branchlets and leaves glabrous; at least the lower teeth of the leaves widely spreading and very acute.

Petioles about 2 mm. long; leaf-blade about 8 mm. wide . . . . 59. ***E. chilca.***

Petioles 8–10 mm. long; leaf-blade about 1.6 cm. wide. 60. *E. affine.*

Branchlets and leaves puberulent or pubescent, sometimes glandular-puberulent . . . . 61. ***E. heptanthum.***

a. Heads 12–∞-flowered *c*.

c. Leaves sessile, but the blade gradually narrowed to a petiole-like base, thickish; involucral scales rounded at tip, grayish-velvety . . . . 39. ***E. Commersonii.***

c. Leaves on wingless usually slender though sometimes short petioles *d*.

d. Leaves entire, large, 8–10 cm. wide . . . . 62. ***E. uber.***

d. Leaves crenate to serrate, rarely more than 6 cm. wide *e*.

e. Leaves acute or acuminate at the base *f*.

f. Leaves 3-nerved from or somewhat above the base.

Heads 13–14-flowered; leaves 3–4.5 cm. long, on petioles 4–6 mm. in length . . . . 63. ***E. cuzcoesne.***

Heads 20–25-flowered; leaves at maturity 5–8 cm. long, on petioles 1–2.5 cm. in length.

Scales of the involucre loosely ciliate toward the the tip with jointed hairs; petiole pubescent, 1–2.5 cm. long; serratures 5–12 on each side of the leaf-blade . . . . 64. ***E. tambillense.***

Scales nearly or quite glabrous toward the tip; petiole minutely puberulent and sparingly muriculate, at most about 1 cm. long; serratures 11–23 on each side of the leaf-blade . . 65. . ***E. stictophyllum.***

f. Leaves pinnately veined.

Involucral scales linear, attenuate; leaves membranaceous . . . . 66. ***E. simulans.***

Involucral scales narrowly oblong, obtuse to rounded at tip; leaves coriaceous, with prominent reticulation on both surfaces . . . . 57. *E. fastigiatum.*

e. Leaves obtuse to cordate at base *g*.

g. Leaves pinnately veined . . . . 27. ***E. endytum.***

g. Leaves 3–5-nerved from or somewhat above the base *h*.

h. Stems villous to hispid-pubescent with widely spreading and jointed hairs *i*.

i. Petioles rarely over 2 mm. long, about one-tenth the length of the leaf-blade . . . . . . 67. *E. cutervense.*

i. Petioles 5–25 mm. long, mostly one-sixth to two-thirds the length of the leaf-blade *j*.

j. Heads 20–40-flowered; pubescence of the stem sparingly if at all gland-tipped.
Leaves crenate-dentate, more than two-thirds as wide as long.
Leaves deltoid-ovate, acute, mostly 2–3 cm. long . . . . 68. *E. vallincola.*
Leaves suborbicular-ovate, acuminate, mostly 4–6 cm. long . . . . 69. *E. articulatum.*
Leaves serrate, about half as wide as long. 64. *E. tambillense.*
j. Heads 80–100-flowered; pubescence of stem gland-tipped and viscid . . . . 70. *E. probum.*
h. Stems sparingly to densely puberulent *k.*
k. Heads paniculate; inflorescence usually ovoid to pyramidal, the branches widely spreading *l.*
l. Heads subglomerate at the tips of the divergent branches of the panicle; leaves crenate; crenatures 13–22 on each side . . . . 49. *E. Cookii.*
l. Heads loosely disposed in the panicle *m.*
m. Leaves nerved essentially from the base.
Stem covered with a dark and dense gland-tipped puberulence; leaves crenate-serrate; teeth about ten on each side. 71. *E. choricephaloides.*
Stem covered with white or gray incurved scarcely or not at all glandular puberulence; leaves remotely cuspidate-dentate; teeth 4–6 on each side . . . . 72. *E. flexile.*
m. Leaves nerved from a point 5–10 mm. above the base.
Petiole 8 mm. long; achenes glabrous; leaves sharply serrate; teeth 11–19 on each side . . . . 72a. *E. Dombeyanum.*
Petiole 10–25 mm. long; achenes somewhat roughened between the ribs; leaves mucronate-serrate; teeth 5–12 on each side . . . . 64. *E. tambillense.*
k. Heads (tending to be crowded) in flattish or round-topped compound corymbs, or (when few) irregularly cymose on chiefly erect or ascending branches *n.*
n. Leaves for the most part conspicuously unequal-sided at the base; internodes very long (10–14 cm. in length) . . . . 74. *E. Sodiroi.*
n. Leaves essentially symmetrical at the base; internodes usually 2–5 (rarely 8) cm. long *o.*
o. Leaves small, 1–1.5(–2) cm. long, typically ovate, finely crenate-serrate; teeth about 5–7 on each side; slender-stemmed copiously branched shrub . . . . 75. *E. scopulorum.*
o. Leaves (at least the mature cauline) larger, 2–7 cm. long *p.*
p. Heads small, about 5 mm. high; leaves very shallowly mucronulate-serrate; stems nearly smooth but slightly and obscurely villous toward the summit; the hairs very slender, short, mostly straight, not glandular . . . . 73. *E. Gilbertii.*

g. Heads 7–9 mm. high; leaves more coarsely serrate; stems closely crisped- or glandular-puberulent.
Heads numerous in dense strongly convex or rounded corymbs; leaves broadly ovate, mostly rounded at the base (though sometimes with a short acumination at the point of attachment), serrate-dentate, the teeth numerous, usually 15–19 on each side, commonly blunt..........76. *E. Sternbergianum*.
Heads few, in flattish open corymbs or loosely cymose; leaves deltoid-ovate or -lanceolate, commonly subtruncat or subcordate at base, coarsely and unequally crenate-serrate, the teeth mostly 5–10 on each side.
Heads about 7 mm. high......76a. *E. glechonophyllum*.
Heads slightly larger, about 9 mm. high .... 77. *E. azangaroense*.
h. Stems glabrous.
Heads about 5 mm. high; leaf-blade about 4 times as long as the petiole...............73. *E. Gilbertii*.
Heads 7–10 mm. high; leaf-blade 8–20 times as long as the petiole
Leaves rounded at base; petioles glabrous; involucral scales subscarious at tip. 72.a. *E. Dombeyanum*.
Leaves subcordate at base; petioles spreading-puberulent; involucral scales attenuate to a sharp non-scarious tip..............78. *E. isillumense*.

57. E. FASTIGIATUM HBK. Nov. Gen. et Spec. iv. 125, t. 347 (1820); Robinson, Proc. Am. Acad. liv. 307 (1918).— PIURA: in the cooler regions of the mountains between Guancabamba [Huancabamba] and the Paramo de Guamani, *Humboldt & Bonpland*, no. 3524 (Par., phot. Gr.).

This species is still somewhat problematic, being known only from the type material with which to date it has been impossible to identify precisely any of the modern collections. The species is clearly very close in habit, inflorescence, leaf-texture and leaf-venation to the variable *E. exserto-venosum* Klatt, but the leaves are more obovate or oblanceolate than in any as yet described variety of the latter plant. Moreover, the leaves are rather conspicuously and bluntly cuspidate at the tip. Finally Kunth, l. c., in describing the species states that the heads are about 12-flowered, that is to say have half again to twice as many florets as are found in *E. exserto-venosum* Klatt. Material approximating *E. fastigiatum* has been collected in Colombia (see Robinson, l. c.) but its identity must be subject to doubt until the type can be re-examined.

58. E. EXSERTO-VENOSUM Klatt, Abh. Naturh. Ges. Halle, xv.

324 (1882); Robinson, Proc. Am. Acad. liv. 358 (1918).— Variable in leaf-form as follows:

Var. α. **crenatum** (Hieron.), comb. nov. Leaves elliptical, obtuse, rounded at the base, shallowly crenate, 1.7–4.5 cm. long, 1.2–2.4 cm. wide.— *E. pseudofastigiatum*, var. *crenata* Hieron. in Engl. Bot. Jahrb. xxxvi. 468 (1905).— CAJAMARCA: near Cutervo, *von Jelski*, no. 789 (Berl., fragm. Gr.). PERU WITHOUT LOCALITY: *Mathews* (fragm. Gr.).

Var. β. PSEUDOFASTIGIATUM (Hieron.) Robinson. Leaves ovate-lanceolate, 3–5 cm. long, 1.6–2.6 cm. wide, acute or acutish at the apex, acute at base, finely serrate-dentate, the teeth subacute.— Proc. Am. Acad. liv. 359 (1918). *E.? fastigiatum* Benth. Pl. Hartw. 135 (1844), not HBK. *E. loxense* Hieron. in Engl. Bot. Jahrb. xxi. 331 (1895), not Klatt. *E. pseudofastigiatum* Hieron. l. c. xxxvi. 467 (1905).— PERU WITHOUT LOCALITY: *Mathews* (N. Y.). [Southern Ecuador.]

Var. γ. **crenato-dentatum** (Hieron.), comb. nov. Leaves elliptic-lanceolate, acute or acutish at the apex, rounded at base, 5–8 cm. long, crenate-dentate or crenate-serrate, the teeth slightly coarser than in the preceding.— *E. pseudofastigiatum*, var. *crenato-dentata* Hieron. l. c. 468.— CAJAMARCA: near Cutervo, *von Jelski*, no. 730 (Berl., fragm. Gr.).

Var. δ. **lanceolatum** (Hieron.), comb. nov. Leaves lance-oblong, 9 cm. long, 3.2 cm. wide, acutish at the apex, cuneate at the base, crenate-serrate, the teeth again sparingly mucronulate-serrulate.— *E. pseudofastigiatum*, var. *lanceolatum* Hieron. l. c.— CAJAMARCA: near Tambillo, *von Jelski*, no. 729 (Berl., fragm. Gr.).

It is possible that this species, highly variable in its leaf-form, will ultimately be found to merge with *E. fastigiatum* HBK., but that is said to have heads about 12-flowered and its leaves, although possessing closely the pinnate venation and reticulation of *E. exserto-venosum*, differ in being oblanceolate, bluntly mucronate at the apex and narrowly cuneate at base. Much more ample material of these forms is essential before they can be classified with confidence. With the present fragmentary representation they can neither be reduced with certainty nor maintained with much satisfaction. But, as they differ, it seems best to keep them up until evidence of intergradation becomes available.

59. E. CHILCA HBK. Glabrous somewhat viscid shrub; branches opposite, erect, or curved-ascending, leafy when young, at maturity apt to be denudated; leaves opposite, rather narrowly lanceolate,

broadest about a sixth of the length above the entire rather rapidly narrowed base, gradually attenuate to an acutish tip, about 3 cm. long, 7–8 mm. wide, thickish-membranaceous and of rather firm texture, 3-ribbed from near the base, serrate from the broadest part to the tip, teeth about 8 on each side slightly curved outward, the lowest narrow and often slightly longer than the rest; petiole about 2 mm. long; corymb compound, many-headed, round-topped, rather dense, 6–9 cm. in diameter, leafy-bracted; heads sessile or short-pedicelled, about 4-flowered, 4 mm. long; corollas white, glabrous, slightly enlarged upward, fragrant; style-branches filiform, slightly thickened at the summit; pappus-bristles hispid-pubescent.— Nov. Gen. et Spec. iv. 125 (1820). *Ophryosporus chilca* (HBK.) Hieron. in Engl. Bot. Jahrb. xxii. 706 (1897).— CAJAMARCA: at the base of Mt. Sta. Polonia, near the city of Cajamarca, *Humboldt & Bonpland*, no. 3682 (Par., phot. Gr., Berl., phot. Gr.).

This and the two following species are very closely related and with the Bolivian *E. eleutheranthcrum* Rusby form a group doubtfully intermediate between *Eupatorium* and *Ophryosporus*. Their final disposition must await better and far more copious material than is yet available.

60. E. AFFINE HBK. Glabrous shrub, closely resembling the preceding; differing chiefly in its larger leaves (4 cm. long and 1.6 cm. wide) and longer petioles (8–10 mm. in length), also in its more loosely branched panicle, and 5–6-flowered heads of somewhat greater size (6 mm. long).— Nov. Gen. et Spec. iv. 126 (1820).— CAJAMARCA: thought to have been collected with the preceding, *Humboldt & Bonpland* (Par., phot. Gr.).

61. E. HEPTANTHUM Sch. Bip. Shrub, closely related to the two preceding, but the young branches and peduncles glandular-pubescent; leaves lanceolate or ovate-lanceolate, 1.5–2.5 cm. long, 5–12 mm. wide, acute at the apex, varying from obtuse to attenuate at the base, glandular-puberulent on both surfaces; petiole short and pubescent; heads in densish corymbs, about 7-flowered; involucral scales 7–8, subequal, linear, acutish.— Bonplandia, iv. 54 (1856), without descript.; Wedd. Chlor. And. i. 217 (1857), where first described; Sch. Bip. Bull. Soc. Bot. Fr. xii. 82 (1865), without char.; not, however, Rusby, Bull. N. Y. Bot. Gard. iv. 378 (1907).— PUNO: mountains about Azangaro, *Lechler*, no. 1751; on stony slopes among herbaceous plants and scattered shrubs, Puno, alt. 3600 m., 19 Aug. 1902, *Weberbauer* no. 1366 (Berl., fragm. Gr.). AREQUIPA: on sparsely covered ground, on the west slope of the Volcano Misti, near

Arequipa, alt. 3200–3600 m., 8 Sept. 1902, *Weberbauer*, no. 1426 (Berl., fragm. Gr.). TACNA (debated region): Cordillera of Tacora, *Weddell.*

62. E. UBER Robinson (p. 37). Robust shrub 4 mm. high, with luxuriant foliage; branches grayish-brown, granular-puberulent, leafy; leaves opposite, deltoid-ovate, acuminate, slightly undulate-denticulate or quite entire, abruptly narrowed and subcuneate at base, 16–19 cm. long, 6–9 cm. wide, above glabrous green, the reticulated veinlets depressed, beneath sordid-puberulent or -tomentellous; chief lateral nerves mostly 2 pairs leaving the midnerve about 1 cm. above the base; petiole 3–5 cm. long, granular-puberulent; panicle corymbiform, as much as 2 dm. in diameter, flattish or moderately convex, rather dense; heads excessively numerous, shortly pedicelled, about 13-flowered, about 9 mm. long; involucre narrowly campanulate; scales about 16, linear-oblong, acutish, subequal, erose-ciliolate, dorsally granular; corollas 5 mm. long, white, gradually and moderately enlarged upward; achenes grayish-brown, 2.5 mm. long, tapering downward, hispidulous on the angles; pappus-bristles about 31, dirty-white, barbellate, nearly equalling the corolla.— ANCACHS: woods, below Pampa Romas, between Samanco and Caraz, alt. 2100 m., *Weberbauer*, no. 3184 (Berl., phot. and fragm. Gr.).

63. E. CUZCOENSE Hieron. Shrubby, 1 m. high; stems terete, grayish-brown, smooth; branches opposite, ascending, finely puberulent, hairs crisped, incurved or appressed, not glandular; leaves opposite, lanceolate, attenuate, obtusish at base, serrate (teeth 5–12 on each side), firmly membranaceous, 3–4.5 cm. long, 1.3–2 cm. wide, glabrous, above dull-green, beneath somewhat paler, finely reticulated and punctate, subtrinervate from a point 1–2 mm. above the base, the lateral nerves reaching about to the middle; veinlets not prominulent; petiole 2–5 mm. long; corymbs terminal, strongly convex to semiglobose, rather dense, 3–6 cm. in diameter, disposed in a leafy-bracted compound corymb; pedicels slender, 5–14 mm. long; obscurely puberulent, usually bearing a filiform bractlet; heads about 12–14-flowered, about 9 mm. long; involucre subcylindric-campanulate; scales subequal (except 2–4 of the outermost smaller ones), lance- or oblong-linear, acutish to acuminate, dorsally puberulent and sparsely glandular, ciliolate, mostly 2-costulate and 3-nerved; corollas purple-tinged, 5 mm. long, beset with sessile glandular granules; proper tube about 1.8 mm. long, slender; throat perceptibly enlarged, subcylindric, 3.2 mm. long; achenes brownish-black, at full maturity about 3 mm. long, tapering somewhat toward the base, closely beset with sessile and short-stiped capitate glands;

pappus-bristles about 25, white, scarcely roughened, slightly united into an annulus at the summit of the achene.— Hieron. in Engl. Bot. Jahrb. xl. 376 (1908).— CUZCO: near the city of Cuzco, *Squier* (Gr.), *Dr. & Mrs. J. N. Rose*, no. 19,028 (Gr., U. S., N. Y.); on Sacsahuamán Hill, in grassy and bushy places, alt. 3500–3600 m., *Weberbauer*, no. 4852; Ollantaytambo, alt. about 3000 m., *Cook & Gilbert*, no. 534 (U. S.).

64. E. TAMBILLENSE Hieron. Branches round, pubescent, at length glabrate and brown; internodes up to 8 cm. long; leaves opposite, ovate or ovate-lanceolate, acuminate, serrate except at the acute apex and rounded or shortly cuneate base (teeth 5–12 on each side, at most 1.5 mm. high, 7 mm. broad, mucronate), firmly membranaceous, bright yellowish-green, 3-nerved from a point 5–10 mm. above the base (nerves connected by subparallel transverse veins), attaining 8 cm. in length and 4 cm. in breadth, above puberulent on the nerves, the reticulated veinlets depressed, beneath pubescent on the nerves; petiole 1–2.5 cm. long; panicle leafy-bracted; pedicels as much as 1.5 cm. long, puberulent, bracteolate; heads 20–25-flowered; involucre narrowly campanulate; scales about 15, lance-linear, acutish, 3-nerved, about 6 mm. long, 2–3 of the outermost shorter; corollas about 4 mm. long, with more or less differentiated proper tube and throat, glabrous except on the limb, where sparingly beset with jointed hairs; achenes dark-brown, 1.75 mm. long (scarcely mature), smooth and shining on the faces, scabrid on the concolorous angles; pappus-bristles about 25, yellowish-white, not thickened toward the tip.— Hieron. in Engl. Bot. Jahrb. xl. 380 (1908).— CAJAMARCA: near Tambillo, 11 Aug. 1878, *von Jelski*, no. 668. No material of this has been seen by the writer. The description is here compiled from the original character.

65. E. STICTOPHYLLUM Robinson (p. 36). Shrubby, 1 m. high; stem subterete, costulate (when dried), purplish-brown, when young obscurely puberulent, opposite-branched, leafy up to or into the broad and flattish or moderately convex compound corymb; leaves opposite, ovate-lanceolate, caudate-acuminate at apex, abruptly subacuminate at the roundish base, serrate (teeth 11–23 on each side, mucronulate, often again toothed), firmly membranaceous, glabrous, above dull-green, beneath paler green, finely reticulate-veined and dark-punctate (veinlets not prominulent), 6–8 cm. long, half as wide, 3-nerved from a point about 5 mm. above the base, the lateral nerves often dividing shortly after leaving the midnerve (or occasionally replaced by 2 adjacent and pinnately disposed nerves on one side);

petiole 7–11 mm. long, puberulent and sparingly muriculate; partial inflorescences rounded moderately dense fastigiately branched corymbs; heads about 24-flowered, 1 cm. long; pedicels filiform, 6–10 mm. long, bearing a filiform bractlet; involucral scales about 24, narrowly lance-linear, unequal but little imbricated, acute, mostly 3-nerved and 2-costulate, purplish-green, dorsally granular-pulverulent; corollas about 7.8 mm. long, deep rose-colored, gradually and but little enlarged upward, glabrous; style-branches subuniformly filiform, not distinctly clavate; achenes (still immature) 3 mm. long, beset with sessile or subsessile glands; pappus-bristles about 27, white, scabrid, nearly equalling the corolla.— PUNO: Cuyocuyo, Prov. Sandia, in bushy places, alt. 3100 m., *Weberbauer*, no. 860 (Berl., phot. and fragm. Gr.).

66. E. SIMULANS Robinson (p. 31). Herbaceous or shrubby, 1–2 m. high, exceedingly similar in habit and foliage to the preceding; leaves smaller, 5–7 cm. long, 1.3–2.5 cm. wide, acute at base, scarcely or not at all punctate beneath, feather-veined, some of the lower lateral veins (often 2–3 on each side) somewhat more prominent than the others; petiole not at all muriculate; involucral scales mostly 1-ribbed, the upper part of softer texture, the outer scales somewhat pubescent and ciliate toward the base.— ANCACHS: among bushes in the gorge of a brook on the slopes of the Cordillera Blanca, above Caraz, alt. 3200–3600 m., 9 June, 1903, *Weberbauer*, no. 3253 (Berl., phot. and fragm. Gr.); among small bushes on a brook, above Ocros, Prov. Cajatambo, alt. 3300 m., 2 Apr. 1903, *Weberbauer*, no. 2766 (Berl., phot. and fragm. Gr.).

67. E. CUTERVENSE Hieron. Low, much branched, scarcely lignescent, 1.5–4 dm. high; stems round, covered with dark articulated stiffish hairs; leaves opposite, subsessile or on very short petioles (scarcely 2 mm. in length), broadly ovate, cordate, short-acuminate, chartaceous, somewhat shallowly crenate, dull yellowish-green drying dark, when young sparingly beset on both surfaces chiefly on the nerves with rigid hairs, later glabrate above, somewhat 5-nerved, at most 2 cm. long, 18 mm. wide; corymbs or cymes loose, few-headed; pedicels sometimes as much as 5 cm. long; heads 30–35-flowered; involucre campanulate; the scales subequal, little imbricated, about 20, dull-green, lanceolate, acuminate, mucronate, villous-ciliate, sparingly villous dorsally on the upper part, the hairs jointed, flexuous, hyaline; corollas yellowish-white after drying, glabrous outside, 4.5 mm. long, the slender proper tube about equalling the enlarged throat; achenes dark, roughened on the upper part of the concolorous

ribs, scarcely 3.5 mm. long; pappus-bristles about 20, whitish.—Hieron. in Engl. Bot. Jahrb. xl. 383 (1908).—CAJAMARCA: near Cutervo, *von Jelski*, nos. 637, 701; near Tambillo, *von Jelski*, no. 612. [Southern Ecuador? see p. 10.]

This species, said to be near the Colombian *E. sotarense* Hieron., has not been seen by the writer and the above description is condensed from the original of Prof. Hieronymus. The species from character appears clearly distinct from other Peruvian Eupatoriums of its affinity by its small broadly ovate and nearly sessile leaves.

68. E. VALLINCOLA DC. Shrubby, erect or nearly so, 5 dm. high; stems (when young) and branches spreading-villous, the hairs jointed; internodes 2–3(–9) cm. long; leaves opposite, often proliferous in the axils, deltoid-ovate, acute, subtruncate to shallowly and openly cordate (shortly acuminate at the attachment), 2–3.5 cm. long and wide, 3-nerved from the very base, coarsely and often somewhat doubly crenate-dentate, above sparingly pubescent, beneath loosely villous especially on the nerves and veins (the hairs slender, flexuous, jointed); petiole 1.4–2.5 cm. long; corymbs terminal, fastigiately branched, rounded, few–many-headed; pedicels 6–9 mm. long; heads 20–25-flowered, 8–10 mm. high; involucre campanulate, the scales subequal, linear, acute, pubescent, mostly 2-ribbed; corollas white, 4.5 mm. long, hispid on the limb, with slender proper tube and moderately enlarged throat; achenes (immature) 1.6 mm. long, slightly contracted at the summit and tapering to the base, hispidulous on the angles; pappus-bristles very few, often about 10, delicate, white.—Prod. v. 168 (1836).—LIMA: mountains of Barranco near Lima, on calcareous rock, alt. 300–600 m., 23 Oct. 1902, *Weberbauer*, no. 1650 (Berl., fragm. Gr.). PERU WITHOUT EXACT LOCALITY: *Haenke* (DC., phot. Gr.).

Var. α. TYPICUM. Hairs of the stem, petioles, pedicels, and sometimes on the lower surface of the leaves long, slender, spreading, more or less conspicuously jointed.—Lit. and exsicc., as above.

Var. β. BREVIPILUM Robinson (p. 39). Puberulent to tomentellous, the hairs mostly short, neither gland-tipped nor conspicuously jointed; otherwise closely like var. α.—LIMA: on the Lima and Oroya Railroad, between Matucana and Tambo de Viso, on rocks, alt. 2370–2650 m., 26 Dec. 1901, *Weberbauer*, no. 103 (Berl., fragm. Gr.); on slope of eruptive rock, alt. 2370 m., 24 Dec. 1901, *Weberbauer*, no. 66 (Berl.).

69. E. ARTICULATUM Sch. Bip. ex Hieron. in Engl. Bot. Jahrb. xl. 385 (1908); Robinson, Proc. Am. Acad. liv. 316, 342 (1918).—CAJAMARCA: between Choto and Cutervo, *von Jelski*, no. 674, acc. to Hieron.

l. c. The following are with some doubt referred here: LORETO: Tarapoto, 1835, *Mathews*, no. 1417 (K.), and AMAZONAS: Prov. Chachapoyas, *Mathews* (K.).

Persistent effort has thus far failed to disclose in North American herbaria any material which can with entire confidence be placed in this species. It is hard to understand just why Hieronymus takes the trouble to distinguish the plant from the geographically remote and as to habit dissimilar *E. prunellaefolium* HBK. of Mexico, yet appears to feel no obligation to point out the much-needed distinctions between *E. articulatum* and the closely related *E. vallincola* DC. and *E. pichinchense* HBK. of similar Andean distribution. While it is impossible at present to unite these species, the types of which appear to differ in several minor features, the characters thus far known to separate them are exceedingly trifling, such as pubescence, length of the petiole, number and size of the teeth of the leaves, etc., matters in which considerable variation has already been observed.

70. E. PROBUM N. E. Brown. Herbaceous (at least above), much branched and very leafy, covered throughout with soft gland-tipped and viscid hairs; stems round, weak, pithy; leaves opposite or the upper subalternate, ovate, acute to acuminate, the cauline rounded to truncate or open-cordate at base, serrate (teeth about 7 on each side), thin, green and thinly pubescent on both surfaces, 3-nerved from the very base (the nerves villous beneath), 2–4.5 cm. long, 1.3–3 cm. wide, delicately membranaceous; petiole 1.5–2.5 cm. long, glandular-pubescent; corymbs terminal, fastigiately branched, flattish topped, sometimes 1 dm. in diameter, 12–15-headed, sometimes much smaller and only 1–5-headed; pedicels (in greenhouse material) 2–5 cm. long; heads 80–100-flowered, 12 mm. high and thick; involucre about 2-seriate, campanulate, the scales about 20, lance-linear, acute, thin, green and pubescent toward the mostly 2-ribbed base, scarious on the margin and toward the tip; corollas white, slightly puberulent on the short limb, the tube about 1.7 mm. long, slender, the throat subcylindric-campanulate, 3 mm. long; achenes upwardly hispid on the angles, about 2.5 mm. long, crowned by a shallow stramineous saucer-shaped disk; pappus-bristles bright white, delicate, nearly smooth.— Gard. Chron. ser. 3, vii. 321, fig. 48 (1890).— PERU: introduced into horticulture about 1870 by Mr. Wilson Saunders, who raised it from seed collected presumably in Peru by Mr. Farris. The cultivated specimen from which the above character is drawn is in the herbarium at Kew. It has thus far proved impossible to match it with any material collected in Peru.

71. E. CHORICEPHALOIDES Robinson (p. 8). Apparently herbaceous (the base unknown); stems and branches terete, flexuous, densely puberulent, with short brownish gland-tipped spreading hairs; internodes sometimes 1 dm. long; leaves opposite, deltoid-ovate, acuminate, subtruncate or shallowly and broadly cordate at base, serrate- or crenate-dentate (teeth unequal, rounded to subacute, about 10 on each side, the largest 2.5 mm. high, 5 mm. broad), membranaceous, above densely puberulent, beneath puberulent and on the nerves and chief veins spreading-villous, about 6 cm. long, about 4.5 cm. wide, 3-nerved from the base; panicle very loose and open, leafy-bracted, 4 dm. high, 3 dm. in diameter, densely glandular-puberulent throughout; pedicels filiform, 1–2.6 cm. long, naked or inconspicuously bracteolate; heads separate, about 7 mm. high, about 25–30-flowered; involucre campanulate; scales subequal, narrowly oblong or oblanceolate, erose toward the acute scarious tip, green and 2–3-ribbed in the middle, puberulent on the back, about 4 mm. long; corollas white, glabrous, the proper tube about equalling the subcylindric distinctly enlarged throat; teeth very short, 0.3 mm. in length; achenes light-brown, 1.8 mm. long, hispidulous on the angles, smooth on the concave faces; pappus-bristles about 12, whitish, minutely scabridulous.— AMAZONAS: Province of Chachapoyas, *Mathews* (K., phot. and fragm. Gr.).

This species in its loose inflorescence recalls the Mexican *E. choricephalum* Robinson.

72. E. FLEXILE Robinson (p. 14). Suffruticose, slender, tending to climb, 2 m. high; stems terete, flexuous, softly and shortly dirty-tomentellous; leaves ovate, caudate-acuminate, cordate at base, 5–7-nerved, thin, puberulent above, below especially on the nerves grayish-pubescent, about 6 cm. long, 3 cm. wide, slightly and remotely toothed; petiole 1 cm. long, slender; panicle ample, pyramidal, 2–4 dm. high, 1.5–2.5 dm. thick, loose, leafy-bracted; pedicels filiform, grayish-tomentellous; heads about 20-flowered, 7 mm. high, 6 mm. in diameter; involucre campanulate; scales about 16, linear-oblong, obtusish but sharply mucronate, mostly about 5 mm. long; corollas pale greenish-yellow, tubular, slightly and gradually enlarged upward; achenes grayish, 2.7 mm. long, hispid on the angles; pappus-bristles about 32, white, scarcely barbellate, about equalling the corolla.— ANCACHS: in woods near a river at Caraz, alt. 2200 m., *Dr. Weberbauer*, no. 3027 (Berl., phot. and fragm. Gr.).

A species recalling *E. solidaginoides* HBK. but with larger, more numerously flowered heads, longer achenes, and various minor differences.

72a. E. Dombeyanum DC. Prod. v. 167 (1836). It seems by no means improbable that this species (discussed in some detail on page 11) was originally collected in Peru. It reached DeCandolle devoid of data beyond the fact that it had been gathered in South America by Dombey (DC., phot. Gr.), whose explorations were chiefly in Peru.

73. E. Gilbertii Robinson (p. 16). Slender and nearly glabrous perennial herb, or perhaps shrub; stems greenish straw-colored, round, about 2 mm. thick, flexuous, at maturity entirely glabrous, when young sparingly provided with an inconspicuous pubescence of minute straightish non-glandular hairs; these becoming somewhat more abundant on the inflorescence; leaves opposite, ovate, acute or slightly acuminate, mucronulate-serrate or -crenate (the teeth inconspicuous, about 0.6 mm. high and 5 mm. apart), thin, membranaceous, above green and puberulent on the nerves, beneath somewhat paler and glabrous, 6–7 cm. long, 3–3.5 cm. wide, 3(–5)-nerved essentially from the rounded or subcordate entire base; petiole slender, about 1.7 cm. long, puberulent above; corymbs compound, flattish, rather loose; pedicels 4–10 mm. long; heads small, scarcely 5 mm. high or thick, about 26-flowered; scales of the campanulate involucre about 17, nearly equal, oblong-lanceolate, greenish-stramineous, acutish, scarious-margined, ciliate, sparingly pubescent on the back; corollas white, about 3.3 mm. long, the proper tube glabrous, about equalling the campanulate throat, limb hispidulous; achenes (very immature) 1.2 mm. long, apparently glabrous; pappus-bristles about 20, white.— Cuzco: San Miguel, Urubamba Valley, alt. about 1800 m., *Cook & Gilbert*, no. 1115 (U. S., phot. and fragm. Gr.).

74. E. Sodiroi Hieron. in Engl. Bot. Jahrb. xxix. 12 (1900); Robinson, Proc. Am. Acad. liv. 362 (1918). *E. Sternbergianum* Ball; Jour. Linn. Soc. xxii. 43 (1885), as to plant first mentioned.— Lima: in the middle region of the Andes in the upper valley of the Rimac River, alt. 1830–3355 m., April, *Ball* (Gr.). Ball, l. c., remarks that the plant grows to a height of several feet. [Ecuador.]

75. E. scopulorum Wedd. Fruticulose, much branched, 2–5 dm. high; stems terete, slender, often decumbent, dark, nodose below after the early fall of the leaves; lower internodes 5–10 mm., the upper often 3–5 cm. long; leaves ovate, acute, rounded or subcordate at base, crenate-serrate (teeth about 5 on each side), membranaceous, 3-nerved from the base, slightly puberulent on the nerves, 8–18 mm. long, 5–12 mm. wide, scarcely paler beneath; petiole slender, 3–6 mm. long; corymbs terminal, simple and 3–5-headed or compound and

about 8–12-headed; pedicels 1–2 cm. long; heads 6–8 mm. high and equally thick, about 46-flowered; involucre campanulate; scales about 20, lance-linear, attenuate, slightly puberulent, mostly 3-nerved and 2-costulate; corollas purplish (Weddell) or white (Weberbauer), with slender proper tube (1.7 mm. long) and perceptibly enlarged cylindrical throat (3.6 mm. long); achenes 2.5 mm. long, hispidulous on the angles and faces; pappus-bristles about 18, delicate, white, barbellate.— Chlor. And. i. 216, t. 40, f. B (1857).— JUNIN: near Oroya, *Dr. & Mrs. J. N. Rose*, nos. 18,711 (N. Y.), 18,712 (N. Y.). PUNO: on moist cliffs around Lake Titicaca, alt. 3900 m., *Weddell.* ANCACHS: in open grassy formation between Samanco and Caraz, alt. 3700 m., *Dr. Weberbauer*, no. 3054 (Berl., fragm. Gr.). [Bolivia.]

76. E. STERNBERGIANUM DC. Apparently herbaceous, 4 dm. or more in height; stem terete, at first minutely puberulent, at maturity glabrate, dark-purple; leaves opposite, ovate, acuminate, rounded at the base, incisely serrate-dentate or deeply crenate-dentate (the teeth numerous, unequal, often again toothed), membranaceous, 3-nerved from the base, sparingly puberulent to glabrous above, scarcely paler and somewhat puberulent beneath, 3–5 cm. long, 2.5–4.5 cm. wide; petiole about 1 cm. long; corymbs dense, rounded; heads about 28-flowered, pedicellate, about 6 mm. high; involucre campanulate, the scales subequal, linear, acutish, about 2-seriate, sparingly puberulent or subglabrous; corollas white to reddish (Weberbauer), with proper tube nearly equalling the cylindric throat, nearly glabrous; achenes hispid.— Prod. v. 167 (1836).— ANCACHS: at Tallenga, alt. 3600–3800 m., *Dr. Weberbauer*, no. 2876 (Berl., fragm. Gr.). LIMA: at Obrajillo, *Wilkes Exp.* (Gr.); near Huarochirí, alt. 2100–3000 m., *Hrdlicka* (U. S.). JUNIN: near Oroya, alt. 3750 m., *Dr. & Mrs. J. N. Rose*, no. 18,685 (Gr., N. Y.). CUZCO: near Tinta, alt. about 3500 m., *Cook & Gilbert*, no. 213 (U. S.); Ollantaytambo, alt. about 3000 m., *Cook & Gilbert*, no. 331 (U. S.). DEPARTMENT NOT ASCERTAINED: among hills in the cordilleras of Peru, *Haenke* (DC.); at Pachacaya, *C. H. T. Townsend*, no. 1505 (U. S.).

This species according to Dr. Weberbauer is locally called *hualmi-hualmi*. He also states that the fresh roots, softened in lukewarm water, are employed as an abortive, and that a tea prepared from the leaves is used for kidney and bladder troubles.

[76a. E. GLECHONOPHYLLUM Less. Linnaea, vi. 105 (1831); Robinson, Proc. Am. Acad. liv. 363 (1918).— This species has not been seen from Peru, but as it occurs in the Chilean valleys and also without apparent distinction of form on the mountains of Ecuador, it seems more than likely that further exploration will show its presence in the intermediate country of Peru.]

77. E. AZANGAROENSE Sch. Bip. ex Wedd. Chlor. And. i. 217 (1857); Robinson, Proc. Am. Acad. liv. 315 (1918).— PUNO: on calcareous rock, Azangaro, alt. 4000 m., 28 Feb. 1902, *Dr. Weberbauer*, no. 468 (Berl., fragm. Gr.). WITHOUT INDICATION OF DEPARTMENT: *de Castelnau* acc. to Weddell, l. c. [Bolivia, Ecuador, Venezuela acc. to Weddell.] A species doubtfully distinct from the preceding.

78. E. ISILLUMENSE Robinson (p. 20). Slender subscandent shrub, glabrous except on the petioles and base of the nerves where puberulent, nigrescent in drying; stem terete, pale-brown, internodes 3–8 cm. long; leaves opposite, ovate-oblong (the uppermost lanceolate), acuminate, slightly cordate at the base, mucronate-serrate, thin, 3–5-nerved from the base (the nerves connected by transverse veins), 10–12 cm. long, 3–4.5 cm. wide; petiole slender, viscid-tomentellous, 4–6 mm. long; panicle ovoid, loose, leafy-bracted below; pedicels filiform, glabrous, 8–15 mm. long, often with scale-like bractlets; heads about 52-flowered, 7 mm. high, 9 mm. in diameter; involucre broadly campanulate; scales about 30, subequal (1–3 of the outermost shorter), linear, very acute, glabrous, faintly 1–3-nerved; corollas 3 mm. long, white, glabrous, slightly and gradually enlarged upward; immature achenes 2 mm. long, apparently glabrous; pappus-bristles about 28, white, delicate, nearly smooth.— PUNO: in woods near the Tambo Isilluma, between Sandia and Chunchusnago, alt. 1000 m., *Dr. Weberbauer*, no. 1206 (Berl., phot. and fragm. Gr.). In habit and foliage recalling the Brazilian *E. laeve* DC.

Sect. IV. PRAXELIS (Cass.) Benth. (See Robinson, Proc. Am. Acad. liv. 318.)

KEY TO SPECIES.

Cauline leaves ovate, petiolate . . . . .79. *E. pauciflorum*.
Cauline leaves lanceolate to linear, sessile or nearly so . . . . . .80. *E. kleinioides*.

79. E. PAUCIFLORUM HBK. Nov. Gen. et Spec. iv. 120 (1820); Robinson, Proc. Am. Acad. liv. 319, 343 (1918).— CUZCO: Santa Ana, alt. 900 m., *Cook & Gilbert*, no. 1606 (U. S.). PERU WITHOUT INDICATION OF DEPARTMENT: *Mathews*, no. 3081 (Gr.).

80. E. KLEINIOIDES HBK. Nov. Gen. et Spec. iv. 120 (1820); Robinson, Proc. Am. Acad. liv. 319 (1918).

Var. **typicum** (Hieron.), comb. nov. Hispid throughout; leaves lanceolate.— *E. kleinioides*, forma *typica* Hieron. in Engl. Bog. Jahrb. xxii. 782 (1897).— PERU WITHOUT LOCALITY: according to Hieronymus, l. c. 783. [Brazil, Venezuela, Argentina, Paraguay.]

Var. SUBGLABRATUM Hieron. Stems subglabrous or below sparingly and shortly pilose; leaves linear-lanceolate, glabrate except on the scabrid edge.— Hieron. l. c. 782, as *subglabrata*. *Ooclinium paucidentatum* DC. Prod. v. 134 (1836).— PERU WITHOUT LOCALITY: *Haenke* according to DeCandolle, l. c. [Argentina, southern Brazil.]

Sect. V. HEBECLINIUM (DC.) Benth. (See Robinson, Proc. Am. Acad. liv. 327.)

KEY TO SPECIES.

Heads 200–300-flowered; leaves on winged auriculate-based petioles . . . . 81. *E. nemorosum.*
Heads 50–75-flowered; petioles not winged . . . . 82. *E. macrophyllum*

81. E. NEMOROSUM Klatt in Engl. Bot. Jahrb. viii. 35 (1887); Robinson, Proc. Am. Acad. liv. 327, 366 (1918). *E. pteropodum* Hieron. in Engl. Bot. Jahrb. xxix. 15 (1900).— CAJAMARCA: near Tambillo, *von Jelski*, nos. 737, 738, according to Hieronymus, l. c., as *E. pteropodum*. [Costa Rica, Colombia, Ecuador, Bolivia.]

82. E. MACROPHYLLUM L. Sp. Pl. ed. 2, ii. 1175 (1763); Bak. in Mart. Fl. Bras. vi. pt. 2, 345, t. 92 (1876); Robinson, Proc. Am. Acad. liv. 329 (1918).— CUZCO: Santa Ana, alt. 900 m., *Cook & Gilbert*, no. 1443 (U. S.). PERU WITHOUT LOCALITY: according to Baker, l. c. [Widely distributed in tropical and subtropical America.]

DOUBTFUL OR TRANSFERRED SPECIES AND VARIETIES.

E. ADENOPHORUM, var. PERUVIANUM Hieron. in Engl. Bot. Jahrb. xxxvi. 470 (1905). This problematic plant, collected in Cajamarca, near Tambillo, by *von Jelski*, no. 661 (Berl., sterile fragm. Gr.), was described by Hieronymus, l. c., as a possible variety of *E. adenophorum* Spreng. Syst. iii. 420 (1826). Sprengel's species, however, was a mere renaming of the Mexican *E. glandulosum* HBK. Nov. Gen. et Spec. iv. 122 (1820), on account of the earlier homonym of Michaux. As the latter is universally relegated to synonymy, there appears at present no reason, according to the International Rules, to reject *E. glandulosum* HBK., a name which was amply characterized and put forward in all good faith. The plant of von Jelski, however, possesses a round-ovate instead of triangular-ovate leaf, and the indumentum, which Hieronymus finds similar, appears to the writer very different. In *E. glandulosum* the hairs are short, dense, and gland-tipped, in the von Jelski plant on the other hand they are flaccid, slender, jointed, and for the most part not gland-tipped. From the sterile fragment, kindly supplied to the writer at the Royal Gardens in Berlin during his visit in 1905, it would appear that the plant is certainly distinct from the Mexican *E. glandulosum* HBK. (*E. adenophorum* Spreng.), but until fertile specimens are available it is quite impossible to give the plant definite disposition.

E. AROMATICUM L., a species of Atlantic North America, extending from Massachusetts to Florida, was recorded as also from Peru by Lamarck, Encyc. ii. 406 (1786), on the basis of a specimen from Joseph de Jussieu. The plant was stated to be smaller than the North American and to have shorter petioles as well as other minor differences. Just what species Lamarck thus identified has not been ascertained, but there is no likelihood of its having been conspecific with the North American plant.

E. CANNABINUM L., the well-known European species, and the medicinal E. TRIPLINERVE Vahl (under the later name of *E. Ayapana* Vent.) were reported by Martinet, Enum. Jard. Med. Lima, 352 (1873), as cultivated in the Botanic Garden of the Medical Faculty at Lima; but there is little likelihood and certainly no evidence that either has at any time escaped or become established in Peru.

E. GLUTINOSUM Lam. Encyc. ii. 408 (1786); Robinson, Proc. Am. Acad. liv. 349 (1918). Described from a specimen in the Peruvian herbarium of Joseph de Jussieu, this species has been traditionally attributed to that country, though all its subsequent collections appear to have been in northern-central Ecuador. It is to be remembered that the boundaries of Peru in the middle of the 18th Century included what is now Ecuador and Bolivia, so it is pertinent to inquire where Joseph de Jussieu collected. According to Lasègue, Mus. Bot. Delessert, 484 (1845), Joseph de Jussieu went to South America as a botanist accompanying an astronomical expedition, which reached Quito in 1756 by way of Guayaquil. Thus de Jussieu must have been in the very region of Ecuador where *E. glutinosum* is now known to be frequent. It is true that he later traveled both in Peru proper and in what is now Bolivia, but the fact that he was also in the Ecuadorian habitat of *E. glutinosum* makes it more than probable that it was there that he obtained the species. Certainly under these circumstances the fact that Lamarck's label mentions Peru as the place of origin can in no sense be taken as evidence that the plant came from what is now Peru. While it is by no means impossible, nor even very unlikely, that *E. glutinosum* may ultimately be found in Peru as now delimited, there is as yet no good basis for its inclusion in the Peruvian flora.

E. Kuntzei Hieron. in Engl. Bot. Jahrb. xxii. 766 (1897). This species, carefully studied from a portion of the material originally collected near Cochabamba by Kuntze (U. S.) and in better specimens subsequently secured in Southern Bolivia by Fiebrig (no. 3150, Gr.), proves to have the anthers destitute of apical appendages and the style-tips rather abruptly thickened, bluntish, and dark. It is unquestionably of the Subtribe *Piquerinae* and belongs to *Ophryosporus* § *Ophryochaeta*. When placed in its proper affinity, it has been found to match in all significant details OPHRYOSPORUS MACRODON Griseb. Abh. Goett. xxiv. 173 (1879), a species heretofore known only from the Nevado del Castillo, Prov. of Salta, in northern Argentina, a locality within about 300 km. of Fiebrig's Bolivian station. To the writer the species appears to have no close resemblance to the well known and widely distributed *Eupatorium inulaefolium* HBK. to which Hieronymus regarded it most nearly related.

E. piquerioides DC. Prod. v. 175 (1836), from the mountains of Peru, is OPHRYOSPORUS PIQUERIOIDES (DC.) Benth. ex Bak. in Mart. Fl. Bras. vi. pt. 2, 188 (1876); Robinson, Proc. Am. Acad. xlii. 23 (1906).

E. SALICINUM Lam. Encyc. ii. 409 (1786); Robinson, Proc. Am. Acad. liv. 286, 348 (1918). Although credited to Peru originally by Lamarck and by various subsequent authors (including the writer) following his lead, the type of this species presumably came from northern-central Ecuador, through which Joseph de Jussieu, its collector, passed, a region where the plant has since been collected on several occasions so that its presence in some abundance there seems likely. Ecuador had not then been set off as a separate country. To date the writer has found no satisfying record of *E. salicinum* from within the present limits of Peru.

E. stramineum DC. Prod. v. 150 (1836). This species, supposed to have been originally collected in Peru by Haenke, has hitherto been represented, so far as known, by a single branch in the Prodromus Herbarium at Geneva. However, there is a photograph of this type in the Gray Herbarium, and this on careful microscopic study proves to be unquestionably an *Helogyne* probably identical with *H. Weberbaueri* Robinson, Proc. Am. Acad. xlii. 32 (1906). The plant therefore should bear the name **Helogyne straminea** (DC.), comb. nov.

E. VAUTHIERIANUM DC. Prod. v. 159 (1836). By Baker in Mart. Fl. Bras. vi. pt. 2, 305 (1876), this species is said to extend from Peru to Panama and Nicaragua. Baker, l. c., cites certain specimens including Hayes's no. 589 from Panama and one of Tate's from Nicaragua. These are subsequently cited by Hemsley, Biol. Cent.-Am. Bot. ii. 102 (1881), under *E. vitalba*[*e*] DC. As *E. vitalbae* is a species well known and widely distributed from Peru to Nicaragua, while *E. Vauthierianum* is decidedly a plant of Atlantic Brazil, by no one else recorded in the Andean countries, there can be no doubt that Baker's note was intended not for *E. Vauthierianum* but for *E. vitalbae*, as confirmed by Hemsley's subsequent placing of the same Central American exsiccatae.

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A STEP FORWARD IN THE METHODOLOGY OF NATURAL SCIENCE (AN INTRODUCTION TO: THE FUNCTIONAL RELATION OF ONE VARIABLE TO EACH OF A NUMBER OF CORRELATED VARIABLES DETERMINED BY A METHOD OF SUCCESSIVE APPROXIMATION TO GROUP AVERAGES. BY GEORGE F. McEWEN AND ELLIS L. MICHAEL).

BY WM. E. RITTER.

A STEP FORWARD IN THE METHODOLOGY OF NATURAL SCIENCE (AN INTRODUCTION TO: THE FUNCTIONAL RELATION OF ONE VARIABLE TO EACH OF A NUMBER OF CORRELATED VARIABLES DETERMINED BY A METHOD OF SUCCESSIVE APPROXIMATION TO GROUP AVERAGES. BY GEORGE F. McEWEN AND ELLIS L. MICHAEL).

By Wm. E. Ritter.

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Modern biology, as the phrase is generally understood, is a development of laboratory and experimental methods. This development has been unprecedentedly rapid and rich. No one can deny this. But neither can any one fail to see, if he faces the situation squarely, that such development has rather narrow and wholly insurmountable limitations.

Only a relatively small part of all the phenomena of living nature can be brought into the confines of the laboratory or by any means whatever subjected to control. The sciences of organic nature, botany and zoology, are in like case with those of inorganic nature, astronomy, geology, physical geography, meteorology, etc., as regards controllability. This is only an illustrative way of expressing the general truth that every science is able to study the phenomena of its province to only a relatively limited extent in the laboratory, or by experiment in the manipulative sense. As to the overwhelmingly vaster part of nature, those who would investigate it in very fact must go where it is, as far as this is possible, and where this is impossible must reach it by such indirect means as may be devised. This is so patent a truth as hardly to need mention: if the astronomer would investigate the stars of the southern sky he must go to the southern hemisphere; that is, must go where he can see those stars, and must study them as he finds them, not as he might wish to by manipulating them in a laboratory or on an experimental plot of the heavens. Similar conditions and limitations are imposed upon the biologist. If he is to study the starfishes of the southern hemisphere, he must go to oceans in that part of the earth and resort to such means as he can to find and examine the creatures where nature has placed them. True, the biologist has one great advantage over the astronomer: not only can he actually get starfishes into his hands, but he can take many of them — or the cadavers of them — home with him. In a word, the biologist has the advantage of being able to study in his laboratory, and by experimentation, the bodies themselves which are the subject-matter of his science.

These general reflections lead to a still more general reflection on the character of the various sciences, which may be introduced by the question: How is it that physics and chemistry are so largely sciences of the laboratory and of experiment as to make them always stand as types of the experimental sciences? The reply is that these sciences are not natural sciences in the full sense; that is, in the sense of dealing, each in itself, with a delimited province of nature. They are sciences which concern themselves with certain elements and attributes of *all* nature, but not exhaustively with *any portion* of nature. Especially they do not deal with forms and changes in the time series which all natural bodies undergo. They are not natural *history* sciences. Gravitation is one and the same to the physicist whether manifested by a human body or an iceberg. Light is light, so far as fundamentals go, whether its source be a lighthouse, a firefly, or a sun. Similarly, it is all the same to the chemist whether his sample of potassium, provided it is pure, is extracted from a kelp plant or a crystal of feldspar.

On the other hand, the several natural history sciences aim to deal exhaustively with all the phenomena presented in their respective domains of nature.

These remarks appertain to such common-places in modern science that the making of them would not be justifiable but for certain implications they bear, which have not received due recognition in the methodology of natural knowledge. The one of these implications which concerns us at present may be expressed thus: The peculiarities of the two groups of science, as indicated, namely, the group characterized by dealing with definitive portions of nature only, and that characterized by dealing with particular attributes only of all bodies, brings it to pass that the two groups supplement and depend upon each other in a more fundamental way than has been fully recognized by the methods actually used in either group. That the natural history sciences can reach full rounding-out only by supplementing their own particular discoveries and methods by those of physics and chemistry has received more recognition than has the fact that physics and chemistry must use the discoveries and methods of the natural history sciences in order to round themselves out. But physics, under such general designations as geo-physics and celestial physics, seems now to be moving rapidly toward a clear perception of its proper relation to, and dependence upon, the natural history sciences. The notable recent achievements in terrestrial magnetism, geodesy, meteorology, oceanography, and stellar distribution and growth, may be specially noted in illustration of this movement.

Although chemistry is considerably behind physics in discovering its interdependence with natural history, astronomical spectroscopy, taxonomic bio-chemistry, and especially the discoveries in radio activity, in so far as these are revealing the evolutional changes and phyletic affinities of chemical substances, are highly suggestive as to what the future may have in store for chemistry. It seems that chemistry has reached a stage in which it recognizes itself as no longer justified in assuming, on the basis of any evidence it possesses, "that the elements of to-day were eons ago the same substances and preserved their properties unaltered."[1] This by itself is an important step toward converting chemistry into a genuinely historical science.

Now comes the kernel of this communication. The drawing into more vital mutual dependence of the two groups of science, the exact sciences, formerly so-characterized, and the natural or descriptive sciences, formerly so-called,[2] might be expected to enrich both groups.

For that is exactly what all natural drawing together does. And expectation is being realized. The paper on method here presented has grown out of the joint labors of a mathematical physicist working at oceanography as a branch of geo-physics, and a systematic zoologist working at the distribution of animals as an aspect of the broader problem of the relation of organisms to their natural environments, the two investigators having been brought together in the enterprise of gaining as much knowledge as possible of the pelagic life of a particular, restricted area of the Pacific Ocean.

Specifically the problem is: Given the requisite taxonomic knowledge of a natural group (an order, say, with its several genera and species) of pelagic animals, and the requisite facts as to the vertical distribution of these animals through diurnal and annual cycles;

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[1] *"Old Age" of Chemical Elements*, by Ingo W. D. Hackh, Science, April 4, 1919, p. 328.

[2] As though a flock of seven wild geese were not physical and the counting of them were not exact; and as though a discharge of electricity between clouds were not natural and could not be or did not need to be, described.

and given further the requisite factual knowledge of the temperature, salinity, light, etc., of the waters inhabited by the animals, how do the observed changes of the several environmental elements operate as casual factors in the distribution of the animals, this operation being inferred from such correlations as may be discovered in the two series of quantities, biologic and oceanographic?

Obviously, the immediate problem is one of applied statistics; that is, of dealing with long numerical series of natural phenomena, which phenomena have been measured.

Obviously, too, the method is one of dealing with phenomena as they occur in nature, as contrasted with the treatment of phenomena which may occur in a laboratory, or under conditions of manual experimentation. Particular attention is called to the fact that this last statement is equivalent to saying that the method is primarily inductive rather than deductive. And attention is called to the further facts that the case is illustrative of the very wide truth that so far as concerns the interpretation of actual nature, both animate and inanimate, laboratory and experimental methods are necessarily deductive for the most part; and that such interpretation can be made inductively only by carrying research into the "field" and putting quantitative determinations on a statistical basis.

Incidentally it may be pointed out that should the method prove practicable and trustworthy, it would be highly useful since there is a wide range of similar problems, many of them exceedingly important. So far as the principle is concerned, its applicability would be to the entire expanse of living nature, because all organisms, man with the rest, are subject to natural environments of some sort, and the very essence of the method is its effort to bring together data pertaining to organisms and their environments thus taken.

Indeed, the method is applicable to very many phenomena of nature outside the organic realm, as to those of the atmosphere, of the land masses, and of the waters of the earth.

THE FUNCTIONAL RELATION OF ONE VARIABLE TO EACH OF A NUMBER OF CORRELATED VARIABLES DETERMINED BY A METHOD OF SUCCESSIVE APPROXIMATION TO GROUP AVERAGES: A CONTRIBUTION TO STATISTICAL METHODS.[1]

BY GEORGE F. MCEWEN AND ELLIS L. MICHAEL.

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1. REMARKS ON METHODS OF ACQUIRING KNOWLEDGE.

One method of acquiring knowledge, the deductive, is to formulate fundamental concepts and principles that are simple but comprehensive, and to attempt to deduce therefrom the sequences and other relationships observed in nature. This implies that nature conforms to a logical system and, consequently, discovery of the few basic elements of the system, together with suitable logical treatment, furnishes a description of the observed phenomena. Considered quantitatively, this method of acquiring knowledge is the classical one of *applied* mathematics.

Another method is the "inductive" or, better, the empirical method. One becomes directly aware of innumerable facts by means of sense perceptions. Many are acquired without particular effort and seem trivial, while special attention is directed toward acquisition of others of more apparent significance. Closely associated with this process of observation is that of description and classification, which makes possible a comparison with the results obtained by other observers. Moreover, classification is the first step toward the important object of determining the uniformities and other relationships exhibited by the mass of facts in question. From definitions of the individual facts, and their classification into groups we pass, by induction, to a description of the observed system as a whole, *i. e.*, to an empirical law Though the observed system is only a fragmentary sample of the "universe" it represents, the empirical laws are just as real, insofar as they describe that sample, as are the individual facts. The gain in simplicity and conciseness is made, of course, at the expense of detail, as is true of any summary. Further, experience shows that, as a rule, there is not in nature a one-to-one correspondence between observations of one kind and those of another kind. A plurality of causes, influences, factors, or whatever one may call them, must, in general, be considered, and their mutual relations taken into account. Considered quantitatively, this method of acquiring knowledge is *statistical* mathematics,[1] the ideal of which is attained when the empirical laws and assignment of a value to each kind of quantity but one serves to determine the latter.

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[1] Presented in abstract before the San Francisco Section of The American Mathematical Society, April 7, 1917.

In former times, when the fundamentals inherent in the deductive method were largely the result of introspection, attention of scholars was directed mainly to these fundamentals and their logical consequences. Initial concepts and deductions from them were regarded as the realities of nature, while direct evidence of the senses was discredited. In spite of the downfall of scholasticism four centuries ago there still is a strong tendency to regard observations as secondary in importance, and even of no importance when they have no apparent bearing upon some dominant theory or fail to fit in with a prevailing practice. Such a tendency does not inhere in the empirical method, which not only yields results as free as possible from personal bias and preconceived opinion, but, when conscientiously applied, affords the only basis for certain knowledge concerning any objective phenomenon.

After the objective phenomena constituting the subject-matter of any particular scientific inquiry have been observed and described with thoroughness and classified into empirical laws and generalizations, it is proper and desirable to employ the deductive method and ascertain the extent to which that particular natural system conforms to a logical one. But, it not infrequently happens that the specialist extends theories that have proved useful in his restricted researches to classes of phenomena with which he is unfamiliar without first making a critical examination as to their applicability. This tendency is manifested especially by the isolated or individual investigator whose attention is necessarily restricted to the limited observations he can make and to relevant data others may have gathered. When confronted with a problem calling for extensive observation in the field he is prone to carry it into the laboratory in an attempt, on the basis of theory and carefully conducted experiments, to reach an explanation of phenomena never observed. The deductive method thus becomes a process of inventing facts to fit theories, instead of theories to fit facts. The empirical method, on the other hand, being concerned primarily with direct observation, often demands that investigation be carried on by an organized group of individuals working in coöperation for the purpose of obtaining as wide a range of relevant observations as possible. Each method has its place in all investigations, and each has its limitations; and it is only by combining all relevant observations with induction and deduction that one can use his full powers of cognition and approach complete solution of any problem in natural science.

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[1] The phrase "mathematical statistics" might have been better, since it is in common use, were it not the prevailing practice, especially among biologists, to base the mathematical reasoning, more or less unconsciously, upon a preconception to which the statistical data treated do not necessarily, or evenly usually, conform, *e. g.*, the Gaussian law of error. Such a practice is essentially an application of the deductive method to statistical problems; statistical mathematics, on the other hand, is strictly empirical since the mathematical logic is based solely upon the statistical data at hand.

2. GENERAL STATEMENT OF PROBLEM AND MODE OF ATTACK.

How can the values of a variable, for example the yield of wheat per acre of a given region, measured at equal intervals of time, say annually, be used for predicting the yield for the ensuing year? The frequency with which the wheat yield has been observed to fall within given limits, divided by the total number of observations, is the empirical probability that the next yield of wheat will fall within those limits. Obviously, the more frequently the wheat yield has been observed to fall within given limits the larger will be this empirical probability, and the greater the total number of observations the nearer will this empirical probability approximate the true probability, *i. e.*, the frequency ratio that would have resulted had the number of observations been infinite. This is one way of making predictions.

Again, suppose observations show that the yield of wheat either increases or decreases on the average with respect to time. or that, within some definite period, there is a cycle or typical variation that is repeated in approximately the same manner in each period. In such cases information regarding the general trend or cycles affords a more satisfactory basis for prediction, *i. e.*, it results in prediction within smaller limits for a given probability than is possible by the simpler frequency method. But, suppose some other phenomenon is also measured, for example, the rainfall during a given season of each year. If it be observed that, in general, a large rainfall is followed by a large yield of wheat, knowledge of the former could also be used to improve prediction of the latter. Still another improvement would be expected if the temperature during the growing season were also measured, and so on.

Prediction, as thus illustrated, implies a lag of the quantity predicted (dependent variable) behind the remaining quantities (independent variables). Considering the case in general this lag may be of any magnitude between any of the selected independent variables and the dependent one, or all may vary simultaneously. But, the problem of determining the empirical relations is the same, and, as more factors are measured and as the number of observations increases, approximation is had to the ideal of precise determination. However, under the most favorable conditions, some deviations between observed and computed values always remain, and these are called accidental or "chance" variations. Even in laboratory experiments, where the idea of artificial control over the independent variables is dominant, it is often necessary, in order to obtain best results, to correct the dependent variable for unavoidable fluctuations due to variables beyond control. In any *natural* problem, however, the factors involved are all necessarily variable and, as a rule, mutually correlated, so that, in any given case, one is confronted with the difficulty of selecting the most important factors, and the necessity of determining the approximate functional relation of the dependent variable to each of the mutually correlated independent ones.

If the functional relation between the variables is known to be approximately linear, or can be made so by introducing suitable functions, the usual method of multiple correlation may be applied. Again, in case the form of the functions expressing the relation of the dependent to each independent variable is known, the method of least squares or the method of moments may be used to determine the numerical values of the constants appearing in the mathematical expressions. But in many, if not most, cases in practice the forms of the functions are quite unknown and must be determined solely from the data at hand.

The object of this investigation is to devise a *general* method of obtaining the relation between a dependent variable and each of the mutually correlated independent ones without being compelled to employ an assumed or predetermined mathematical function. This is accomplished by applying to the observed values of the dependent variable successive corrections based upon each value of all the independent variables. In this way is obtained a series of averages of the dependent variable corresponding to a series of averages of each one of the independent variables in turn, and corrected to a constant value of each of the remaining ones. Perhaps this will be more intelligible if stated in the concrete terms of the wheat problem. In this particular case the method is that of obtaining a series of averages of the wheat yield, corrected to a constant rainfall, corresponding to a series of temperature averages; and a similar series of averages of the wheat yield, corrected to a constant temperature, corresponding to a series of rainfall averages. The averages thus obtained define, approximately, the functional relation desired.

The idea of defining a function by means of a series of corresponding values of dependent and independent variables is utilized in certain problems of higher mathematics (Fredholm, 1900; 1903; Bôcher, 1909). But, in pure mathematics, it is possible to pass to the limit and obtain an infinite series of pairs of corresponding values, which defines the functional relation uniquely. In objective science this is impossible, and, although various well-known methods of interpolation are available for approximating thereto, one is between the two horns of a dilemma. It is obvious that the effect of accidental variations is reduced to a minimum for any given number of observations when the number entering into each average is a maximum, but this also reduces to a minimum the number of averages upon which definition of the functional relation depends. Stated otherwise, the greater the number of averages for a given number of observations the more precisely will the functional relation be determined; but, owing to the larger effect of accidental variations, the less reliable will be the result. One must therefore use his judgment in classifying the data, and should test the reliability of the results.

In closing this section, it may be of interest to mention how the method happened to be devised. It had its origin in our collaboration on problems concerning the quantitative relation between variations in the number of certain marine organisms and fluctuations in the elements of their environmental complexes. Attempts to eliminate the effects of correlation between the environmental elements by the method of multiple correlation and that of least squares, combined with various subsidiary expedients, proved highly unsatisfactory. The reason is that, at the outset of the mathematical reasoning, assumptions of either a linear or some other definite type of regression, or of the functional form of the observation equations must be introduced for which no justification is afforded by the data themselves. After a fairly exhaustive study of the literature, which failed to provide any practicable and rigorous way of handling such problems, we were led to devise one which culminated, in part, in this method of successive approximation to group averages. Although the central idea is the product of our collaboration, the mathematical demonstration and the practical process of making the computations are primarily due to the senior author. Furthermore, the particular problems whose study led to developing this method are too complex to afford suitable means of illustration. For this reason the simpler problem of the relation between temperature, precipitation, and yield of wheat in South Dakota is used, a study of which, by means of multiple linear correlation, has been published by Blair (1918).

The mathematical demonstration, while close and rigorous, is neither abstruse nor difficult. In the case when variability within the group is neglected (section 3A) it involves nothing beyond the elements of algebra. But, in the case when this variability is taken into account (section 3B) the demonstration also presupposes knowledge of linear regression, so that some readers may prefer to follow through the concrete process of computation given in section 4 illustrating the first case, before turning attention to the analytic demonstration in the second case.

3. MATHEMATICAL DEMONSTRATION:

A. THE CASE WHEN VARIABILITY WITHIN THE GROUP IS NEGLECTED.

When the *change* in the dependent variable, w, corresponding to a given change in one independent variable, say x, is negligibly influenced by the magnitude of the constant values to which the remaining independent variables y, z, etc., are reduced (see p. 128), the expression for w takes the form

$$w = f_1(x) + f_2(y) + f_3(z) + \dots \tag{1}$$

where f_1, f_2, f_3, etc., denote the unknown functional relations of w to x, y, z, etc. The problem is to determine each of these unknown functions from the numerical data.

In laboratory experiments it is usually possible so to control the independent variables as to hold all but one, say x, at constant values, $\mathbf{y}$, $\mathbf{z}$, etc. In such instances the difference between any two values of the series $f_1(\mathbf{x}_1)$, $f_1(\mathbf{x}_2)$, $f_1(\mathbf{x}_3)$, etc., can be readily found, where $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, etc., are averages of x in each of a series of groups formed in succession from the values of x arranged in ascending order of magnitude. The purpose of taking averages is to eliminate so far as possible effects of accidental variations due to variables beyond control. The corresponding averages, $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, etc., of the observed values of w, therefore are

$$\begin{aligned} \mathbf{w}_1 &= f_1(\mathbf{x}_1) + M \\ \mathbf{w}_2 &= f_1(\mathbf{x}_2) + M \\ \mathbf{w}_3 &= f_1(\mathbf{x}_3) + M \\ &\dots\dots\dots\dots \\ \mathbf{w}_n &= f_1(\mathbf{x}_n) + M \end{aligned} \tag{2}$$

where

$$M = f_2(\mathbf{y}) + f_3(\mathbf{z}) + \dots \tag{3}$$

is a constant since y, z, etc., are constant. Similarly, the relation of w to y, w to z, etc., may be thus determined.

But it is only under the *artificial* conditions of the laboratory that this simple way of determining the unknown functions is valid; and, even so, there is no guarantee that the same functional relations will hold good under *natural* conditions. In nature one is limited to observing what is actually taking place; all influences are beyond control; all vary simultaneously; and all are more or less correlated. Differences between successive values of $\mathbf{w}$ (equations 2) are therefore not due, in general, to the fluctuation in x alone, but also to fluctuations in the remaining variables, y, z, etc. Moreover, the effect of these remaining variables often is large enough to produce serious errors in the relations indicated by this simple mode of procedure.

Such errors must be eliminated. To accomplish this, say in the relation of w to x, corrections are computed for the purpose of reducing each value of w in the (w, x) series as nearly as possible to the value it would have had if y, z, etc., had constant values arbitrarily chosen. Regarding the average of the original values in each group of the (w, x) series as a first approximation, the second is obtained by applying to each value of w, corrections derived from the relation of w to y, w to z, etc., indicated by the original series of group averages,[1] $(\mathbf{w}, \mathbf{y})$, $(\mathbf{w}, \mathbf{z})$, etc. A second approximation to the relation of w to y is then obtained by introducing corrections based upon the second approximation in the (w, x) series and first approximations in the remaining series. The process is thus continued until second approximations are obtained to the relation of w to each of the remaining independent variables. By means of these second approximations in the (w, y), (w, z), etc., series a third approximation to the functional relation of w to x is obtained, and so on. It seems reasonable that such successive approximations would result in convergence to values of w corresponding to a variation of only one independent variable at a time. This is confirmed by the following analytical demonstration, which also yields a practicable method of making and checking the computations.

For clearness the analytical demonstration is given for the special case of three independent variables and three groups of each, but the same reasoning applies to the general case of any number of variables and groups. Arrange the values of the independent variable x in ascending order, segregate them into three groups, and let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, be the average of x in the three groups respectively. Let $\mathbf{A}^i$, $\mathbf{B}^i$, and $\mathbf{C}^i$ be the corresponding original averages of the dependent variable w, and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be the required values of these averages corresponding to constant values of the two independent variables, y and z. Denote the number of entries per group by N_1, N_2, and N_3. This notation, together with that for the y and z variables, is presented in tabular form as follows:

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[1] All values of the independent variable, y, in any one y-group are thus assumed to equal the average within that group, and similarly for the remaining independent variables, z, etc. To state it otherwise, in correcting for the effects of any variable, y, each value of w in each y-group is assumed to correspond to the average value of y within that group (see p. 105).

TABLE I.

General Notation Employed.

$\mathbf{A}^i\ \mathbf{B}^i\ \mathbf{C}^i$	$\mathbf{D}^i\ \mathbf{E}^i\ \mathbf{F}^i$	$\mathbf{G}^i\ \mathbf{H}^i\ \mathbf{I}^i$	Original group averages of dependent variable, w.
$\mathbf{A}\ \ \mathbf{B}\ \ \mathbf{C}$	$\mathbf{D}\ \ \mathbf{E}\ \ \mathbf{F}$	$\mathbf{G}\ \ \mathbf{H}\ \ \mathbf{I}$	Required group averages of dependent variable, w.
$\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3$	$\mathbf{y}_4\ \ \mathbf{y}_5\ \ \mathbf{y}_6$	$\mathbf{z}_7\ \ \mathbf{z}_8\ \ \mathbf{z}_9$	Group averages of the values of the independent variable arranged in ascending order.
$N_1\ N_2\ N_3$	$N_4\ N_5\ N_6$	$N_7\ N_8\ N_9$	Number of entries per group.

In determining the relation of w to x, let all values of w be corrected for the variable y to its middle value $\mathbf{y}_5$, and for the variable z to its middle value $\mathbf{z}_8$. Likewise, in determining the relation of w to y, let all values of w be corrected for x to its middle value $\mathbf{x}_2$, and for z to its middle value $\mathbf{z}_8$. Similarly, in determining the relation of w to z, let all values of w be corrected for x to $\mathbf{x}_2$ and for y to $\mathbf{y}_5$. Correction to these arbitrarily chosen "standard" values requires the following differences between the group averages:

$$\left.\begin{array}{l} \mathbf{B}^i - \mathbf{A}^i = a^i,\ \mathbf{B}^i - \mathbf{C}^i = c^i \\ \mathbf{B} - \mathbf{A} = a,\ \mathbf{B} - \mathbf{C} = c \end{array}\right\} \qquad (4)$$

$$\left.\begin{array}{l} \mathbf{E}^i - \mathbf{D}^i = d^i,\ \mathbf{E} - \mathbf{F}^i = f^i \\ \mathbf{E} - \mathbf{D} = d,\ \mathbf{E} - \mathbf{F} = f \end{array}\right\} \qquad (5)$$

$$\left.\begin{array}{l} \mathbf{H}^i - \mathbf{G}^i = g^i,\ \mathbf{H}^i - \mathbf{I}^i = i^i \\ \mathbf{H} - \mathbf{G} = g,\ \mathbf{H} - \mathbf{I} = i \end{array}\right\} \qquad (6)$$

Denote by A^i any observed value of w in the (w, x_1) group; by B^i any observed value of w in the (w, x_2) group, and so on to I^i for any observed value of w in the (w, z_9) group. Then one particular value of A^i is identical with some particular value of w in the (w, y) series and also in the (w, z) series, *i. e.*, it is identical with a particular value of D^i, E^i, or F^i, and also of G^i, H^i, or I^i. Suppose it is in the (w, x_1),

(w, y_4), and (w, z_8) groups, which, for convenience, will be referred to as groups 1, 4, and 8. The correction $\mathbf{E} - \mathbf{D} = d$ (see equations 5) must be added to reduce this particular value of w to that corresponding to the standard value $\mathbf{y}_5$ of y, but no correction need be applied for z, since it is in group 8 which was selected as the standard. If each value of w in group 1 be corrected in this way, the mean of the corrected values is, by definition, equal to the required value $\mathbf{A}$. A similar procedure with respect to all the other groups gives the remaining required values $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{I}$. Evidently, this is equivalent to adding to the average of the observed values A^i the average of all the corrections. That is

$$\begin{aligned}\mathbf{A} &= \frac{\Sigma A^i}{N_1} + \frac{1}{N_1}(n_{14}\,d + n_{16}\,f + n_{17}\,g + n_{19}\,i) \\ &= \mathbf{A}^i + \frac{1}{N_1}(n_{14}\,d + n_{16}\,f + n_{17}\,g + n_{19}\,i)\end{aligned} \tag{7}$$

where

n_{14} = number of observations common to groups 1 and 4
n_{16} = " " " " " " 1 and 6
n_{17} = " " " " " " 1 and 7
n_{19} = " " " " " " 1 and 9

In the same manner equations (8) to (15) are obtained.

$$\mathbf{B} = \mathbf{B}^i + \frac{1}{N_2}(n_{24}d + n_{26}f + n_{27}g + n_{29}i) \tag{8}$$

$$\mathbf{C} = \mathbf{C}^i + \frac{1}{N_3}(n_{34}d + n_{36}f + n_{37}g + n_{39}i) \tag{9}$$

$$\mathbf{D} = \mathbf{D}^i + \frac{1}{N_4}(n_{41}a + n_{43}c + n_{47}g + n_{49}i) \tag{10}$$

$$\mathbf{E} = \mathbf{E}^i + \frac{1}{N_5}(n_{51}a + n_{53}c + n_{57}g + n_{59}i) \tag{11}$$

$$\mathbf{F} = \mathbf{F}^i + \frac{1}{N_6}(n_{61}a + n_{63}c + n_{67}g + n_{69}i) \tag{12}$$

$$\mathbf{G} = \mathbf{G}^i + \frac{1}{N_7}(n_{71}a + n_{73}c + n_{74}d + n_{76}f) \tag{13}$$

$$\mathbf{H} = \mathbf{H}^i + \frac{1}{N_8}(n_{81}a + n_{83}c + n_{84}d + n_{86}f) \tag{14}$$

TABLE 2.

PROCESS OF SOLUTION BY SUCCESSIVE APPROXIMATION.

Substitute	in equations	and obtain			Li
Second approximation determined directly					
$d^{i}\quad f^{i}\quad g^{i}\quad i^{i}$	7, 8, and 9	$\Delta\,\mathbf{A}^{ii}\,\Delta\,\mathbf{B}^{ii}\,\Delta\,\mathbf{C}^{ii}$	$\mathbf{A}^{ii}\quad\mathbf{B}^{ii}\quad\mathbf{C}^{ii}$	$a^{ii}\,c^{ii}\,\Delta^1a^{ii}\,\Delta^1c^{ii}$	2
$a^{ii}\quad c^{ii}\quad g^{i}\quad i^{i}$	10, 11, and 12	$\Delta\,\mathbf{D}^{ii}\,\Delta\,\mathbf{E}^{ii}\,\Delta\,\mathbf{F}^{ii}$	$\mathbf{D}^{ii}\quad\mathbf{E}^{ii}\quad\mathbf{F}^{ii}$	$d^{ii}\,f^{ii}\,\Delta^1d^{ii}\,\Delta^1f^{ii}$	3
$a^{ii}\quad c^{ii}\quad d^{ii}\quad f^{ii}$	13, 14, and 15	$\Delta\,\mathbf{G}^{ii}\,\Delta\,\mathbf{H}^{ii}\,\Delta\,\mathbf{I}^{ii}$	$\mathbf{G}^{ii}\quad\mathbf{H}^{ii}\quad\mathbf{I}^{ii}$	$g^{ii}\,i^{ii}\,\Delta^1g^{ii}\,\Delta^1i^{ii}$	
Third approximation determined directly and checked by first differences Δ^1					
$d^{ii}\quad f^{ii}\quad g^{ii}\quad i^{ii}$	7, 8, and 9	$\Delta\,\mathbf{A}^{iii}\Delta\,\mathbf{B}^{iii}\Delta\,\mathbf{C}^{iii}$	$\mathbf{A}^{iii}\quad\mathbf{B}^{iii}\quad\mathbf{C}^{iii}$	$a^{iii}c^{iii}\Delta^1a^{iii}\Delta^1c^{iii}$	4
$\Delta^1d^{ii}\,\Delta^1f^{ii}\,\Delta^1g^{ii}\,\Delta^1i^{ii}$	7, 8, and 9	$\Delta^1\mathbf{A}^{iii}\Delta^1\mathbf{B}^{iii}\Delta^1\mathbf{C}^{iii}$	$\Delta\,\mathbf{A}^{iii}\Delta\,\mathbf{B}^{iii}\Delta\,\mathbf{C}^{iii}$	$\Delta^1a^{iii}\Delta^1c^{iii}$	5
$a^{iii}\quad c^{iii}\quad g^{ii}\quad i^{ii}$	10, 11, and 12	$\Delta\,\mathbf{D}^{iii}\Delta\,\mathbf{E}^{iii}\Delta\,\mathbf{F}^{iii}$	$\mathbf{D}^{iii}\quad\mathbf{E}^{iii}\quad\mathbf{F}^{iii}$	$d^{iii}f^{iii}\Delta^1d^{iii}\Delta^1f^{iii}$	6
$\Delta^1a^{iii}\Delta^1c^{iii}\Delta^1g^{ii}\,\Delta^1i^{ii}$	10, 11, and 12	$\Delta^1\mathbf{D}^{iii}\Delta^1\mathbf{E}^{iii}\Delta^1\mathbf{F}^{iii}$	$\Delta\,\mathbf{D}^{iii}\Delta\,\mathbf{E}^{iii}\Delta\,\mathbf{F}^{iii}$	$\Delta^1d^{iii}\Delta^1f^{iii}$	7
$a^{iii}\quad c^{iii}\quad d^{iii}\quad f^{iii}$	13, 14, and 15	$\Delta\,\mathbf{G}^{iii}\Delta\,\mathbf{H}^{iii}\Delta\,\mathbf{I}^{iii}$	$\mathbf{G}^{iii}\quad\mathbf{H}^{iii}\quad\mathbf{I}^{iii}$	$g^{iii}i^{iii}\Delta^1g^{ii}\Delta^1i^{iii}$	8
$\Delta^1a^{iii}\Delta^1c^{iii}\Delta^1d^{iii}\Delta^1f^{iii}$	13, 14, and 15	$\Delta^1\mathbf{G}^{iii}\Delta^1\mathbf{H}^{iii}\Delta^1\mathbf{I}^{iii}$	$\Delta\,\mathbf{G}^{iii}\Delta\,\mathbf{H}^{iii}\Delta\,\mathbf{I}^{iii}$	$\Delta^1g^{iii}\Delta^1i^{iii}$	9
Fourth approximation determined by first differences and checked by second differences Δ^2					
$\Delta^1a^{iii}\Delta^1f^{iii}\Delta^1g^{iii}\Delta^1i^{iii}$	7, 8, and 9	$\Delta^1\mathbf{A}^{iv}\Delta^1\mathbf{B}^{iv}\Delta^1\mathbf{C}^{iv}$	$\Delta\,\mathbf{A}^{iv}\Delta\,\mathbf{B}^{iv}\Delta\,\mathbf{C}^{iv}$	$\Delta^1a^{iv}\Delta^1c^{iv}$	10
$\Delta^2d^{iii}\Delta^2f^{iii}\Delta^2g^{iii}\Delta^2i^{iii}$	7, 8, and 9	$\Delta^2\mathbf{A}^{iv}\Delta^2\mathbf{B}^{iv}\Delta^2\mathbf{C}^{iv}$	$\Delta^1\mathbf{A}^{iv}\Delta^1\mathbf{B}^{iv}\Delta^1\mathbf{C}^{iv}$	$\Delta^1a^{iv}\Delta^1c^{iv}\Delta^2a^{iv}\Delta^2c^{iv}$	11
$\Delta^1a^{iv}\Delta^1c^{iv}\Delta^1g^{iii}\Delta^1i^{iii}$	10, 11, and 12	$\Delta^1\mathbf{D}^{iv}\Delta^1\mathbf{E}^{iv}\Delta^1\mathbf{F}^{iv}$	$\Delta\,\mathbf{D}^{iv}\Delta\,\mathbf{E}^{iv}\Delta\,\mathbf{F}^{iv}$	$\Delta^1d^{iv}\Delta^1f^{iv}$	12
$\Delta^2a^{iv}\Delta^2c^{iv}\Delta^2g^{iii}\Delta^2i^{iii}$	10, 11, and 12	$\Delta^2\mathbf{D}^{iv}\Delta^2\mathbf{E}^{iv}\Delta^2\mathbf{F}^{iv}$	$\Delta^1\mathbf{D}^{iv}\Delta^1\mathbf{E}^{iv}\Delta^1\mathbf{F}^{iv}$	$\Delta^1d^{iv}\Delta^1f^{iv}\Delta^2d^{iv}\Delta^2f^{iv}$	13
$\Delta^1a^{iv}\Delta^1c^{iv}\Delta^1d^{iv}\Delta^if^{iv}$	13, 14, and 15	$\mathbf{A}^1\mathbf{G}^{iv}\Delta^1\mathbf{H}^{iv}\Delta^1\mathbf{I}^{iv}$	$\Delta\,\mathbf{G}^{iv}\Delta\,\mathbf{H}^{iv}\Delta\,\mathbf{I}^{iv}$	$\Delta^1g^{iv}\Delta^1i^{iv}$	14
$\Delta^2a^{iv}\Delta^2c^{iv}\Delta^2d^{iv}\Delta^2f^{iv}$	13, 14, and 15	$\Delta^2\mathbf{G}^{iv}\Delta^2\mathbf{H}^{iv}\Delta^2\mathbf{I}^{iv}$	$\Delta^1\mathbf{G}^{iv}\Delta^1\mathbf{H}^{iv}\Delta^1\mathbf{I}^{iv}$	$\Delta^1g^{iv}\Delta^1i^{iv}\,\Delta^2g^{iv}\Delta^2i^{iv}$	15

Supplementary explanations

$$\mathbf{A}^{ii}=\mathbf{A}^{i}+\Delta\mathbf{A}^{ii}$$
$$\mathbf{A}^{iii}=\mathbf{A}^{i}+\Delta\mathbf{A}^{iii}$$
$$\mathbf{A}^{iv}=\mathbf{A}^{i}+\Delta\mathbf{A}^{iv}$$
etc.

$$\mathbf{B}a^{ii}=\mathbf{B}^{ii}-\mathbf{A}^{ii}=\mathbf{B}^{i}-\mathbf{A}^{i}+\Delta\mathbf{B}^{ii}-\Delta\mathbf{A}^{ii}$$
$$=a^{i}+\Delta\mathbf{B}^{ii}-\Delta\mathbf{A}^{ii}$$
$$a^{iii}=a^{i}+\Delta\mathbf{B}^{iii}-\Delta\mathbf{A}^{iii}$$
$$a^{iv}=a^{i}+\Delta\mathbf{B}^{iv}-\Delta\mathbf{A}^{iv}$$
etc.

$$d^{ii}-d^{i}=\Delta^1d^{ii}=\Delta^1\mathbf{E}^{ii}-\Delta^1\mathbf{D}^{ii}$$
$$d^{iii}-d^{ii}=\Delta^1d^{iii}=\Delta^1\mathbf{E}^{iii}-\Delta^1\mathbf{D}^{iii}$$
$$d^{iv}-d^{iii}=\Delta^1d^{iv}=\Delta^1\mathbf{E}^{iv}-\Delta^1\mathbf{D}^{iv}$$
etc.

$$\Delta^1\mathbf{A}^{iii}=\Delta\mathbf{A}^{iii}-\Delta\mathbf{A}^{ii}$$
$$\Delta^1\mathbf{A}^{ii}=\Delta\mathbf{A}^{iv}-\Delta\mathbf{A}^{iii}$$
etc.

$${}^1d^{iii}-\Delta^1d^{ii}=\Delta^2d^{iii}=\Delta^2\mathbf{E}^{iii}-\Delta^2\mathbf{D}^{iii}$$
$${}^1d^{iv}-\Delta^1d^{iii}=\Delta^2d^{iv}=\Delta^2\mathbf{E}^{iv}-\Delta^2\mathbf{D}^{iv}$$
etc.

$$\Delta^2\mathbf{A}^{iv}=\Delta^1\mathbf{A}^{iv}-\Delta^1\mathbf{A}^{iii}$$
$$\Delta^2\mathbf{A}^{v}=\Delta^1\mathbf{A}^{v}-\Delta^1\mathbf{A}^{iv}$$
etc.

$$\mathbf{I} = \mathbf{I}^i + \frac{1}{N_9}(n_{91}a + n_{93}c + n_{94}d + n_{96}f) \qquad (15)$$

where $n_{41} = n_{14}$, $n_{34} = n_{43}$, etc.

These nine equations, together with equations (4), (5), and (6) defining the quantities a, c, d, f, g, and i, determine the nine unknowns, **A**, **B**, **C**, etc. They can be solved simultaneously, but as a rule, labor is saved by employing a method of successive approximation, the details of which are given in table 2.

If the process of successive approximation be continued, as indicated by lines 1, 2, 3, and 4, 6, 8, and results in convergence [1] to definite limiting values, these values will evidently satisfy equations (4) to (15). In the third approximation the procedure indicated by lines 5, 7, and 9, involving first differences, affords a numerical check on the computation of $\Delta\mathbf{A}^{iii}$, $\Delta\mathbf{B}^{iii}$, etc., and Δ^1a^{iii}, Δ^1c^{iii}, etc., of lines 4, 6, and 8. It is possible to continue checking each computation in this way until convergence is attained. But, beginning with the fourth approximation a further saving of labor is effected by computing first differences (lines 10, 12, and 14) and checking these results by second differences (lines 11, 13, and 15), since all differences converge to zero. For a numerical illustration see page 113.

B. THE CASE WHEN VARIABILITY WITHIN THE GROUP IS TAKEN INTO ACCOUNT.

As stated on page 102, the variability, for example in a (w, x) group, due to the range in value of x in that group, and the correlation between w and x in that group, is neglected in the foregoing solution. Justification for this neglect depends upon the nature and magnitude of the variability, which, in turn, depends upon the range in value of the independent variable; magnitude of the change in the dependent variable due to a given change in the independent one; degree of correlation between the independent variables; and number of groups. When a large amount of data is at hand it is usually possible to classify it into a correspondingly large number of groups with respect to each independent variable, in which event little gain in accuracy is made by taking account of variability within the group. But, when the number of groups is small, this is not generally true. In such cases variability within the group may be significantly decreased by introducing corrections based upon an assumed linear regression of the dependent on the independent variable, *e. g.*, a linear regression of w on x in a (w, x) group. By this means each value of the dependent variable may be approximately reduced to what it would have been had the independent variable remained at its constant average value, *e. g.* each value of w in the (w, x_1) group may be reduced to a value corresponding to $x = \mathbf{x}_1$. If the central idea of group averages has been made clear it will be obvious that the error introduced by an assumption of linear regression *within the group* is negligible. Accordingly, after applying this correction, the outstanding variability is legitimately attributed to "chance" and, after selecting the independent variables, can be further reduced only by increasing the number of groups and the number of observations in each.

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[1] No general criterion for convergence has been worked out, but it evidently depends upon the closeness of correlation between the independent variables. Of the ten problems to which the method has thus far been applied, ranging from the relation between dew point, humidity, and minimum air temperatures to the relation between body length, tail length, and foot length in mice, the greatest number of approximations required was fifteen.

For clearness, the analytical demonstration is given for the special case of two independent variables and three groups of each, but the same reasoning applies to the general case of any number of variables and groups, as well as to the case in which regressions are run in some of the groups and not in others. The notation is presented in table 3.

TABLE 3.

GENERAL NOTATION EMPLOYED.

$\mathbf{A}^i$ $\mathbf{B}^i$ $\mathbf{C}^i$	$\mathbf{D}^i$ $\mathbf{E}^i$ $\mathbf{F}^i$	Original group averages of dependent variable, w.
$\mathbf{A}$ $\mathbf{B}$ $\mathbf{C}$	$\mathbf{D}$ $\mathbf{E}$ $\mathbf{F}$	Required group averages of dependent variable, w.
$\mathbf{x}_1$ $\mathbf{x}_2$ $\mathbf{x}_3$	$\mathbf{y}_4$ $\mathbf{y}_5$ $\mathbf{y}_6$	Group averages of the values of the independent variable arranged in ascending order.
N_1 N_2 N_3	N_4 N_5 N_6	Number of entries per group.
R_1^i R_2^i R_3^i	R_4^i R_5^i R_6^i	Original regression coefficients.
R_1 R_2 R_3	R_4 R_5 R_6	Required regression coefficients.

The following equations, except for the terms involving R, are derived as before:

$$\mathbf{A} = \mathbf{A}^i + \frac{1}{N_1}\{n_{14}d + n_{16}f\} - \frac{1}{N_1}\{R_4\Sigma(y - \mathbf{y}_4)_1 + R_5\Sigma(y - \mathbf{y}_5)_1 + R_6\Sigma(y - \mathbf{y}_6)_1\} \tag{16}$$

$$\mathbf{B} = \mathbf{B}^i + \frac{1}{N_2}\{n_{24}d + n_{26}f\} - \frac{1}{N_2}\{R_4\Sigma(y - \mathbf{y}_4)_2 + R_5\Sigma(y - \mathbf{y}_5)_2 + R_6\Sigma(y - \mathbf{y}_6)_2\} \tag{17}$$

$$\mathbf{C} = \mathbf{C}^i + \frac{1}{N_3}\{n_{34}d + n_{36}f\} - \frac{1}{N_3}\{R_4\Sigma(y - \mathbf{y}_4)_3 + R_5\Sigma(y - \mathbf{y}_5)_3 + R_6\Sigma(y - \mathbf{y}_6)_3\} \tag{18}$$

$$\mathbf{D} = \mathbf{D}^i + \frac{1}{N_4}\{n_{41}a + n_{43}c\} - \frac{1}{N_4}\{R_1\Sigma(x - \mathbf{x}_1)_4 + R_2\Sigma(x - \mathbf{x}_2)_4 + R_3\Sigma(x - \mathbf{x}_3)_4\} \tag{19}$$

$$\mathbf{E} = \mathbf{E}^i + \frac{1}{N_5}\{n_{51}a + n_{53}c\} - \frac{1}{N_5}\{R_1\Sigma(x - \mathbf{x}_1)_5 + R_2\Sigma(x - \mathbf{x}_2)_5 + R_3\Sigma(x - \mathbf{x}_3)_5\} \tag{20}$$

$$\mathbf{F} = \mathbf{F}^i + \frac{1}{N_6}\{n_{61}a + n_{63}c\} - \frac{1}{N_6}\{R_1\Sigma(x - \mathbf{x}_1)_6 + R_2\Sigma(x - \mathbf{x}_2)_6 + R_3\Sigma(x - \mathbf{x}_3)_6\} \tag{21}$$

where,

n_{14} = number of observations of w common to group 1 and 4
n_{16} = " " " " " " " " 1 and 6

and so on, and where

$$\mathbf{B} - \mathbf{A} = a,\ \mathbf{B} - \mathbf{C} = c,\ \mathbf{E} - \mathbf{D} = d, \text{ and } \mathbf{E} - \mathbf{F} = f.$$

In order to correct for the position of w with respect to y in groups 1, 2, and 3, and for w with respect to x in groups 4, 5, and 6, the terms involving the regressions R are added. They are readily derived. Consider, for example, the values of w common to groups 1 and 4 (equation 16): $\frac{n_{14}\,d}{N_1}$ is the correction to $\mathbf{A}^i$ on the assumption that each value of w corresponds to the average value of y, *i. e.*, to $\mathbf{y}_4$. But, when the variability in y is taken account of, the difference between any particular value of y and its average value $(y - \mathbf{y}_4)$ must be multiplied by the coefficient expressing the regression of w on y in group 4, *i. e.*, by R_4, and subtracted. Hence $- R_4\Sigma(y - \mathbf{y}_4)_1$ is the total correction due to the position in group 4 of the n_{14} values of w common to groups 1 and 4, where the subscript outside of the bracket signifies that only the values of y in group 4 corresponding to values of w in group 1 are to be summed. Similarly $- R_5\Sigma(y - \mathbf{y}_5)_1$, and $- R_6\Sigma(y - \mathbf{y}_6)_1$ are the total corrections due to the n_{15} values of w common to groups 1 and 5, and the n_{16} values of w common to groups 1 and 6. Finally, the sum of all these corrections divided by their number, $N_1 = n_{14} + n_{15} + n_{16}$, gives the correction to the average $\mathbf{A}^i$, which is the second term in brackets of equation (16). In the same way the corresponding expressions of equations (17) to (21) are obtained.

Introduction of the six unknown regression coefficients, however, requires six additional equations, which are readily obtained. Any regression coefficient, for example R_1, is by definition

$$R_1 = \frac{\Sigma(A - \mathbf{A})\,(x - \mathbf{x}_1)}{\Sigma(x - \mathbf{x}_1)^2} \tag{22}$$

where A denotes that each observed value of the dependent variable in group 1 is corrected to a constant value of the remaining independent variables. Since, in this demonstration, y is the only remaining independent variable, this correction for any particular value of w common to groups 1 and 4 is evidently given by $d - R_4(y - \mathbf{y}_4)_1$ for the corresponding value of y, whence the part of the numerator of equation (22) due to all values of w common to groups 1 and 4 is

$$\Sigma\{A^i + [d - R_4(y - \mathbf{y}_4)_1] - \mathbf{A}\}_4\,\{x - \mathbf{x}_1\}_4.$$

Likewise the part of the numerator due to all values of w common to groups 1 and 5, and also to groups 1 and 6 are respectively

$$\Sigma\{A^i + [0 - R_5\,(y - \mathbf{y}_5)_1] - \mathbf{A}\}_5\,\{x - \mathbf{x}_1\}_5 \quad \text{and}$$
$$\Sigma\{A^i + [f - R_6\,(y - \mathbf{y}_6)_1] - \mathbf{A}\}_6\,\{x - \mathbf{x}_1\}_6.$$

Rearranging and combining these three terms of the numerator, equation (22) becomes

$$R_1 = \frac{\begin{array}{r}\{\Sigma(A^i - \mathbf{A})\ (x - \mathbf{x}_1) + d\Sigma(x - \mathbf{x}_1)_4 + f\Sigma(x - \mathbf{x}_1)_6 \\ - R_4\Sigma(y - \mathbf{y}_4)_1\ (x - \mathbf{x}_1)_4 - R_5\Sigma(y - \mathbf{y}_5)_1(x - \mathbf{x}_1)_5 \\ - R_6\Sigma(y - \mathbf{y}_6)_1(x - \mathbf{x}_1)_6\}\end{array}}{\Sigma(x - \mathbf{x}_1)^2} \tag{23}$$

Similarly equations (24) to (28) are derived

$$R_2 = \{\Sigma(B^i - \mathbf{B})\ (x - \mathbf{x}_2) + d\Sigma(x - \mathbf{x}_2)_4 + f\Sigma(x - \mathbf{x}_2)_6 - R_4\Sigma(y - \mathbf{y}_4)_2$$
$$\frac{(x - \mathbf{x}_2)_4 - R_5\Sigma(y - \mathbf{y}_5)_2\ (x - \mathbf{x}_2)_5 - R_6\Sigma(y - \mathbf{y}_6)_2\ (x - \mathbf{x}_2)_6\}}{\Sigma(x - \mathbf{x}_2)^2} \tag{24}$$

$$R_3 = \{\Sigma(C^i - \mathbf{C})\ (x - \mathbf{x}_3) + d\Sigma(x - \mathbf{x}_3)_4 + f(x - \mathbf{x}_3)_6 - R_4\Sigma(y - \mathbf{y}_4)_3$$
$$\frac{(x - \mathbf{x}_3)_4 - R_5\Sigma(y - \mathbf{y}_5)_3\ (x - \mathbf{x}_3)_5 - R_6\Sigma(y - \mathbf{y}_6)_3\ (x - \mathbf{x}_3)_6\}}{\Sigma(x - \mathbf{x}_3)^2} \tag{25}$$

$$R_4 = \{\Sigma(D^i - \mathbf{D})\ (y - \mathbf{y}_4) + a\Sigma(y - \mathbf{y}_4)_1 + c\Sigma(y - \mathbf{y}_4)_3 - R_1\Sigma(x - \mathbf{x}_1)_4$$
$$\frac{(y - \mathbf{y}_4)_1 - R_2\Sigma(x - \mathbf{x}_2)_4\ (y - \mathbf{y}_4)_2 - R_3\Sigma(x - \mathbf{x}_3)_4\ (y - \mathbf{y}_4)_3\}}{\Sigma(y - \mathbf{y}_4)^2} \tag{26}$$

$$R_5 = \{\Sigma(E^i - \mathbf{E})\ (y - \mathbf{y}_5) + a\Sigma(y - \mathbf{y}_5)_1 + c\Sigma(y - \mathbf{y}_5)_3 - R_1\Sigma(x - \mathbf{x}_1)_5$$
$$\frac{(y - \mathbf{y}_5)_1 - R_2\Sigma(x - \mathbf{x}_2)_5\ (y - \mathbf{y}_5)_2 - R_3\Sigma(x - \mathbf{x}_3)_5(y - \mathbf{y}_5)_3\}}{\Sigma(y - \mathbf{y}_5)^2} \tag{27}$$

$$R_6 = \{\Sigma(F^i - \mathbf{F})\ (y - \mathbf{y}_6) + a\Sigma(y - \mathbf{y}_6)_1 + c\Sigma(y - \mathbf{y}_6)_3 - R_1\Sigma(x - \mathbf{x}_1)_6$$
$$\frac{(y - \mathbf{y}_6)_1 - R_2\Sigma(x - \mathbf{x}_2)_6\ (y - \mathbf{y}_6)_2 - R_3\Sigma(x - \mathbf{x}_3)_6\ (y - \mathbf{y}_6)_3\}}{\Sigma(y - \mathbf{y}_6)^2} \tag{28}$$

For brevity, let

$$\begin{aligned} M_{14} &= \Sigma(x - \mathbf{x}_1)_4,\ M_{16} = \Sigma(x - \mathbf{x}_1)_6, \text{ etc.} \\ P_{41} &= \Sigma(y - \mathbf{y}_4)_1,\ P_{43} = \Sigma(y - \mathbf{y}_4)_3, \text{ etc.} \\ K_{14} &= \Sigma(x - \mathbf{x}_1)_4\ (y - \mathbf{y}_4)_1,\ K_{15} = \Sigma(x - \mathbf{x}_1)_5\ (y - \mathbf{y}_5)_1, \text{ etc.} \\ L_1^2 &= \Sigma(x - \mathbf{x}_1)^2,\ L_2^2 = \Sigma(x - \mathbf{x}_2)^2, \text{ etc.} \end{aligned}$$

Finally, in group 1, for example, the observed average of w, $\mathbf{A}^i$, may be substituted for the required average $\mathbf{A}$, in the expression $\Sigma(A^i - \mathbf{A})$ $(x - \mathbf{x}_1)$ because the sum of the deviations $(x - \mathbf{x}_1)$ is zero. In other words $\Sigma(A^i - \mathbf{A})\,(x - \mathbf{x}_1) = \Sigma(A^i - \mathbf{A}^i)\,(x - \mathbf{x}_1)$. Introducing these equivalents into equations (16) to (21) and (24) to (28), equations (29) to (40) are obtained

$$\mathbf{A} = \mathbf{A}^i + \frac{1}{N_1}\{n_{14}d + n_{16}f\} - \frac{1}{N_1}\{R_4P_{41} + R_5P_{51} + R_6P_{61}\} \tag{29}$$

$$\mathbf{B} = \mathbf{B}^i + \frac{1}{N_2}\{n_{24}d + n_{26}f\} - \frac{1}{N_2}\{R_4P_{42} + R_5P_{52} + R_6P_{62}\} \tag{30}$$

$$\mathbf{C} = \mathbf{C}^i + \frac{1}{N_3}\{n_{34}d + n_{36}f\} - \frac{1}{N_3}\{R_4P_{43} + R_5P_{53} + R_6P_{63}\} \tag{31}$$

$$\mathbf{D} = \mathbf{D}^i + \frac{1}{N_4}\{n_{41}a + n_{43}c\} - \frac{1}{N_4}\{R_1M_{14} + R_2M_{24} + R_3M_{34}\} \tag{32}$$

$$\mathbf{E} = \mathbf{E}^i + \frac{1}{N_5}\{n_{51}a + n_{53}c\} - \frac{1}{N_5}\{R_1M_{15} + R_2M_{25} + R_3M_{35}\} \tag{33}$$

$$\mathbf{F} = \mathbf{F}^i + \frac{1}{N_6}\{n_{61}a + n_{63}c\} - \frac{1}{N_6}\{R_1M_{16} + R_2M_{26} + R_3M_{36}\} \tag{34}$$

$$R_1 = R_1^i + \frac{M_{14}d + M_{16}f - R_4K_{14} - R_5K_{15} - R_6K_{16}}{L_1^2} \tag{35}$$

$$R_2 = R_2^i + \frac{M_{24}d + M_{26}f - R_4K_{24} - R_5K_{25} - R_6K_{26}}{L_2^2} \tag{36}$$

$$R_3 = R_3^i + \frac{M_{34}d + M_{36}f - R_4K_{34} - R_5K_{35} - R_6K_{36}}{L_3^2} \tag{37}$$

$$R_4 = R_4^i + \frac{P_{41}a + P_{43}c - R_1K_{14} - R_2K_{24} - R_3K_{34}}{L_4^2} \tag{38}$$

$$R_5 = R_5^i + \frac{P_{51}a + P_{53}c - R_1K_{15} - R_2K_{25} - R_3K_{35}}{L_5^2} \tag{39}$$

$$R_6 = R_6^i + \frac{P_{61}a + P_{63}c - R_1K_{16} - R_2K_{26} - R_3K_{36}}{L_6^2} \tag{40}$$

where R_1^i, R_2^i, R_3^i, etc., are coefficients of regression of the dependent on the independent variable for groups 1, 2, 3, etc., respectively, computed from the original uncorrected values. The part added to this coefficient in each case is the correction which must be added to the

TABLE 4.

Process of Solution by Successive Approximation.

in equations	and obtain			Line
Second approximation determined directly				
38, 39, and 40	$\Delta R_4^{ii}\ \Delta R_5^{ii}\ \Delta R_6^{ii}$	$R_4^{ii}\ \ R_5^{ii}\ \ R_6^{ii}$		1
29, 30, and 31	$\Delta \mathbf{A}^{ii}\ \Delta \mathbf{B}^{ii}\ \Delta \mathbf{C}^{ii}$	$\mathbf{A}^{ii}\ \ \mathbf{B}^{ii}\ \ \mathbf{C}^{ii}$	$a^{ii}\ c^{ii}\ \Delta^1 a^{ii}\ \Delta^1 c^{ii}$	2
35, 36, and 37	$\Delta R_1^{ii}\ \Delta R_2^{ii}\ \Delta R_3^{ii}$	$R_1^{ii}\ \ R_2^{ii}\ \ R_3^{ii}$		3
32, 33, and 34	$\Delta \mathbf{D}^{ii}\ \Delta \mathbf{E}^{ii}\ \Delta \mathbf{F}^{ii}$	$\mathbf{D}^{ii}\ \ \mathbf{E}^{ii}\ \ \mathbf{F}^{ii}$	$d^{ii}\ f^{ii}\ \Delta^1 d^{ii}\ \Delta^1 f^{ii}$	4
proximation determined directly and checked by first differences Δ^1				
38, 39, and 40	$\Delta R_4^{iii}\Delta R_5^{iii}\Delta R_6^{iii}$	$R_4^{iii}\ \ R_5^{iii}\ \ R_6^{iii}$		5
38, 39, and 40	$\Delta^1 R_4^{iii}\Delta^1 R_5^{iii}\Delta^1 R_6^{iii}$	$\Delta R_4^{iii}\Delta R_5^{iii}\Delta R_6^{iii}$		6
29, 30, and 31	$\Delta \mathbf{A}^{iii}\Delta \mathbf{B}^{iii}\Delta \mathbf{C}^{iii}$	$\mathbf{A}^{iii}\ \ \mathbf{B}^{iii}\ \ \mathbf{C}^{iii}$	$a^{iii}c^{iii}\Delta^1 a^{iii}\Delta^1 c^{iii}$	7
29, 30, and 31	$\Delta^1 \mathbf{A}^{iii}\Delta^1 \mathbf{B}^{iii}\Delta^1 \mathbf{C}^{iii}$	$\Delta \mathbf{A}^{iii}\Delta \mathbf{B}^{iii}\Delta \mathbf{C}^{iii}$	$\Delta^1 a^{iii}\Delta^1 c^{iii}\Delta^2 a^{iii}\Delta^2 c^{iii}$	8
35, 36, and 37	$\Delta R_1^{iii}\Delta R_2^{iii}\Delta R_3^{iii}$	$R_1^{iii}\ \ R_2^{iii}\ \ R_3^{iii}$		9
35, 36, and 37	$\Delta^1 R_1^{iii}\Delta^1 R_2^{iii}\Delta^1 R_3^{iii}$	$\Delta R_1^{iii}\Delta R_2^{iii}\Delta R_3^{iii}$		10
32, 33, and 34	$\Delta \mathbf{D}^{iii}\Delta \mathbf{E}^{iii}\Delta \mathbf{F}^{iii}$	$\mathbf{D}^{iii}\ \ \mathbf{E}^{iii}\ \ \mathbf{F}^{iii}$	$d^{iii}f^{iii}\Delta^1 d^{iii}\Delta^1 f^{iii}$	11
32, 33, and 34	$\Delta^1 \mathbf{D}^{iii}\Delta^1 \mathbf{E}^{iii}\Delta^1 \mathbf{F}^{iii}$	$\Delta \mathbf{D}^{iii}\Delta \mathbf{E}^{iii}\Delta \mathbf{F}^{iii}$	$\Delta^1 d^{iii}\Delta^1 f^{iii}\Delta^2 d^{iii}\Delta^2 f^{iii}$	12
ation determined by first differences and checked by second differences Δ^2				
38, 39, and 40	$\Delta^1 R_4^{iv}\Delta^1 R_5^{iv}\Delta^1 R_6^{iv}$	$\Delta R_4^{iv}\Delta R_5^{iv}\Delta R_6^{iv}$		13
38, 39, and 40	$\Delta^2 R_4^{iv}\Delta^2 R_5^{iv}\Delta^2 R_6^{iv}$	$\Delta^1 R_4^{iv}\Delta^1 R_5^{iv}\Delta^1 R_6^{iv}$		14
29, 30, and 31	$\Delta^1 \mathbf{A}^{iv}\Delta^1 \mathbf{B}^{iv}\Delta^1 \mathbf{C}^{iv}$		$\Delta^1 a^{iv}\Delta^1 c^{iv}\Delta^2 a^{iv}\Delta^2 c^{iv}$	15
29, 30, and 31	$\Delta^2 \mathbf{A}^{iv}\Delta^2 \mathbf{B}^{iv}\Delta^2 \mathbf{C}^{iv}$	$\Delta^1 \mathbf{A}^{iv}\Delta^1 \mathbf{B}^{iv}\Delta^1 \mathbf{C}^{iv}$	$\Delta^2 a^{iv}\Delta^2 c^{iv}$	16
35, 36, and 37	$\Delta^1 R_1^{iv}\Delta^1 R_2^{iv}\Delta^1 R_3^{iv}$	$\Delta R_1^{iv}\Delta^1 R_2^{iv}\Delta R_3^{iv}$		17
35, 36, and 37	$\Delta^2 R_1^{iv}\Delta^2 R_2^{iv}\Delta^2 R_3^{iv}$	$\Delta^1 R_1^{iv}\Delta^2 R_2^{iv}\Delta^1 R_3^{iv}$		18
32, 33, and 34	$\Delta^1 \mathbf{D}^{iv}\Delta^1 \mathbf{E}^{iv}\Delta^1 \mathbf{F}^{iv}$		$\Delta^1 d^{iv}\Delta^1 f^{iv}\Delta^2 d^{iv}\Delta^2 f^{iv}$	19
32, 33, and 34	$\Delta^2 \mathbf{D}^{iv}\Delta^2 \mathbf{E}^{iv}\Delta^2 \mathbf{F}^{iv}$	$\Delta^1 \mathbf{D}^{iv}\Delta^1 \mathbf{E}^{iv}\Delta^1 \mathbf{F}^{iv}$	$\Delta^2 d^{iv}\Delta^2 f^{iv}$	20

Supplementary explanations

$R_1^{ii} - R_1^{i} = \Delta R_1^{ii}$	$\Delta^2 R_1^{iii} = \Delta^1 R_1^{iii} - \Delta^1 R_1^{ii}$	For similar explanations of averages see Table 2.
$R_1^{iii} - R_1^{ii} = \Delta R_1^{iii} - \Delta R_1^{ii}$	$\Delta^2 R_1^{iv} = \Delta^1 R_1^{iv} - \Delta^1 R_1^{iii}$	
$R_1^{iv} - R_1^{iii} = \Delta R_1^{iv} - \Delta R_1^{iii}$	$\Delta^2 R_1^{v} = \Delta^1 R_1^{v} - \Delta^1 R_1^{iv}$	
etc.	etc.	

value computed, without taking into account the correlation between the independent variables. These twelve equations (29) to (40), together with the four equivalents $\mathbf{B} - \mathbf{A} = a$, $\mathbf{B} - \mathbf{C} = c$, $\mathbf{E} - \mathbf{D} = d$, and $\mathbf{E} - \mathbf{F} = f$ determine the six required averages and the six required regression coefficients, and may be most conveniently solved by a process of successive approximation similar to that already presented for the simpler case in which variability within the group is neglected. The details of this process are given in table 4.

As in the simpler case (see p. 105), if the process of successive approximation, indicated by lines 1, 2, 3, 4, and 5, 7, 9, and 11 results in convergence to definite limiting values, these values will satisfy equations (29) to (40). In the third approximation, the procedure indicated by lines 6, 8, 10, and 12, involving first differences, affords a numerical check on the computation of ΔR_4^{iii}, ΔR_5^{iii}, etc., and $\Delta\mathbf{A}^{iii}$, $\Delta\mathbf{B}^{iii}$, etc., of lines 5, 7, 9 and 11, and beginning with the fourth approximation, second differences (lines 14, 16, 18, and 20), afford a check against the first differences. For a numerical illustration see page 122.

If the reader has followed the reasoning thus far, he will doubtless feel that, although the regression method is formally complete, the labor involved in its application would, in many instances, be so great as to make its use impracticable. To meet this objection, a *slope* method has been devised which takes account of the variability within the group in nearly as accurate a manner, but one that eliminates half of the equations, namely, those similar to (23) to (28). The basis of this method is the fact that the slope of a chord of a simple curve is approximately equal to that of the tangent at the point midway between the extremities of the chord. Accordingly, the slope

$\dfrac{\mathbf{B} - \mathbf{A}}{\mathbf{x}_2 - \mathbf{x}_1} = S_{1-2}$ of the chord whose extremities are $(\mathbf{x}_1, \mathbf{A})$ and $(\mathbf{x}_2, \mathbf{B})$ is approximately that of the tangent at the point whose abscissa is $\dfrac{\mathbf{x}_1 + \mathbf{x}_2}{2}$. Similarly, for the point midway between $(\mathbf{x}_2, \mathbf{B})$ and $(\mathbf{x}_3, \mathbf{C})$, the slope of the tangent is approximately $\dfrac{\mathbf{C} - \mathbf{B}}{\mathbf{x}_3 - \mathbf{x}_2} = S_{2-3}$, and so on.

But the slopes at the points $(\mathbf{x}_1, \mathbf{A})$, $(\mathbf{x}_2, \mathbf{B})$, and $(\mathbf{x}_3, \mathbf{C})$ are required. That at $(\mathbf{x}_2, \mathbf{B})$ is readily obtained by utilizing the rate of change in slope as a means of interpolating between S_{1-2} and S_{2-3}. Thus

$$\frac{S_{2-3} - S_{1-2}}{\frac{\mathbf{x}_3 + \mathbf{x}_2}{2} - \frac{\mathbf{x}_2 + \mathbf{x}_1}{2}} = \frac{2(S_{2-3} - S_{1-2})}{\mathbf{x}_3 - \mathbf{x}_1}$$ is the rate of change in slope,

whence the slope at any point of abscissa x between ($\mathbf{x}_1$, **A**) and ($\mathbf{x}_2$, **B**) is $S_{1-2} + \left[x - \left(\frac{\mathbf{x}_2 + \mathbf{x}_1}{2}\right)\right] \left[\frac{2(S_{2-3} - S_{1-2})}{\mathbf{x}_3 - \mathbf{x}_1}\right]$ and putting $x = \mathbf{x}_2$ the slope at ($\mathbf{x}_2$, **B**) is

$$S_2 = S_{1-2} + \frac{\mathbf{x}_2 - \mathbf{x}_1}{\mathbf{x}_3 - \mathbf{x}_1}(S_{2-3} - S_{1-2}) \tag{41}$$

Similarly, the slope at any point of abscissa x between ($\mathbf{x}_2$, **B**) and ($\mathbf{x}_3$, **C**) is $S_{2-3} + \left[x - \frac{(\mathbf{x}_3 + \mathbf{x}_2)}{2}\right] \left[\frac{2(S_{2-3} - S_{1-2})}{\mathbf{x}_3 - \mathbf{x}_1}\right]$ and putting $x = \mathbf{x}_2$, the slope at ($\mathbf{x}_2$, **B**) is

$$S_2 = S_{2-3} + \frac{\mathbf{x}_2 - \mathbf{x}_3}{\mathbf{x}_3 - \mathbf{x}_1}(S_{2-3} - S_{1-2}) \tag{42}$$

In the same way the slope corresponding to each abscissa between the extremes $\mathbf{x}_1$ and $\mathbf{x}_n$ is found, where $\mathbf{x}_n$ denotes the last group average. In the particular case at hand x is divided into but three groups so that $\mathbf{x}_n = \mathbf{x}_3$, whence the slopes S_1 and S_3 must be determined by utilizing the rate of change in slope between S_{1-2} and S_{2-3} as a means of exterpolating beyond S_{1-2} and S_{2-3}. Thus

$$S_1 = S_{1-2} + \frac{\mathbf{x}_1 - \mathbf{x}_2}{\mathbf{x}_3 - \mathbf{x}_1}(S_{2-3} - S_{1-2}) \tag{43}$$

and

$$S_3 = S_{2-3} + \frac{\mathbf{x}_3 - \mathbf{x}_2}{\mathbf{x}_3 - \mathbf{x}_1}(S_{2-3} - S_{1-2}) \tag{44}$$

Similarly equations (45) to (48) are obtained.

$$S_4 = S_{4-5} + \frac{\mathbf{y}_4 - \mathbf{y}_5}{\mathbf{y}_6 - \mathbf{y}_4}(S_{5-6} - S_{4-5}) \tag{45}$$

$$S_5 = S_{4-5} + \frac{\mathbf{y}_5 - \mathbf{y}_4}{\mathbf{y}_6 - \mathbf{y}_4}(S_{5-6} - S_{4-5}) \tag{46}$$

$$S_5 = S_{5-6} + \frac{\mathbf{y}_5 - \mathbf{y}_6}{\mathbf{y}_6 - \mathbf{y}_4}(S_{5-6} - S_{4-5}) \tag{47}$$

$$S_6 = S_{5-6} + \frac{\mathbf{y}_6 - \mathbf{y}_5}{\mathbf{y}_6 - \mathbf{y}_4}(S_{5-6} - S_{4-5}) \tag{48}$$

Equations (23) to (28) are thus replaced by equations (41) to (48) whose solution depends solely upon the relation between the averages **A**, **B**, **C**, etc. Thus, if S_4, S_5, and S_6 be substituted for R_4, R_5, and R_6 in equation (16) the last expression will be found to involve only d and f as unknowns. For S_{4-5} and S_{5-6} of equations (45) to (48) are respectively defined as $\frac{\mathbf{E} - \mathbf{D}}{\mathbf{y}_5 - \mathbf{y}_4} = \frac{d}{\mathbf{y}_5 - \mathbf{y}_4}$ and $\frac{\mathbf{E} - \mathbf{F}}{\mathbf{y}_6 - \mathbf{y}_5} = \frac{-f}{\mathbf{y}_6 - \mathbf{y}_5}$.

4. ILLUSTRATION OF METHOD BY SOLUTION OF A PARTICULAR PROBLEM CONCERNING THE RELATION BETWEEN TEMPERATURE, PRECIPITATION, AND YIELD OF WHEAT IN SOUTH DAKOTA.[1]

From 1891 to 1917 the mean air temperature for the month of June in South Dakota varied from 60.4° F to 73.4° F, while the total precipitation during the months of May and June varied from 3.5 to 11.6 inches, and the yield of wheat (harvested in August) varied from 4.0 to 17.0 bushels per acre. In attempting to ascertain what effect, if any, temperature and precipitation had upon the yield of wheat, Blair (1913; 1915) applied the method of simple linear correlation to the portion of the data then available, and found a strong negative correlation between temperature and yield, and a somewhat smaller positive correlation between precipitation and yield. But, he also found a high negative correlation between temperature and precipitation, which, in 1918, led him to bring the data up to date and to consider the question: "how much of the apparent relation between precipitation and yield is really due to the influence of precipitation, and how much is due to the simultaneous influence of temperature; and, similarly, how much of the apparent relation between temperature and yield is due to precipitation." (Blair 1918, p. 71). He applied the method of multiple linear correlation and found that, after eliminating the effect of precipitation, correlation between temperature and yield was reduced from − 0.62 to − 0.48, and, likewise, that, after eliminating the influence of temperature, correlation between precipitation and yield was reduced from + 0.49 to + 0.22. The functional relation he obtained between yield, temperature and precipitation is, in his notation,

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[1] We desire to express our obligation to Mr. H. H. Collins who has made the computations involved in this illustrative problem and in many others. Without his aid publication would have been materially delayed.

$$y = 11.2 - 0.48 \frac{\sigma_y}{\sigma_t} (t^\circ - 65.9^\circ) + 0.22 \frac{\sigma_y}{\sigma_p} (\mathrm{p} - 6.8) \qquad (49)$$

where $\sigma_y = 3.0$, $\sigma_t = 3.0$, and $\sigma_p = 2.1$

In applying the method of successive approximation to these data it is not our purpose to discuss, except incidentally, the results obtained, but to give a simple, concrete illustration of the process actually followed, first, in the case when variability within the group is neglected, and second, in the case when this variability is taken into account. In both instances the same notation is employed as in the analytic demonstrations. It should be noted, however, that, in the case when variability within the group is neglected, three independent variables, x, y, and z, are used in the analytic demonstration, while, in this illustrative problem, only two are involved.

In the first instance, then, the initial step, as shown in table 5, is to group the data with respect to temperature, arranging the twenty seven entries according to its ascending order of magnitude, and similarly, to group the data with respect to precipitation, arranging the twenty-seven entries according to its increase. Secondly, each series is divided into three groups of nine entries each (see p. 127), *i. e.* groups 1, 2, and 3 of the data arranged with respect to temperature, and groups 4, 5, and 6 of the data arranged with respect to precipitation. Thirdly, opposite each entry in groups 1, 2, and 3 is entered the number 4, 5, or 6 designating which precipitation group (y-group) the entry is in, and, similarly, opposite each entry in groups 4, 5, and 6 is entered the number 1, 2, or 3 designating which temperature group (x-group) the entry is in. Lastly, the average wheat yield for each group ($\mathbf{A}^i$, $\mathbf{B}^i$, $\mathbf{C}^i$, $\mathbf{D}^i$, $\mathbf{E}^i$, and $\mathbf{F}^i$), the average temperature for each of groups 1, 2, and 3 ($\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$), the average precipitation for each of groups 4, 5, and 6 ($\mathbf{y}_4$, $\mathbf{y}_5$, and $\mathbf{y}_6$), and the number of entries common to groups 1, 2, or 3, and 4, 5, or 6 ($n_{14} = n_{41}$, $n_{15} = n_{51}$, etc.) are determined. Each of these steps is indicated in table 5.

TABLE 5.

Data Concerning Yield of Wheat in South Dakota from 1891 to 1917 (Blair 1918, p. 73) Tabulated as Required by the Method of Successive Approximation in the Case when Variability within the Group is Neglected.

Grouped with respect to temperature, x				Grouped with respect to precipitation, y			
Yield	x		y-group	Yield	y		x-group
17.0 bu. per acre	60°.4	F Group 1	6	6.9 bu. per acre	3.5	inches Group 4	3
6.3	61°.5		6	4.0	3.6		3
12.2	62°.6		4	6.6	3.7		3
14.0	62°.7		4	14.2	3.8		2
12.8	63°.7		6	12.8	3.9		3
12.0	63°.7		5	8.5	4.5		3
12.5	63°.9		6	8.0	4.6		2
13.4	63°.9		6	14.0	5.3		1
11.2	64°.2		5	12.2	6.0		1
111.4 = total 12.38 = $\mathbf{A}^i$	566.6 = total 62.95 = $\mathbf{x}_1$		$n_{14}=2$ $n_{15}=2$ $n_{16}=5$ $N_1=9$	87.2 = total 9.69 = $\mathbf{D}^i$	38.9 = total 4.32 = $\mathbf{y}_4$		$n_{41}=2$ $n_{42}=2$ $n_{43}=5$ $N_4=9$
15.2	64.2	Group 2	5	9.0	6.0	Group 5	3
13.7	64.4		6	15.2	6.5		2
9.6	64.5		5	9.6	6.5		2
14.2	64.8		4	11.2	6.6		3
8.0	65.0		4	12.4	6.8		3
13.8	65.0		5	12.0	6.9		1
12.9	66.3		6	13.8	7.0		2
10.7	66.4		5	11.2	7.7		1
14.1	66.9		6	10.7	8 1		2
112.2 = total 12.47 = $\mathbf{B}^i$	587.5 = total 65.27 = $\mathbf{x}_2$		$n_{24}=2$ $n_{25}=4$ $n_{26}=3$ $N_2=9$	105.1 = total 11.68 = $\mathbf{E}^i$	62.1 = total 6.90 = $\mathbf{y}_5$		$n_{51}=2$ $n_{52}=4$ $n_{53}=3$ $N_5=9$
11.2	67.0	Group 3	5	12.9	8.1	Group 6	2
12.4	67.3		5	9.0	8.1		3
9.0	67.5		6	6.3	8.2		1
12.8	68.3		4	13.4	8.4		1
6.9	69.4		4	14.1	9.0		2
9.0	69.6		5	17.0	9.0		1
8.5	70.3		4	12.5	9.5		1
6.6	70.6		4	12.8	10.0		1
4.0	73.4		4	13.7	11.6		2
80.4 = total 8.93 = $\mathbf{C}^i$	623.4 = total 69.27 = $\mathbf{x}_3$		$n_{34}=5$ $n_{35}=3$ $n_{36}=1$ $N_3=9$	111.7 = total 12.41 = $\mathbf{F}^i$	81.9 = total 9.10 = $\mathbf{y}_6$		$n_{61}=5$ $n_{62}=3$ $n_{63}=1$ $N_6=9$

Selecting the middle group of each series (groups 2 and 5) as the "standard" (see p. 103) the equations, derived as on page 104 for determining the corrected averages, are

$$\mathbf{A} = 12.38 + \frac{1}{9}(2d + 5f) \tag{50}$$

$$\mathbf{B} = 12.47 + \frac{1}{9}(2d + 3f) \tag{51}$$

$$\mathbf{C} = 8.93 + \frac{1}{9}(5d + 1f) \tag{52}$$

$$\mathbf{D} = 9.69 + \frac{1}{9}(2a + 5c) \tag{53}$$

$$\mathbf{E} = 11.68 + \frac{1}{9}(2a + 3c) \tag{54}$$

$$\mathbf{F} = 12.41 + \frac{1}{9}(5a + 1c) \tag{55}$$

where

$$a = \mathbf{B} - \mathbf{A} \tag{56}$$

$$c = \mathbf{B} - \mathbf{C} \tag{57}$$

$$d = \mathbf{E} - \mathbf{D} \tag{58}$$

$$f = \mathbf{E} - \mathbf{F} \tag{59}$$

Following the method of solution given in table 2, the first approximations to d and f[1] ($d^i = 11.68 - 9.69 = 1.99$, $f^i = 11.68 - 12.41 = -0.73$) substituted in the second members of equations (50), (51) and (52) give

$$\Delta\mathbf{A}^{ii} = \frac{1}{9}[2 \times 1.99 + 5\,(-0.73)] = 0.037 \tag{60}$$

$$\Delta\mathbf{B}^{ii} = \frac{1}{9}[2 \times 1.99 + 3\,(-0.73)] = 0.199 \tag{61}$$

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[1] In order to save labor and reduce the number of required approximations to a minimum, correction should be made first for the variable having the greatest apparent effect. In this particular case the apparent effects of temperature and precipitation are essentially the same, and first approximations to d and f are used rather than those to a and c merely because it conforms to the procedure in table 2.

$$\Delta \mathbf{C}^{ii} = \frac{1}{9}[5 \times 1.99 + 1\,(-\,0.73)] = 1.024 \qquad (62)$$

which are second approximations to the quantities that must be added to $\mathbf{A}^{i}$, $\mathbf{B}^{i}$, and $\mathbf{C}^{i}$, respectively, to equal **A**, **B**, and **C**. Substituting the original averages $\mathbf{A}^{i}$, $\mathbf{B}^{i}$, and $\mathbf{C}^{i}$ plus the quantities $\Delta\mathbf{A}^{ii}$, $\Delta\mathbf{B}^{ii}$, and $\Delta\mathbf{C}^{ii}$ respectively into equations (56) and (57) gives second approximations (a^{ii} and c^{ii}) to a and c. That is

$$\begin{aligned} a^{ii} &= (12.47 + 0.199) - (12.38 + 0.037) = \\ &\quad a^{i} + (\Delta\mathbf{B}^{ii} - \Delta\mathbf{A}^{ii}) = a^{i} + \Delta^{1}a^{ii} = 0.09 + 0.162 = 0.252 \quad (63) \end{aligned}$$

$$\begin{aligned} c^{ii} &= (12.47 + 0.199) - (8.93 + 1.024) = \\ &\quad c^{i} + (\Delta\mathbf{B}^{ii} - \Delta\mathbf{C}^{ii}) = c^{i} + \Delta^{1}c^{ii} = 2.715 \quad (64) \end{aligned}$$

Substituting a^{ii} and c^{ii} for a and c in equations (53), (54), and (55) gives

$$\Delta\mathbf{D}^{ii} = \frac{1}{9}[2 \times 0.252 + 5 \times 2.715] = 1.564 \qquad (65)$$

$$\Delta\mathbf{E}^{ii} = \frac{1}{9}[2 \times 0.252 + 3 \times 2.715] = 0.961 \qquad (66)$$

$$\Delta\mathbf{F}^{ii} = \frac{1}{9}[5 \times 0.252 + 1 \times 2.715] = 0.442 \qquad (67)$$

which are second approximations to the quantities that must be added to $\mathbf{D}^{i}$, $\mathbf{E}^{i}$, and $\mathbf{F}^{i}$ to give **D**, **E**, and **F**. Substituting the original averages $\mathbf{D}^{i}$, $\mathbf{E}^{i}$, and $\mathbf{F}^{i}$ plus the quantities $\Delta\mathbf{D}^{ii}$, $\Delta\mathbf{E}^{ii}$, and $\Delta\mathbf{F}^{ii}$ into equations (58) and (59) gives second approximations (d^{ii} and f^{ii}) to d and f. That is

$$d^{ii} = d^{i} + (\Delta\mathbf{E}^{ii} - \Delta\mathbf{D}^{ii}) = d^{i} + \Delta^{1}d^{ii} = 1.99 - 0.603 = 1.387 \qquad (68)$$

$$f^{ii} = f^{i} + (\Delta\mathbf{E}^{ii} - \Delta\mathbf{F}^{ii}) = f^{i} + \Delta^{1}f^{ii} = -\,0.73 + 0.519 = -\,0.211 \qquad (69)$$

If this process be continued, the successive approximations will converge to the required corrections $\Delta\mathbf{A}$, $\Delta\mathbf{B}$, $\Delta\mathbf{C}$, $\Delta\mathbf{D}$, $\Delta\mathbf{E}$, and $\Delta\mathbf{F}$, and to the required differences a, c, d, and f. But, unless two computers are available to duplicate each other's work, a second process of computation is needed to check the results obtained by the first process. Beginning with the third approximation, this is accomplished by computing directly the differences $\Delta^{1}\mathbf{A}^{iii}$, $\Delta^{1}\mathbf{B}^{iii}$, $\Delta^{1}\mathbf{C}^{iii}$ etc., that must be added to $\Delta\mathbf{A}^{ii}$, $\Delta\mathbf{B}^{ii}$, $\Delta\mathbf{C}^{ii}$, etc., to obtain $\Delta\mathbf{A}^{iii}$,

$\Delta\mathbf{B}^{iii}$, $\Delta\mathbf{C}^{iii}$, etc., and so on. For example, the third approximations found by substituting d^{ii} and f^{ii}, given by equations (68) and (69), for d and f in equations (50), (51) and (52) are

$$\Delta\mathbf{A}^{iii} = 0.191 \tag{70}$$

$$\Delta\mathbf{B}^{iii} = 0.238 \tag{71}$$

$$\Delta\mathbf{C}^{iii} = 0.747 \tag{72}$$

But by substituting the differences $\Delta^1 d^{ii} = d^{ii} - d^i = -0.603$ and $\Delta^1 f^{ii} = f^{ii} - f^i = 0.519$ for d and f in equations (50), (51) and (52) we obtain

$$\Delta^1\mathbf{A}^{iii} = 0.154 \tag{73}$$

$$\Delta^1\mathbf{B}^{iii} = 0.039 \tag{74}$$

$$\Delta^1\mathbf{C}^{iii} = -0.277 \tag{75}$$

which when added to $\Delta\mathbf{A}^{ii} = 0.037$, $\Delta\mathbf{B}^{ii} = 0.199$, and $\Delta\mathbf{C}^{ii} = 1.024$ (equations 60 to 62) give $\Delta\mathbf{A}^{iii} = 0.191$, $\Delta\mathbf{B}^{iii} = 0.238$, and $\Delta\mathbf{C}^{iii} = 0.747$, thus checking the results obtained in equations (70), (71), and (72).

In the same way the values of $\Delta\mathbf{D}^{iii}$, $\Delta\mathbf{E}^{iii}$, and $\Delta\mathbf{F}^{iii}$ are obtained directly and checked by first differences Δ^1, and, beginning with the fourth approximation, computation of first differences is checked by that of second differences Δ^2, as indicated in table 2. The values obtained in the second and third approximations computed by the processes just described, the values obtained in the fourth and higher approximations computed by first differences and checked by second differences, the final limiting values obtained in the seventh approximation. and everything required for making each computation is given in table 6, which is for the most part self-explanatory. Suffice it merely to call attention to the fact that the coefficients 2 and 5 of equation (50) multiplied respectively by the *first* approximation to d and f (1.99 and $-$ 0.73), by the *second* approximation to $\Delta^1 d$ and $\Delta^1 f$ ($-$ 0.603 and 0.519), and by the *third* approximation to $\Delta^2 d$ and $\Delta^2 f$ (0.533 and $-$ 0.410), and divided by 9, gives the *second* approximation, 0.037, to $\Delta\mathbf{A}$, the *third* approximation, 0.154, to $\Delta^1\mathbf{A}$, and the *fourth* approximation, $-$ 0.109, to $\Delta^2\mathbf{A}$, and so on. But the coefficients 2 and 5 of equation (53) multiplied respectively by the *second* approximation to a and c (0.252 and 2.715), the *third* approximation to $\Delta^1 a$ and $\Delta^1 c$ ($-$ 0.115 and 0.316), and the *fourth* approximation to

$\Delta^2 a$ and $\Delta^2 c$ (0.091 and − 0.268) and divided by 9, gives the *second* approximation, 1.564, to $\Delta\mathbf{D}$, the *third* approximation, 0.150, to $\Delta^1\mathbf{D}$, and the *fourth* approximation, − 0.129, to $\Delta^2\mathbf{D}$, and so on. Finally, each limiting value, for example, $\Delta\mathbf{A} = 0.245$, is obtained by adding to $\Delta\mathbf{A}^{ii} = 0.037$ the sum of all approximations to the first differences $\Delta^1\mathbf{A}$, *i. e.*, 0.154 + 0.045 + 0.008 + 0.001.

TABLE 6.

NUMERICAL SOLUTION BY METHOD OF SUCCESSIVE APPROXIMATION OF SOUTH DAKOTA WHEAT PROBLEM IN THE CASE WHEN VARIABILITY WITHIN THE GROUP IS NEGLECTED.

Second and third approximations and final values (7th approximation)										
d	f	$\Delta\mathbf{A}$	$\Delta\mathbf{B}$	$\Delta\mathbf{C}$	a	c	$\Delta\mathbf{D}$	$\Delta\mathbf{E}$	$\Delta\mathbf{F}$	Number of Approximation
1.99	−0.73	——	——	——	——	——	——	——	——	I
1.387	−0.211	0.037	0.199	1.024	0.252	2.715	1.564	0.961	0.442	II
——	——	0.191	0.238	0.747	0.137	3.031	1.714	1.041	0.413	III
1.305	−0.081	0.245	0.263	0.716	0.108	3.087	1.738	1.053	0.404	VII
Third and succeeding approximations to first differences										
$\Delta^1 d$	$\Delta^1 f$	$\Delta^1\mathbf{A}$	$\Delta^1\mathbf{B}$	$\Delta^1\mathbf{C}$	$\Delta^1 a$	$\Delta^1 c$	$\Delta^1\mathbf{D}$	$\Delta^1\mathbf{E}$	$\Delta^1\mathbf{F}$	
−0.603	0.519	——	——	——	——	——	——	——	——	II
−0.070	0.109	0.154	0.039	−0.277	−0.115	0.316	0.150	0.080	−0.029	III
−0.010	0.019	0.045	0.021	−0.027	−0.024	0.048	0.021	0.011	−0.008	IV
−0.002	0.002	0.008	0.004	−0.003	−0.004	0.007	0.003	0.001	−0.001	V
−0.000	0.000	0.001	0.000	−0.001	−0.001	0.001	0.000	0.000	−0.000	VI
−0.000	0.000	0.000	0.000	−0.000	−0.000	0.000	0.000	0.000	−0.000	VII
Fourth and succeeding approximations to second differences										
$\Delta^2 d$	$\Delta^2 f$	$\Delta^2\mathbf{A}$	$\Delta^2\mathbf{B}$	$\Delta^2\mathbf{C}$	$\Delta^2 a$	$\Delta^2 c$	$\Delta^2\mathbf{D}$	$\Delta^2\mathbf{E}$	$\Delta^2\mathbf{F}$	
0.533	−0.410	——	——	——	——	——	——	——	——	III
0.060	−0.090	−0.109	−0.018	0.250	0.091	−0.268	−0.129	−0.069	0.021	IV
0.008	−0.017	−0.037	−0.017	0.023	0.020	−0.041	−0.018	−0.009	0.007	V
0.002	−0.002	−0.008	−0.004	0.003	0.003	−0.006	−0.003	−0.001	0.001	VI
0.000	−0.000	−0.001	−0.000	0.001	0.001	−0.001	−0.000	−0.000	0.000	VII
Coefficients used in multiplying by *d, f, a,* and *c*, etc.										
2	5 of *d* and *f* in (50)				2	5 of *a* and *c* in (53)				
2	3 of *d* and *f* in (51)				2	3 of *a* and *c* in (54)				
5	1 of *d* and *f* in (52)				5	1 of *a* and *c* in (55)				

Substituting the final values,

$\Delta\mathbf{A} = \frac{1}{9}(2d + 5f) = 0.245$, $\Delta\mathbf{B} = \frac{1}{9}(2d + 3f) = 0.263$, etc., carried to two decimals, in equations (50) to (55) gives the corrected averages

$$\mathbf{A} = 12.38 + 0.245 = 12.62 \tag{76}$$

$$\mathbf{B} = 12.47 + 0.263 = 12.73 \tag{77}$$

$$\mathbf{C} = 8.93 + 0.716 = 9.65 \tag{78}$$

$$\mathbf{D} = 9.69 + 1.738 = 11.43 \tag{79}$$

$$\mathbf{E} = 11.68 + 1.053 = 12.73 \tag{80}$$

$$\mathbf{F} = 12.41 + 0.404 = 12.81 \tag{81}$$

and these substituted in equations (56) to (59) give 0.11 for a, 3.08 for c, 1.30 for d, and -0.08 for f, which agree to the nearest hundredth with the final values of these differences entered in table 6. An additional check upon the computations is $\mathbf{B} = \mathbf{E}$, which should be the case since each of these two averages correspond to the standard values of temperature and precipitation.

As stated on page 99, the functional relations found by this method are defined by the series of corresponding averages of dependent and independent variables. Many ways of utilizing these relations will occur to the reader. The most precise and, perhaps, the most desirable way would be to plot the averages of w corresponding to $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and those corresponding to $\mathbf{y}_4$, $\mathbf{y}_5$, and $\mathbf{y}_6$, and so determine the type of equation relating w to x and w to y; and then, by the method of least squares or method of moments, compute the constants from the original data. A more expedient way is to use the relations between the averages directly, and correct for the neglected variability within the group. For this purpose the functional relations may be conveniently expressed as

$$w = f_1(\mathbf{x}) + f_2(\mathbf{y}) = 12.73 + F_1(\mathbf{x}) + F_2(\mathbf{y}) \tag{82}$$

where $\mathbf{x}$ and $\mathbf{y}$ signify any one of the three group averages of x and of y, and where $F_1(\mathbf{x})$ and $F_2(\mathbf{y})$ are defined by the series in table 7.

TABLE 7.

DEFINITION OF FUNCTIONAL RELATIONS.

x		$F_1(\mathbf{x})$	y		$F_2(\mathbf{y})$
mean	limits		mean	limits	
$62.95 = \mathbf{x}_1$	$60.4 - 64.2$	$-0.11 = -a$	$4.32 = \mathbf{y}_4$	$3.5 - 6.0$	$-1.30 = -d$
$65.27 = \mathbf{x}_2$	$64.2 - 66.9$	0	$6.90 = \mathbf{y}_5$	$6.0 - 8.1$	0
$69.27 = \mathbf{x}_3$	$67.0 - 73.4$	$-3.08 = -c$	$9.10 = \mathbf{y}_6$	$8.1 - 11.6$	$0.08 = -f$

In computing a value of w for given values of x and y, say $x = 73.4$ and $y = 3.6$, correction is first made for the group which, from equation (82) and table 7, is

$$w = 12.73 - 3.08 - 1.30 = 8.35 \tag{83}$$

However, this value for w corresponds not to $x = 73.4$ and $y = 3.6$, but, since variability within the group was neglected, to $x = 69.27$ and $y = 4.32$. Correction for this neglected variability may be made by running a linear regression in each group and introducing the regression coefficients into equation (83); by plotting the averages and ascertaining the type of equation as stated above, and then computing the constants from the averages, and replacing equation (83) by the equation so obtained; or by a simple process of interpolation like the following. Since -3.08 is the change in w due to a change in x from $\mathbf{x}_2 = 65.27$ to $\mathbf{x}_3 = 69.27$, the change in w due to a change in x from $\mathbf{x}_3 = 69.27$ to $x = 73.4$ should be approximately proportional. That is the change in w due to the variability of x in group 3 is approximately given by $\dfrac{73.4 - 69.27}{69.27 - 65.27} \times (-3.08) = -3.09$, and, in the same way, the change in w due to the variability of y in group 4 is approximately given by $\dfrac{3.6 - 4.32}{4.32 - 6.90} \times (-1.30) = -0.36$. Introducing these approximate corrections into equation (83) gives

$$w = 8.35 - 3.09 - 0.36 = 4.7 \tag{84}$$

which agrees well with the observed value of 4.0 given in table 5. For comparison, it is well to add that the linear regression equation (49), derived by the method of multiple correlation, is in our notation

$$w = 11.2 - 0.48\,(x - 65.9) + 0.31\,(y - 6.8) \tag{85}$$

Putting $x = 73.4$ and $y = 3.6$, as above, gives $w = 6.6$, a value departing more widely from that observed.

In illustrating the procedure when variability within the group is taken into account, the wheat data are arranged and grouped as shown in table 5. In addition, for the purpose of computing the required regression coefficients, the deviations $A^i - \mathbf{A}^i$, and $x - \mathbf{x}_1$ in group 1, $B^i - \mathbf{B}^i$ and $x - \mathbf{x}_2$ in group 2, and so on to $F^i - \mathbf{F}^i$ and $y - \mathbf{y}_6$ in group 6 are entered. However, in order to save labor in computing $\Sigma(A^i - \mathbf{A}^i)\,(x - \mathbf{x}_1)$, etc., the averages $\mathbf{A}^i$, $\mathbf{x}_1$, $\mathbf{B}^i$, $\mathbf{x}_2$, etc., are replaced by the approximate values $\mathbf{A}^i_1$, $\mathbf{x}'_1$, $\mathbf{B}^i_1$, $\mathbf{x}'_2$, etc., carried only to as many places as the individual entries, and deviations from these values are the ones that are entered. To save space table 5, including these deviations, is not reproduced.

Owing to substitution of the above approximate values for the true averages in equations (29) to (40), certain initial corrections must be applied before they are ready for solution. Let $\mathbf{x}_1 = \mathbf{x}'_1 + \Delta\mathbf{x}'_1$ and $\mathbf{A}^i = \mathbf{A}^i_1 + \Delta\mathbf{A}^i_1$ where $\Delta\mathbf{x}'_1$ and $\Delta\mathbf{A}^i_1$ are the respective corrections that must be added. Substituting these equivalents into the expression

$$\frac{\Sigma(A^i - \mathbf{A}^i)\,(x - \mathbf{x}_1)}{\Sigma(x - \mathbf{x}_1)^2} = R^i_1,$$ as defined on page 108, gives

$$\begin{aligned}
R^i_1 &= \frac{\Sigma\,[A^i - (\mathbf{A}^i_1 + \Delta\mathbf{A}^i_1)]\,[x - (\mathbf{x}'_1 + \Delta\mathbf{x}'_1)]}{\Sigma\,[x - (\mathbf{x}'_1 + \Delta\mathbf{x}_1)]^2} \\
&= \frac{\Sigma(A^i - \mathbf{A}^i_1)\,(x - \mathbf{x}'_1) - \Delta\mathbf{x}'_1\,\Sigma(A^i - \mathbf{A}^i_1) - \Delta\mathbf{A}^i\,\Sigma(x - \mathbf{x}'_1 - \Delta\mathbf{x}'_1)}{\Sigma(x - \mathbf{x}'_1)^2 - 2\Delta\mathbf{x}'_1\,\Sigma(x - \mathbf{x}'_1) + \Sigma(\Delta\mathbf{x}'_1)^2} \\
&= \frac{\Sigma(A^i - \mathbf{A}^i_1)\,(x - \mathbf{x}_1) - N_1(\Delta\mathbf{A}^i_1)\,(\Delta\mathbf{x}_1)}{\Sigma(x - \mathbf{x}'_1)^2 - N_1(\Delta\mathbf{x}'_1)^2}
\end{aligned} \tag{86}$$

since

$$\left.\begin{aligned}
\Sigma(A^i - \mathbf{A}^i_1) &= \Sigma\Delta\mathbf{A}_1 = N_1\Delta\mathbf{A}^i_1 \\
\Sigma(x - \mathbf{x}'_1) &= \Sigma\Delta\mathbf{x}'_1 = N_1\Delta\mathbf{x}'_1 \\
\Sigma(x - \mathbf{x}'_1 - \Delta\mathbf{x}'_1) &= \Sigma(x - \mathbf{x}_1) = 0
\end{aligned}\right\} \tag{87}$$

In the same way corrections for R_2^i, R^i_3, etc., are obtained. Corrections for M, P, and K are found as follows:

$$M_{14} = \Sigma(x - \mathbf{x}_1)_4 = \Sigma(x - \mathbf{x}_1' - \Delta\mathbf{x}_1')_4 = \Sigma(x - \mathbf{x}_1')_4 - n_{41}\Delta\mathbf{x}_1' \qquad (88)$$

and similarly for M_{15}, M_{16}, etc.

$$P_{41} = \Sigma(y - \mathbf{y}_4)_1 = \Sigma(y - \mathbf{y}_4' - \Delta\mathbf{y}_4')_1 = \Sigma(y - \mathbf{y}_4') - n_{14}\Delta\mathbf{y}_4' \qquad (89)$$

and similarly for P_{42}, P_{43}, etc.

$$\begin{aligned} K_{14} &= \Sigma(x - \mathbf{x}_1)_4\,(y - \mathbf{y}_4)_1 = \Sigma(x - \mathbf{x}_1' - \Delta\mathbf{x}_1')_4\,(y - \mathbf{y}_4' - \Delta\mathbf{y}_4')_1 \\ &= \Sigma(x - \mathbf{x}_1')_4\,(y - \mathbf{y}_4')_1 - \Delta\mathbf{y}_4'\Sigma(x - \mathbf{x}_1')_4 - \Delta\mathbf{x}_1'\Sigma(y - \mathbf{y}_4')_1 + \\ &\qquad n_{14}\,(\Delta\mathbf{x}_1')\,(\Delta\mathbf{y}_4') \qquad (90) \end{aligned}$$

and similarly for K_{15}, K_{16}, etc.

Applying the foregoing corrections to the numerical values, and substituting these corrected values in equations (29) to (40), gives

$$\mathbf{A} = 12.3778 + \frac{1}{9}(2d + 5f) - \frac{1}{9}\{2.6644R_4 + 0.8000R_5 - 0.4000R_6\} \qquad (91)$$

$$\mathbf{B} = 12.4667 + \frac{1}{9}(2d + 3f) - \frac{1}{9}\{-0.2356R_4 + 0.5000R_5 + 1.4000R_6\} \qquad (92)$$

$$\mathbf{C} = 8.9333 + \frac{1}{9}(5d + 1f) - \frac{1}{9}\{-2.3890R_4 - 1.3000R_5 - 1.0000R_6\} \qquad (93)$$

$$\mathbf{D} = 9.6889 + \frac{1}{9}(2a + 5c) - \frac{1}{9}\{-0.6290R_1 - 0.7244R_2 + 5.6335R_3\} \qquad (94)$$

$$\mathbf{E} = 11.6778 + \frac{1}{9}(2a + 3c) - \frac{1}{9}\{1.9710R_1 - 0.9488R_2 - 3.9199R_3\} \qquad (95)$$

$$\mathbf{F} = 12.4111 + \frac{1}{9}(5a + 1c) - \frac{1}{9}\{-1.4225R_1 + 1.8134R_2 - 1.7733R_3\} \qquad (96)$$

$$R_1 = 0.2763 + \frac{-0.6290d - 1.4225f + 0.8463R_4 - 0.9956R_5 - 1.9522R_6}{13.2822} \qquad (97)$$

$$R_2 = -0.8264 + \frac{-0.7244d + 1.8134f - 0.1723R_4 - 2.0639R_5 + 3.3689R_6}{7.8556} \tag{98}$$

$$R_3 = -1.2081 + \frac{5.6335d - 1.7733f + 3.3324R_4 - 0.5667R_5 - 1.7667R_6}{33.1200} \tag{99}$$

$$R_4 = 2.0965 + \frac{2.6644a - 2.3890c + 0.8463R_1 - 0.1723R_2 + 3.3324R_3}{5.9156} \tag{100}$$

$$R_5 = 0.1687 + \frac{0.8000a - 1.3000c - 0.9956R_1 - 2.0639R_2 - 0.5667R_3}{3.3200} \tag{101}$$

$$R_6 = 1.0161 + \frac{-0.4000a - 1.0000c - 1.9522R_1 + 3.3324R_2 - 1.7667R_3}{10.5400} \tag{102}$$

These equations are solved by successive approximation exactly as indicated in table 4. The final values so obtained are

$$\mathbf{A} = 12.38 - 0.31 = 12.07 \tag{103}$$
$$\mathbf{B} = 12.47 - 0.29 = 12.18 \tag{104}$$
$$\mathbf{C} = 8.93 - 0.12 = 8.81 \tag{105}$$
$$\mathbf{D} = 9.69 + 2.64 = 12.33 \tag{106}$$
$$\mathbf{E} = 11.68 + 0.50 = 12.18 \tag{107}$$
$$\mathbf{F} = 12.41 + 0.34 = 12.75 \tag{108}$$
$$R_1 = 0.28 + 0.03 = 0.31 \tag{109}$$
$$R_2 = -0.83 + 0.36 = -0.47. \tag{110}$$
$$R_3 = -1.21 + 0.00 = -1.21 \tag{111}$$
$$R_4 = 2.10 - 2.09 = 0.01 \tag{112}$$
$$R_5 = 0.17 - 1.00 = -0.83 \tag{113}$$
$$R_6 = 1.02 - 0.33 = 0.69 \tag{114}$$

Utilizing these values in the most expedient way, the functional relations are conveniently expressed as

$$w = f_1(x) + f_2(y) = 12.18 + F_1(\mathbf{x}) + F_2(\mathbf{y}) + R_x(x - \mathbf{x}) + R_y(y - \mathbf{y}) \tag{115}$$

where $\mathbf{x}$ and $\mathbf{y}$ signify any one of the three group averages of x and y; where R_x represents any one of the three regression coefficients R_1, R_2, and R_3, and R_y any one of R_4, R_5, and R_6; and, where $F_1(\mathbf{x})$ and $F_2(\mathbf{y})$ are defined in table 8.

TABLE 8.

DEFINITION OF FUNCTIONAL RELATIONS.

x		$F(\mathbf{x})$	R^x	y		$F(\mathbf{y})$	R^y
mean	limits			mean	limits		
$\mathbf{x}_1 = 62.95$	60.4–64.2	$-0.11 = -a$	$R_1 = 0.31$	$\mathbf{y}_4 = 4.32$	3.5– 6.0	$0.15 = -d$	$R_4 = 0.01$
$\mathbf{x}_2 = 65.27$	64.2–66.9	0	$R_2 = -0.47$	$\mathbf{y}_5 = 6.90$	6.0– 8.1	0	$R_5 = -0.83$
$\mathbf{x}_3 = 69.27$	67.0–73.4	$-3.37 = -c$	$R_3 = -1.21$	$\mathbf{y}_6 = 9.10$	8.1–11.6	$0.57 = -f$	$R_6 = 0.69$

The value of w corresponding to $x = 73.4$ and $y = 3.6$, may be computed, as in the case just considered, by correcting first for the group, which, from equation (115) and table 8 is

$$w = 12.18 - 3.37 + 0.15 = 8.96 \tag{116}$$

which corresponds to $x = \mathbf{x}_3 = 69.27$ and $y = \mathbf{y}_4 = 4.32$. Introducing the regression coefficients to correct for the position within the group; $R_3(73.4 - \mathbf{x}_3)$ and $R_4\ (3.6 - \mathbf{y}_4)$ must be added; whence

$$w = 8.96 - 5.00 - 0.01 = 3.95 \tag{117}$$

which agrees still better[1] with the observed value of 4.0 than that given by equation (83).

Solution by the *slope* method is effected, after introducing the corrections indicated by equations (88) and (89), by replacing equations (35) to (40) with equations (41) to (48) and substituting S_1, S_2, S_3, S_4, S_5, and S_6 for R_1, R_2, R_3, R_4, R_5, and R_6, in equations (29) to (34).

$$\mathbf{A} = 12.38 + \frac{1}{9}(2d + 5f) - \frac{1}{9}\{2.66(0.597d + 0.246f) \\ + 0.8(0.179d - 0.246f) - 0.4(-0.179d - 0.664f)\} \tag{118}$$

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[1] . . . This agreement between computed and observed values is, of course, accidentally close. Considering the observed series as a whole, the greatest deviation between computed and observed values is 7.5 by Blair's method, 6.3 by the first method of successive approximation, and 5.3 by the second method; while the "probable error" of a single computed value in each case lies between 1 and 2 (see page 132).

$$\mathbf{B} = 12.47 + \frac{1}{9}(2d + 3f) - \frac{1}{9}\{-0.236(0.597d + 0.246f) + 0.5(0.179d - 0.246f) + 1.4(-0.179d - 0.664f)\} \quad (119)$$

$$\mathbf{C} = 8.93 + \frac{1}{9}(5d + 1f) - \frac{1}{9}\{-2.389(0.597d + 0.246f) - 1.3(0.179d - 0.246f) - 1.0(-0.179d - 0.664f)\} \quad (120)$$

$$\mathbf{D} = 9.69 + \frac{1}{9}(2a + 5c) - \frac{1}{9}\{-0.629(0.590a + 0.092c) - 0.724(0.273a - 0.092c) + 5.633(-0.273a - 0.408c)\} \quad (121)$$

$$\mathbf{E} = 11.68 + \frac{1}{9}(2a + 3c) - \frac{1}{9}\{1.971(0.590a + 0.092c) - 0.949(0.273a - 0.092c) - 3.920(-0.273a - 0.408c)\} \quad (122)$$

$$\mathbf{F} = 12.41 + \frac{1}{9}(5a + 1c) - \frac{1}{9}\{-1.422(0.590a + 0.092c) + 1.813(0.273a - 0.092c) - 1.773(-0.273a - 0.408c)\} \quad (123)$$

These equations reduce to

$$\mathbf{A} = 12.38 + 0.0218d + 0.4750f \quad (124)$$

$$\mathbf{B} = 12.47 + 0.2560d + 0.4567f \quad (125)$$

$$\mathbf{C} = 8.93 + 0.720d + 0.0671f \quad (126)$$

$$\mathbf{D} = 9.69 + 0.4562a + 0.8100c \quad (127)$$

$$\mathbf{E} = 11.68 + 0.0029a + 0.1258c \quad (128)$$

$$\mathbf{F} = 12.41 + 0.5402a + 0.0637c \quad (129)$$

which are of the same form as equations (50) to (55). The solution gives $\mathbf{A} = 12.13$, $\mathbf{B} = 12.13$, $\mathbf{C} = 8.60$, $\mathbf{D} = 12.55$, $\mathbf{E} = 12.13$, and $\mathbf{F} = 12.64$; and, from equations (41) to (48), $S_1 = 0.32$, $S_2 = -0.32$, $S_3 = -1.44$, $S_4 = -0.376$, $S_5 = 0.05$, and $S_6 = 0.413$.

Using these relations to compute the value of w corresponding to $x = 73.4$ and $y = 3.6$ gives

$$w = 12.13 - 3.53 + 0.42 - 5.95 + 0.27 = 3.34 \quad (130)$$

a fair approximation to the value 3.95 given when regressions in each group were used.

5. SUPPLEMENTARY CONSIDERATIONS.

For clearness in presenting the central idea of this method of successive approximation, and for convenience in the analytic demonstrations and concrete illustrations, certain matters were barely mentioned that require further consideration. They are: first, manner of grouping; second, basis of mathematical reasoning; third, reliability of results; and fourth, meaning of end results.

Manner of grouping.

It is desirable so to group the values of each independent variable as to equally distribute the effect of "chance" on the corresponding average of the dependent variable. Among other things, this effect of chance increases in any particular group as the number of entries in that group decreases, whence, in order to distribute it equally, each group should contain approximately the same number of entries. But, this effect of chance in any particular group also depends upon the range in value of the independent variable within that group, and, for this reason, the entries should be so grouped as to make this range the same in each group. In practice, however, it is usually impossible to group the data so as to realize, even approximately, both of these conditions. Thus, in the illustrative wheat problem (see table 5), although the first condition is completely realized by dividing the twenty-seven entries into three groups of nine entries each with respect to both temperature and precipitation, the second condition is not, the temperature range being 3.8° F. in group 1, 2.7° F. in group 2, and 6.4° F. in group 3, and the precipitation range being 2.5 inches in group 4, 2.1 inches in group 5, and 3.5 inches in group 6. When regressions are used it is probably best to meet the first requirement at the expense of the second, as is here done, but when variability within the group is neglected, this is not so evident. No general rule can be stated, and choice of the number of groups and number of entries in each is a matter of judgment, just as in fitting a parabolic function

$$w = a_0 + a_1x + a_2x^2 + \dots \quad + b_1y + b_2y^2 + \dots$$

to empirical data, the number of terms to retain is a matter of judgment. In either case the final result is fixed as soon as the choice is made.

After deciding upon the number of groups and number of entries in each, it is not uncommon to find that the last value of the independent variable in one group equals the first value in the succeeding group. This is the case, for example, in the illustrative wheat problem where the highest temperature in group 1 (see table 5) is 64.2° F. which is also the lowest temperature in group 2. In such cases it might seem better to vary the number of entries in the groups so as to bring the point of division between different values of the independent variable, *e. g.*, in the particular case cited, to increase the number of entries in group 1 and decrease that in group 2 by 1 so that the highest temperature in group 1 would be 64.2° F. and the lowest in group 2 would be 64.4° F. However, this is only an apparent advantage, for, in the case when variability within the group is neglected the error introduced depends primarily upon the range in value of the independent variable within the group and not upon the precise point of division. The latter is comparatively insignificant, and the little effect it does have in the end results decreases as the number of entries per group increases and as the range in value of the independent variable decreases. Finally, in the case when regressions are used the effect of a change in the point of division is almost if not quite eliminated.

Basis of mathematical reasoning.

As stated on page 100, the special form of expression, upon which the mathematical demonstration of the method of successive approximation is based, implies that the change in the dependent variable, w, corresponding to a given change in any independent variable is negligibly affected by the magnitude of the constant values to which the remaining independent variables are reduced. Stated in the concrete terms of the wheat problem, the assumption is that a change of say two degrees in temperature increases the wheat yield by essentially the same amount whether the precipitation is 3.5 or 11.6 inches. It is evident that this involves a second assumption; that the change in w is approximately independent of the magnitude of w, or, expressed in terms of the wheat problem, that an increase in temperature, say from 62.9° F. to 65.3° F., increases the wheat yield by approximately 0.11 bushels per acre (see table 7), irrespective of whether the yield corresponding to 62.9° F. is 4 or 17 bushels per acre. Although both of these assumptions are likewise inherent in the method of multiple correlation, which, as a matter of fact, is but a special case of this more general method of successive approximation, the question at once arises: to what extent are these assumptions valid, and how may one proceed in any particular case when they are known not to be valid?

This is not an easy question to answer. However, the validity of the second assumption obviously depends upon distribution of the values of the dependent variable. If they are distributed in approximate accordance with the Gaussian, or normal law of error, the second assumption may be regarded as true, while, if their distribution deviates widely from this law, it is clear that this assumption is not valid. In such cases it is desirable to find some function of the dependent variable that is distributed in accordance with the Gaussian law, and deal with that function. For example, in many, if not most, biological problems, the change in w is not independent of the magnitude of w, but approximately proportional to it, and the logarithm of w, rather than w itself, is distributed in accordance with the Gaussian law. Accordingly, the proper form of expression, upon which to base the mathematical reasoning, becomes $(w + k) = f_1(x) \times f_2(y) \times f_3(z)$ etc., where, for greater generality, the constant k is introduced. But, this expression may be written

$$log(w + k) = log\, f_1(x) + log\, f_2(y) + log\, f_3(z) + \dots \qquad (131)$$

and putting $W = log(w + k)$, $F_1 = log\, f_1$, $F_2 = log\, f_2$, etc., gives

$$W = F_1(x) + F_2(y) + F_3(z) + \dots \qquad (132)$$

which is of the same form as that given by equation (1). In general, then, the nature of the frequency distribution of the dependent variable, affords an empirical criterion for determining the validity of the second assumption, and for suggesting what function of w to use to make this assumption valid.

But this does not wholly eliminate the difficulty. For although the first assumption is, as a rule, valid whenever the second one is, this is not necessarily the case. Furthermore, there seems to be no criterion in the data themselves for determining this. However, in any case, after the "standard" values are selected, the ascertained change in the dependent variable corresponding to a given change in any particular independent one, is the change that would take place under average conditions of the remaining independent variables. It is obvious, then, that, when the range in value of the independent variables is small, the error introduced by this assumption is correspondingly small. Therefore, by classifying the data so as to restrict the range of those independent variables with respect to which the assumption does not accord well with fact, and applying the method of successive approximation separately to each portion of the data thus segregated, the nature and magnitude of the inherent error can be found, and approximately eliminated.

Finally, the method of group averages involves no assumption as to the nature of the correlation between the independent variables. Consequently, the ascertained relation, say of w to x is not influenced by the way x may be correlated with y, z, etc., in the data used, and values of w computed on the basis of these ascertained relations are quite as reliable for any one particular combination of values of the independent variables as for any other combination within the ranges covered by the data. On the other hand, any method based upon assumed forms of functional relations is likely to lead to erroneous conclusions as to the relative importance of the various independent variables unless the assumed forms fit the data well. Values of w computed from results based upon unsuitable functional forms may agree well with those observed when the relation between the independent variables accords well with that occurring on the average in the data from which the results were obtained, but computed and observed values disagree widely when this relation differs materially from that prevailing in the original data. In other words, the error inherent in the initial assumption is so distributed among the various independent variables as to give the best possible fit to the data as a whole for the functions used, and this fit may seem accurate and still be highly artificial.

For example, application of the group method to the wheat problem (see tables 7 and 8) shows clearly that the relation of yield to temperature and precipitation was not linear, and indicates that temperature was a much more important factor than precipitation. Consequently, if those values of the yield corresponding to a narrow temperature range and a wide range of precipitation be computed from temperature data alone the error should be nearly the same as when both temperature and precipitation data are used. In eleven cases the temperature lies between 63.7° F. and 65.0° F., while the precipitation varies from 3.7 to 11.6 inches. The standard deviation of the differences between observed values of the yield and those computed from temperature data alone is 2.03 for the multiple linear correlation method, and 1.96 for the *slope* method. Introducing the correction for precipitation decreases the standard deviation in the first case by only 0.01 and increases it in the second case by only 0.01, thus agreeing with expectation that no significant change would result. But the standard deviation of the differences between observed and computed yields for those data having a large temperature range and a small precipitation range should be materially less when based upon both temperature and precipitation, than when based upon the latter alone. Moreover, reduction of the standard deviation effected by introducing the correction for temperature should be materially greater for the *slope* method than for that of multiple linear correlation. In eight instances the temperature ranges from 62.6° F. to 73.4° F., while the precipitation varies only from 3.5 to 5.3 inches. The standard deviation is reduced by 2.85, or from 4.94 to 2.09, for the *slope* method by introducing the correction for temperature, while for the method of multiple linear correlation, the standard deviation is reduced by only 0.96, or from 3.53 to 2.57. These results are not only in agreement with expectation but clearly indicate that part of the actual effect of temperature on yield is attributed by the method of multiple linear correlation to precipitation. To further test the truth of this indication, each value of the wheat yield given in table 5 was computed on the basis of the temperature relation ascertained by the *slope* method, and a linear correlation run between the residuals and precipitation. As a result the precipitation-yield correlation coefficient is reduced from + 0.22 to + 0.015.

Reliability of results.

Experience proves that, after all practicable efforts are made toward controlling or correcting variations attributable to the independent variables, some deviation between computed and observed values of the dependent variable always remains. Although the magnitude of these deviations is an index of the accuracy with which the empirical relations describe the data at hand, one usually needs an estimate of the reliability with which these empirical relations describe the whole "universe" of which those data are a sample. Accordingly, the complexity of these relations must be taken into account. This need is particularly evident in the method of group averages, for, while the empirical relations describe the data at hand with increasing precision as the number of groups approaches the number of observations, reliability of this description decreases until, in the limiting case, it is no greater than that afforded by the isolated observations themselves. It is evident, then, that in estimating reliability, the actual number of observations should be reduced by an amount depending on the total number of groups chosen to express the empirical relation, and, for this purpose, each group in which a regression is used is equivalent to two groups. An approximate rule is to add the number of independent variables less 1 to the total number of observations and to subtract from this sum the number of groups plus the number of regression coefficients. A similar procedure is followed in the method of least squares when, in determining the "probable error" of a value computed by a formula having m unknown coefficients, the number of observations is decreased by m.

Applying this rule to determine the reliability of the estimated yield of wheat, the reduced number of observations is $27 - 3 = 24$ for the multiple linear correlation method; $27 - 5 = 22$ for the group method used when variability within the group is neglected, and also for the *slope* method; and $27 - 11 = 16$ for the group method in which regressions are used. The reliability for each of the four methods, given by the probable error of a single observation, is

$$0.6745\sqrt{\frac{151.2}{24}} = \pm 1.69,\quad 0.6745\sqrt{\frac{139.6}{22}} = \pm 1.70,$$

$$0.6745\sqrt{\frac{129.6}{22}} = \pm 1.63;\ \text{and}\ 0.6745\sqrt{\frac{123.4}{16}} = \pm 1.87.$$

In case the results given by any one of the three group methods are "smoothed" by any process, a simpler relation is obtained which justifies a corresponding increase in the number of observations on which to base estimates of reliability. For example, in the case when regressions are used, the functional relation is given independently in two ways; first, by the series of corresponding averages of dependent and independent variable; and, second, by the regression coefficients and corresponding values of the independent variable. Accordingly, if the functional relations defined in these two ways be averaged, subtraction of the number of regression coefficients from the total number of observations would not be required to obtain the reduced number for estimating reliability, and the probable error of ± 1.87 would be correspondingly reduced. The *slope* method amounts to averaging these relations at the outset.

Meaning of end results.

In order to visualize the results obtained by this method of successive approximation, each functional relation may be represented graphically. In the case when regressions are computed by the usual method, or estimated from the averages, the functional relation of say w to x is approximately represented by a series of straight lines, having slopes equal to the regression coefficients and drawn through points whose respective coordinates are $\mathbf{w}_1$, $\mathbf{x}_1$; $\mathbf{w}_2$, $\mathbf{x}_2$, etc. As the number of observations and groups increases each point tends to coincide with a point on the curve corresponding to the true functional relation, and each line tends to coincide with the tangent at that point. In case regressions are not used, the result is simply a series of points determining the curve.

Lastly, as is the case in applying any statistical method, the end result gives the relation of dependent variable not to each measured independent one, as such, but to a correlated system of unobserved variables which each measured one represents. In other words each observed variable is an index of a certain complex consisting of a number of unknown elements, and a statistical method merely serves to eliminate the effects of all but one of these complexes at a time and, by synthesis, to determine the effect of any combination of them. The method of successive approximation offers no exception to this rule, and, in order to further disentangle these complexes, additional variables must be measured and the number of observations increased.

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CONTRIBUTION TO THE GENERAL KINETICS OF MATERIAL TRANSFORMATIONS.

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CONTRIBUTION TO THE GENERAL KINETICS OF MATERIAL TRANSFORMATIONS.

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F. JÜTTNER[1] has given a general discussion of the differential equations of chemical dynamics, in so far as they apply to a single chemical reaction taking place in accordance with the law of mass action.

A certain interest attaches to a still more general treatment which should make us independent of the law of mass action, and which should cover the wider field of concurrent reactions of various types. Such a treatment is therefore given below.

The systems with which we have to deal comprise a number of components $S_1\ S_2 \ldots S_n$, whose mass at any instant may be denoted by $X_1\ X_2 \ldots X_n$. Observational data furnish directly or indirectly a system of differential equations[2]

$$\left.\begin{aligned} \frac{dX_1}{dt} &= F_1(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j) \\ \frac{dX_2}{dt} &= F_2(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dX_n}{dt} &= F_n(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j) \end{aligned}\right\} \qquad (1)$$

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[1] Zeitschr. f. phys. Chemie, 1911, v. 77, p. 735. See also R. Marcelin, Ann. de Phys. 1895, v. 3, p. 120.

[2] The system of equations (1) although very general, does not, however, cover cases in which geometric factors (e. g. diffusion effects) play a dominant rôle. Such cases are therefore excluded from our present considerations. Certain reactions in heterogeneous systems, nevertheless, do fall within the scope of this discussion, namely those the velocity of which is small as compared with the rate of diffusion, so that the course of the reaction is determined practically by the reaction velocities alone, irrespective of diffusion velocity.

A class of reactions which, perhaps, may not always fall within the scope of equations of the type (1) and (2) are so-called induced, sympathetic or coupled reactions. In such cases it may be that the velocities $\frac{dX}{dt}$ can not be represented as functions of the X's alone, but that the velocities $\frac{dX}{dt}$ themselves must appear explicitly in these functions.

The quantities $P_1\, P_2 \ldots P_j$ which occur in (1) are certain parameters, such as for example the volume and the temperature[3] of the system in course of transformation. The case commonly considered is that in which these parameters P are kept constant (or, as in radio-active transformations, the case in which pressure, temperature, etc. are without influence upon the transformation). We shall for the present at any rate, adopt this customary convention, which in point of fact represents the case of prime interest both in theory and in practice. The parameters P being constant, we may, for our present purposes, omit them from the functional parentheses in (1), so that we have, in effect,

$$\left.\begin{aligned} \frac{dX_1}{dt} &= F_1(X_1,\, X_2, \ldots X_n) \\ \frac{dX_2}{dt} &= F_2(X_1,\, X_2, \ldots X_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dX_n}{dt} &= F_n(X_1,\, X_2, \ldots X_n) \end{aligned}\right\} \qquad (2)$$

It is expedient to introduce new variables as follows:

Putting

$$F_1 = F_2 = \ldots = F_n = 0 \qquad (3)$$

we obtain[4]

$$\begin{aligned} X_1 &= C_1 \\ X_2 &= C_2 \\ &\ldots\ldots \\ X_n &= C_n \end{aligned} \qquad (4)$$

We will write

$$x_i = X_i - C_i \qquad (5)$$

and introduce x in place of X in (2).

We then have:

$$\frac{dx_i}{dt} = f_i(x_1,\, x_2, \ldots x_n) \qquad (6)$$

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[3] The choice of the parameters P is more or less arbitrary, but in any case the general form of the equations (1), (2) is independent of the particular choice made.

[4] This system of equations will in general have a number of solutions, corresponding to so many different states of equilibrium. In the cases ordinarily considered in physical chemistry, however, one of these solutions is of special or sole interest.

In the expansion of the right-hand member of (6) by Taylor's theorem the absolute term must be zero, in view of (5), hence

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + \dots \\ \frac{dx_2}{dt} &= a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n + \dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{dx_n}{dt} &= a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n + \dots \end{aligned}\right\} \quad (7)$$

A general solution of this system (7) is [5]

$$\left.\begin{aligned} x_1 = {} & \alpha_1' e^{\lambda_1 t} + \alpha_2' e^{\lambda_2 t} + \dots + \alpha_n' e^{\lambda_n t} \\ & + \alpha_{11}' e^{2\lambda_1 t} + \alpha_{22}' e^{2\lambda_2 t} + \dots + \alpha_{nn}' e^{2\lambda_n t} \\ & + \alpha_{12}' e^{(\lambda_1+\lambda_2)t} + \dots \\ x_2 = {} & \alpha_1'' e^{\lambda_1 t} + \alpha_2'' e^{\lambda_2 t} + \dots \end{aligned}\right\} \quad (8)$$

where $\lambda_1, \lambda_2, \dots \lambda_n$ are the n roots [5a] of the equation of n^{th} degree

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \dots a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} \dots\dots & \dots a_{nn}-\lambda \end{vmatrix} = \Delta(\lambda) = 0 \quad (9)$$

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[5] See Picard, Traité d'Analyse, 1908, v. 3, p. 14; Forsyth, Theory of Differential Equations, 1900, v. 3, pp. 2, 8, 9; Königsberger, Lehrbuch der Theorie der Linearen Differentialgleichungen, 1889, p. 283.

[5a] The quantities λ in (8), (9) are subject to certain restrictions. See Picard, loc. cit., pp. 9, 10; 17, 18. The method of determination of certain of the constants α in (8) also breaks down in the special case that two or more of the roots of (9) are equal, or differ only by an integral factor.

We shall not here discuss the case of multiple roots of equations (9), which presents no particular difficulty (see Königsberger loc. cit.; Lotka, Zeitschr. f. Phys. Chemie, 1912, v. 80, p. 161); nor the case in which the series in the right hand members of (7) contain no terms of the first degree. This case does not appear to admit of general treatment. Certain special cases have been treated by Picard, Poincaré, Dulac, Bendixon, Jordan. (See Dulac, Jl. Ec. Polytechn. IX cahier 1904; Bul. Soc. Math. 1906; Bendixon, Acta Math. 24; Jordan, Jl. Math. 1906). It should be noted that if the series (8) begin with the terms of second degree, for example, a change in *sign only* of all the variables x leaves the velocities $\frac{dx}{dt}$ unaltered, for small displacements. Such a state of affairs does not correspond to the nature of the physical systems and processes here under discussion. This special case which presents certain mathematical difficulties, is therefore also a case of no practical interest for our present discussion.

The law of expansion of the determinant Δ (λ) is readily recognized if we use Cauchy's notation, in which a determinant is defined by writing down the terms of its dexter diagonal enclosed in parentheses. In the case that $n = 4$, for example, we have[6]

$$\begin{aligned} &\lambda^4 - \{a_{11} + a_{22} + a_{33} + a_{44}\}\lambda^3 \\ &\quad + \{(a_{11}\,a_{22}) + (a_{11}\,a_{33}) + (a_{11}\,a_{44}) + (a_{22}\,a_{33}) + (a_{22}\,a_{44}) + (a_{33}\,a_{44})\}\lambda^2 \\ &\quad - \{(a_{11}\,a_{22}\,a_{33}) + (a_{11}\,a_{22}\,a_{44}) + (a_{11}\,a_{33}\,a_{44}) + (a_{22}\,a_{33}\,a_{44})\}\lambda \\ &\qquad\qquad + (a_{11}\,a_{22}\,a_{33}\,a_{44}) = 0 \end{aligned} \tag{10}$$

To determine the coefficients α', α'' .... we substitute in (7)

$$\left.\begin{aligned} x_1 &= \alpha'_i \quad e^{\lambda_i t} \\ x_2 &= \alpha''_i \quad e^{\lambda_i t} \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{11}$$

Equating coefficients of homologous terms we then obtain

$$\left.\begin{array}{llll} \alpha'_i\,(a_{11} - \lambda_i) & + \alpha''_i\,a_{12} & + \alpha'''_i\,a_{13} & + \ldots = 0 \\ \alpha'_i\,a_{21} & + \alpha''_i\,(a_{22} - \lambda_i) & + \alpha'''_i\,a_{23} & + \ldots = 0 \\ \alpha'_i\,a_{31} & + \alpha''_i\,a_{32} & + \alpha'''_i\,(a_{33} - \lambda_i) & + \ldots = 0 \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \end{array}\right\} \tag{12}$$

The equations (12) are not independent, seeing that Δ (λ) vanishes according to (9).

The trivial solution (13)

$$\alpha'_i = \alpha''_i = \alpha'''_i = \ldots = 0 \tag{13}$$

of (12) is of no interest. The existence of other solutions is assured by the vanishing of Δ (λ) according to (9). One of the constants α'_i, α''_i, ... remains arbitrary, and the proportion $\alpha'_i : \alpha''_i : \alpha'''_i : \ldots$ is then determined in the well-known manner by the homogeneous linear system of equations (12). That is to say, if we form the determinant of the coefficients of $\alpha'_i, \alpha''_i, \alpha'''_i, \ldots$ in the left-hand members of (12), and if we denote by $\Delta_{\alpha'_i}, \Delta_{\alpha_i}, \Delta_{\alpha'''_i}, \ldots$ the minors of the coefficients of $\alpha'_i, \alpha''_i, \alpha'''_i \ldots$ in any one selected row, then

$$\alpha'_i : \alpha''_i : \alpha'''_i : \ldots\ldots :: \Delta_{\alpha'_i} : \Delta_{\alpha''_i} : \Delta_{\alpha'''_i} : \ldots\ldots \tag{14}$$

---

[6] See, for example, Czuber, Einführung in die höhere Mathematik, p. 178. The symbols enclosed in round parentheses denote the determinants of which these symbols form the dexter diagonal.

The coefficients $\alpha'_{11}, \alpha''_{11}, \ldots$ of the terms of second and higher degree are determined in analogous manner.

Equations (3), which have so far been introduced merely as a mathematical auxiliary, have an obvious physical significance. They represent the conditions for a steady state or an equilibrium.

Now, if $X_1, X_2 \ldots X_n$ are entirely independent, it will be seen that (3) completely determines the equilibrium. Physically this means that in such a case, when the values of the parameters P (e. g. volume, temperature) were once fixed, there would then be no further freedom whatever in fixing the equilibrium. This latter would then, for example be wholly independent of the masses of the several components of the system.

This does not correspond to the actual conditions ordinarily met with in cases of concrete interest. In such cases there are commonly given, in addition to the differential equations (1), which express the general kinetics of the system, a further set of *equations of constraint*[7] of the form

$$\Phi\,(X_1,\ X_2, \ldots X_n) = 0 \qquad (15)$$

In physico-chemical systems the equations of constraint are commonly derived directly from the *reaction equations*, of which, according

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[7] Ordinarily the equations of constraint derived from the reaction equation contain the stoichiometric constants of those equations. However, in the case of certain simultaneous reactions, in which the products of the several reactions are formed in constant proportion, the function Φ will contain also certain other constants defining this proportion and not themselves defined by the stoichiometric properties of the system.

In the case most frequently dealt with in physical chemistry, the case of a system of constant total mass, one of the equations of constraint takes the form

$$\Sigma\, X = \text{const.}$$

Furthermore, the immutability of the chemical elements (in ordinary reactions) furnishes for each element e an equation of constraint.

$$\Sigma\, m_e = \text{constant}$$

where m_e denotes the total mass of the element e present in the system.

In special cases there may be constraints of entirely different character. Thus, in certain technical processes a stream of gas is passed over a catalytic substance in such manner that an approximately constant mass of the products of reaction issue per unit of time.

In such case, in addition to equations of type (1) there will be given one or more equations of the form

$$\frac{dX}{dt} = \text{constant} = F\,(C_1,\ C_2, \ldots C_n)$$

where $C_1, C_2, \ldots C_n$ are the (constant) concentrations of the substances S_1 $S_2, \ldots S_n$ entering the reaction chamber.

Cases of this kind do not fall within the scope of our present reflection. Their theoretical and practical treatment presents, however, no particular difficulties.

to circumstances, there may be one or more applying to the system under consideration. They commonly involve the initial values A_1, $A_2, \ldots A_n$ of X_1, $X_2, \ldots X_n$ as parameters, so that they are of the general form

$$\Phi\,(X_1, X_2, \ldots X_n;\ A_1, A_2, \ldots A_n) = 0 \tag{16}$$

The effect of such equations of restraint is to cause the determinant $\Delta\,(0)$ to vanish.[8]

For, in view of (2), (5), and (7) we may write (15) in the form

$$\varphi\,(\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n) = 0 \tag{17}$$

which implies that

$$J = \frac{\partial\,(\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n)}{\partial\,(x_1, x_2, \ldots x_n)} = 0 \tag{18}$$

But inspection of (7) shows that

$$a_{rs} = \left(\frac{\partial\,\dot{x}_r}{\partial\,x_s}\right)_o \tag{19}$$

and that therefore

$$\Delta\,(0) = J_o = 0 \tag{20}$$[9]

That is to say, the existence of equations of constraint causes the determinant $\Delta\,(0)$ to vanish, as was stated above.

Furthermore, the vanishing of $\Delta\,(0)$ means that at least one of the roots λ of (9) is zero. Since the left-hand members of (8) can not contain an absolute term, it follows that at least one of the coefficients α is thus no longer arbitrary, but is fixed at the value 0.

This loss of an arbitrary constant in (8) must be in some way compensated. This compensation is effected by the equation of constraint (15), which introduces as an arbitary constant the initial value A of one of the variables X.

Similarly it can be seen that m equations of constraint (15) cause the appearance of m zero roots in (9) and of m arbitrary constants in the form of initial values of variables X.

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[8] $\Delta\,(0)$ is the determinant obtained by putting $\lambda = 0$ in the right-hand member of (9).

[9] The subscript o in equation (19) and (20) signifies that the values of $\frac{\partial \dot{x}}{\partial t}$ to be taken are those corresponding to the origin $x_1 = x_2 = \ldots = x_n = 0$

In such case, then, the equilibrium or steady state is no longer fixed by (1) as soon as the values of the parameters $P_1, P_2, \ldots P_j$ are given, but the equilibrium further requires, for its definition, the statement of the initial values $A_1, A_2, \ldots A_m$ of the variables $X_1, X_2, \ldots X_m$. That is to say, the equations of equilibrium (3) take the form

$$\left.\begin{aligned} & F_1 (A_1, A_2, \ldots A_m;\ X_{m+1} \ldots X_n) \\ = {} & F_2 (A_1, A_2, \ldots A_m;\ X_{m+1} \ldots X_n) \\ = {} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ = {} & F_n (A_1, A_2, \ldots A_m;\ X_{m+1} \ldots X_n) = 0 \end{aligned}\right\} \qquad (21)$$

the $X_1, X_2, \ldots X_m$ having been eliminated by virtue of the equations of constraint (15), with introduction, in their place, of the initial constants $A_1, A_2, \ldots A_m$. Equilibrium is now free to be influenced by the arbitrary selection of the initial values of m of the masses X of the components of the system. This is the case of common interest in physico-chemical transformations.

It has been shown that, in the case where equations of constraint exist between the variables X, the polynomical Δ (λ) contains the factor λ, the determinant Δ (0) vanishing. In fact, it is evident that the differential equation of n^{th} order in x which can be derived in well-known manner[10] from the system (7), after rejection of all terms of higher degree than the first, is reducible to an order as much lower than n as there are zero roots for the equation $\Delta (\lambda) = 0$.

However, in practice, instead of operating in this manner upon the primitive system (1) or its equivalents (2) or (6), it is preferable to eliminate from the start m of the variables X by the aid of the equations of constraint (15). There then remain $(n - m)$ dependent variables, which may be either certain of the original variables X

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[10] The elimination of $(n - 1)$ of the variables $x_1 x_2 \ldots x_n$ in the system (7), discarding the terms of second and higher degree, can be effected at sight according to the plan

$$\Delta \left(\frac{d}{dt}\right) x_1 = 0$$

that is to say

$$\begin{vmatrix} a_{11} - \frac{d}{dt}, & a_{12}, & \ldots a_{1n} \\ a_{21}, & a_{22} - \frac{d}{dt}, & \ldots a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots a_{nn} - \frac{d}{dt} \end{vmatrix} x_1 = 0$$

themselves, or certain other variables which it may be convenient to introduce in their stead, and which serve equally to define the state of the system.

Let $Y_1, Y_2, \ldots Y_\nu$ be such variables given that

$$\nu = n - m \tag{22}$$

In place of the primitive system (2) we shall then have a system

$$\left.\begin{aligned}
\frac{dY_1}{dt} &= \Psi_1(Y_1, Y_2, \ldots Y\nu;\ A_1, A_2, \ldots A_m) \\
\frac{dY_2}{dt} &= \Psi_2(Y_1, Y_2, \ldots Y\nu;\ A_1, A_2, \ldots A_m) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{dY_\nu}{dt} &= \Psi_\nu(Y_1, Y_2, \ldots Y_\nu;\ A_1, A_2, \ldots A_m)
\end{aligned}\right\} \tag{23}$$

the $A_1, A_2, \ldots A_m$, initial values of $X_1, X_2, \ldots X_m$, functioning as parameters independent of the time t.

The system (23) yields to a treatment identical with that applied to the system (2) above, and gives under corresponding conditions a general solution of the form (8) containing $(n - m)$ exponential and $(n - m)$ arbitrary multiplicative coefficients, as required by the conditions of the problem.

CONVERGENCE OF THE SERIES (8) AND STABILITY[11] OF THE SYSTEM AT THE ORIGIN OF THE VARIABLES x.

To simplify the discussion let us write

$$\lambda_j = \mu_j + i\nu_j \tag{24}$$

If μ_j is zero, λ_j is a pure imaginary.

We need not discuss the case in which the equation $\Delta\ (\lambda) = 0$ has zero roots, since these can be eliminated by a change of variables by virtue of the equations of constraint.

There remain to be discussed the following cases:

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[11] A detailed discussion of the question of the existence of series solution of equations of the type (1) will be found in Poincaré's memoir "Sur les Courbes Définies par les Equations Différentielles," Jl. mathém. 1886, Ser. 4, v. 2, chapter XVII. See also Encycl. des Sciences Math. t. II, v. 3, fasc. 1.

1. If all the μ's are negative, the series (8) evidently converge for large values of t, and the x's then approach zero as t approaches infinity. In this case, then, the point $x_1 = x_2 = \dots = x_n = 0$ corresponds to a stable condition.

2. If, on the other hand, all the μ's are positive, then the series (8) converge for large values of $-t$, and diverge for large values of t. The origin of the x's in this case corresponds to an unstable condition.

3. If some of the μ's are positive, others negative, the series (8) evidently in general converge neither for large values of t nor for large values of $-t$.

However, in this case we can single out a particular solution

$$x_i = \alpha_i e^{\lambda t} + \alpha_{ii} e^{2\lambda t} + \dots \qquad (25)$$

in which $\mu > 0$; from which it is evident that the state of the system at the origin of the x's is certainly unstable at any rate for certain displacements. Equilibrium at the origin is therefore in this case unstable.

4. If some of the roots λ are pure imaginaries, the presence of even a single positive μ will determine the instability of the equilibrium of the origin. If, on the other hand, those μ's which are not zero, are all negative, then the series may converge for large values of t, and in that case the equilibrium, at the origin of the x's, would be in a certain sense stable; for, after an arbitrary displacement, though the system would not in general return to the origin, it would ultimately travel in a periodic path not containing the origin.[12] That is to say, after the lapse of a certain time the point representing the state of the system would thenceforth never be further away from the origin than a certain finite maximum distance M.

The criterion for the existence of purely imaginary roots is evidently that $\Delta(\lambda)$ and $\Delta(-\lambda)$ have one or more common factors of the form $(\lambda^2 + \rho^2)$, where ρ is a real quantity.

5. If all the λ's are pure imaginaries,[13] (in which case $\Delta(\lambda)$ contains

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[12] To be more exact, the point $x_1, x_2 \dots x_n$ in n — dimensional space, which represents the state of the system, describes such a path.

[13] The existence of purely imaginary roots λ may perhaps seem of theoretical rather than practical interest, since they could occur only when the coefficients in (7) satisfy very particular conditions, and since it appears improbable that these conditions would be satisfied in nature.

However, the case, in no way improbable *a priori*, that the real part μ of λ be very small, in practice will differ but little from the case where $\mu = 0$. For, if μ is sufficiently small, any accidental disturbance of the system (e.g. any accidental change in one of the parameters P) from equilibrium, will set up oscillations with only slight damping, so that the system will be liable to be kept in more or less constant oscillations.

only even powers of λ, with all coefficients of the same sign), then the series (8) may converge for all values of t. In that case the process would be periodic *ab initio*, and the equilibrium at the origin would be stable in the same sense as in case (4) above.

To recapitulate, we may enunciate two propositions, of which one is the converse of the other:

Given that the series (8) are convergent,

1. If none of the real parts μ of the roots λ of the equation $\Delta(\lambda) = 0$ is positive, the system is in stable equilibrium at the origin of the x's.

This stability is absolute if none of the μ's are zero.

It is relative, of the type (4) or (5), if $\Delta(\lambda) = 0$ has one or more roots in which $\mu = 0$.

2. If the system is stable (absolutely or relatively) at the origin of the x's, then the equation $\Delta(\lambda) = 0$ can not have a root λ such that $\mu > 0$.

SPECIAL CASES.

1. Only one dependent variable.

If $n - m = 1$, i. e., if the number of equations of constraint is only one less than the total number of primitive dependent variables X, then the state of the system at any instant is completely defined by statement of the value of one single variable Y, given the values of the initial constants $A_1, A_2, \ldots$ and of the parameters $P_1, P_2, \ldots$ The transformation can in this case be regarded as one single reaction.

The commonest example of this kind is that of one simple chemical reaction, reversible or not, unaccompanied by any simultaneous reaction of any kind (such as side reactions, consecutive reactions, etc.). For the variable Y we may then employ the mass (or concentration) of an arbitrarily selected component, or any convenient multiple thereof.[14]

In this case, if we omit from the functional parenthesis the parameters $A_1, A_2, \ldots$, the system of equations (19) reduces to a single equation.

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[14] In chemical dynamics it is customary to employ for the variable Y in a single reaction the change in concentration, from the origin of time to the instant t, of one or more, chosen arbitrarily, of the components. The relation between the X's, the A's and the Y's, at constant volume, is then

$$Y = A - X.$$

A somewhat different convention is followed by Jüttner loc. cit. See also Mellor, Chemical Statics and Dynamics, 1904, pp. 85, 86; 91, 92.

$$\frac{dY}{dt} = F(Y) \tag{26}$$

or, shifting the origin after the manner of (5), (6),

$$\frac{dy}{dt} = f(y) \tag{27}$$

and hence, by the same procedure as in the case of (7)

$$\frac{dy}{dt} = ay + by^2 + \ldots \tag{28}$$

The solution (8) in this case takes the form

$$y = \alpha_1 e^{\lambda t} + \alpha_{11} e^{2\lambda t} + \alpha_{111} e^{3\lambda t} + \ldots \tag{29}$$

where

$$\lambda = a \tag{30}$$

$$= \frac{\partial \dot{y}}{\partial y} \tag{31}$$

Now if the equilibrium at $y = 0$ is stable, it is evident that we must have

$$\left.\begin{aligned} \frac{\partial \dot{y}}{\partial y} &< 0 \\ a &< 0 \\ \lambda &< 0 \end{aligned}\right\} \tag{32}$$

Hence we conclude that

1. The series (29) converges for large values of t.
2. The variable y ultimately approaches zero asymptotically, since the coefficient a, from the nature of things, can not be complex.

A proof of this asymptotic approach to equilibrium has been given by Jüttner, for the case of a single chemical reaction following the law of mass action. The proof given above is independent of the law of mass action and applies to all cases satisfying only the broad condition that the equilibrium corresponding to $y = 0$ is stable.

SPECIAL CASES (continued).

2. If $n - m > 1$ the transformation can not be regarded as one single reaction. The state of the system at any instant, for given values of the parameters P and A requires for its definition $(n - m)$

$= \nu$ variables $Y_1, Y_2, \ldots Y_\nu$. The process can in this case be regarded as the resultant of $(n - m)$ single transformations proceeding simultaneously, and the state of the system, and in particular the velocities $\frac{dX}{dt}$, are fully defined when we are given the values of the parameters P; the initial values $A_1, A_2, \ldots A_m$ of the variables X (masses of the several components); and the values of the variables $Y_1, Y_2, \ldots Y_\nu$ which tell us how far each reaction has proceeded.

CONSECUTIVE TRANSFORMATIONS.

A special case arises if the $(n - m)$ "single" transformations whose resultant is the actual transformation can be so arranged in order, that the velocity of the first depends solely on Y_1, that of the second on both Y_1 and Y_2, in general that of the j^{th} on $Y_1\ Y_2 \ldots Y_j$. This is the case of what might be termed *purely consecutive* reactions.

The system of equations (7) in this case takes the form:

$$\left.\begin{aligned} \frac{dy_1}{dt} &= a_{11}\, y_1 + \ldots \\ \frac{dy_2}{dt} &= a_{21}\, y_1 + a_{22}\, y_2 + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dy_\nu}{dt} &= a_{\nu 1}\, y_1 + a_{\nu 2}\, y_2 + \ldots + a_{\nu\nu}\, y_\nu + \ldots \end{aligned}\right\} \quad (33)$$

The solution of (33) is here also given by (8); however, it takes on a simplified form owing to certain special properties of the exponential constants λ and the coefficients α, as follows:

A. EXPONENTIAL CONSTANTS.

1. It will be seen that in (33), as compared with (7), every coefficient a_{rs} for which $s > r$ is zero. In consequence of this $\Delta\ (\lambda)$ reduces to the dexter diagonal, and hence

$$\lambda_i = a_{ii}$$

2. Inasmuch as the coefficients a are essentially real, it follows that the λ's also are all real. If therefore equilibrium is established,

the approach to it must be ultimately asymptotic, the presence of trigonometric terms in (8) being excluded by the reality of all the λ's.

Indeed, the purely successive character of the reactions, as defined above, excludes reversible reactions, so that the final mass of certain of the components, when in equilibrium, is zero. Oscillations beyond the equilibrium point are therefore excluded by the physical nature of the variables X (masses), which do not admit of negative values.

3. We have seen that the system will be stable at the origin of the y's, and at the same time the series (8) will converge, provided that all the μ's are negative, i. e. in the present case, provided that

$$a_{ii} < 0 \ (i = 1, 2, \ldots \nu) \tag{34}$$

that is to say, provided that each individual transformation taken by itself tends towards a stable equilibrium (see discussion of equation 32).

B. MULTIPLICATIVE CONSTANTS.

1. The system (33) has the following peculiarity: We may break off at any one of the equations, say the j^{th}, and leave a self-complete system.

From this it is easily seen that the series (8) for y_j must in this case break off at the terms in $\lambda_j = a_{jj}$ or its multiples, and can not, for example contain any terms in $\lambda_{(j+1)}$. The coefficients of these further terms must therefore be zero. This also follows directly by (12).

2. Let $-\lambda_m = -a_{mm}$ be the least of the $-\lambda$'s, all the λ's being negative. Or, in other words, let $|\lambda_m| = |a_{mm}|$ be the least of the $|\lambda|$.

Then it is evident that for sufficiently large values of t all other terms may be neglected in comparison with those in λ_m and of the first degree. In its last phases the process is therefore represented by

$$\left.\begin{array}{l} y_1 = 0 \\ y_2 = 0 \\ \cdots\cdots\cdots\cdots \\ y_{m-1} = 0 \\ y_m = C_o e^{\lambda_m t} \\ y_{m+1} = C_1 e^{\lambda_m t} \\ \cdots\cdots\cdots\cdots \\ y_\nu = C_\nu e^{\lambda m t} \end{array}\right\} \tag{36}$$

so that in the last phases

$$y_1 : y_2 : \ldots : y_{m-1} : y_m : y_{m+1} : \ldots : y_\nu : : 0 : 0 : \ldots : 0 : C_o : C_1 : \ldots : C_\nu \tag{37}$$

That is to say, the quantities y are in constant ratio. Furthermore, if y_m relates to the substance or substances S_m having the least $|\lambda|$, all the substances parent to S_m are entirely absent in the last stages of the transformation.

We recognize here a well-known property of radioactive equilibria, in which the longest-lived substance (e. g. uranium) heads the series, while the other substances are present in constant ratio. However, from the method by which this conclusion has here been reached, it is evident that this property is independent of the particular form of radioactive transformations, but rests on a broader basis and is common to all systems in which is taking place a series of purely successive transformations according to any law whatever.

SPECIAL CASES. (Continued)

Consecutive Reversible[15] *Reactions.*

It may be noted in passing that the conclusions of the last section above are still applicable to a broad class of cases of consecutive reactions, including reversible reactions, provided that at least one among them is irreversible. It will suffice here to point out, by the way of example, one such case, that of five consecutive reactions, of which the second and third only are reversible. It is understood that the reactions considered are otherwise purely consecutive, so that the second, for instance, is wholly independent of the progress of the fourth and fifth.

The determinant Δ (λ) in this case takes the form:

$$\Delta(\lambda) = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & 0 & 0 \\ a_{21} & a_{22}-\lambda & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}-\lambda \end{vmatrix}$$

In this case also it will be found that Δ (λ) reduces to its dexter diagonal, all the roots for λ being in consequence necessarily real. The approach to equilibrium is therefore in this case also asymptotic, and the proportion of the y's is, during the last stages of the process, constant. This last conclusion applies, in fact to all cases in which all the roots λ are real and negative.

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[15] The word "reversible" is here of course not understood in its thermodynamical sense.

GENERAL CASE.

We have considered certain special cases in which Δ (λ) reduces to its dexter diagonal, so that all the roots λ are real. In general some or all of these roots may be complex, so that in general a more or less complicated series of reactions may give rise to oscillatory phenomena.[16]

In conclusion it may be remarked that the treatment set forth above presents an evident analogy to the theory of small oscillations according to Lagrange. However, in the case of the oscillations of a mechanical system if the number of coordinates is n, the general solution contains $2n$ arbitrary multiplicative constants; whereas in the case here studied the number of such constants is equal to the number of variables, namely n.

This point of difference is not without physical significance:

The multiplicative constants are determined by initial conditions. In the case of an oscillating mechanical system it is necessary, in order completely to determine the motion of the system, to know the n initial values of the n coordinates $X_1, X_2, \dots X_n$, and further, the n initial values of the velocities $\frac{dX_1}{dt}\ \frac{dX_2}{dt} \dots \frac{dX_n}{dt}$.

On the other hand, in the case of transformations of the type here considered, not only is it *sufficient* to know the initial values of X_1, $X_2, \dots X_n$, but, when these are given there remains no further freedom of choice for the initial values of the velocities $\frac{dX}{dt}$. We have here a characteristic property of *inertia-free* or completely damped systems, in which velocities are completely determined when the values of the generalized coordinates are given, and in which the accelerations vanish with the velocities.[17]

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[16] See Lotka, Zeitschrift f. phys. Chem., 1910, v. 72, p. 508; 1912, v. 80, p. 159; Hirniak ibid. 1910, v. 75, p. 675.

[17] By differentiating (7) we obtain

$$\frac{d^2x_1}{dt^2} = a_{11}\frac{dx_1}{dt} + a_{12}\frac{dx_2}{dt} + \dots$$

Hence it is evident, provided none of the coefficients a are infinite, that the second and all higher derivatives of $x_1\, x_2 \dots x_n$ vanish with the velocities $\frac{dx}{dt}$. Compare Buckingham, Thermodynamics, 1900, p. 33

Furthermore, in the absence of inertia, the velocities being nevertheless finite, these must be held in check by dissipative resistances. The transformations which we have been considering are accordingly typical *irreversible*[18] *processes*, and the history of the system undergoing such transformation is a typical example of *evolution*. It is, indeed, with a view to preparing the ground for a general investigation of the dynamics of evolution, that the reflections here set forth were conceived along broad and comprehensive lines. Attention may be drawn to the fact that, while we have spoken of the variables X as the masses of the components of a system in chemical reaction, there is nothing in the method followed to restrict us to such interpretation. The X's may be masses of any species of material complexes in any mutual relation of interpendence. For example, they may be the masses of biological species dependent on each other for food and in other ways. The equations developed by Sir Ronald Ross[19] relative to the interpendence of the malaria parasite, the anopheles mosquito and man, fall within the type here considered, and their solution is comprehended in the general solution given above.

SUMMARY

THE differential equations relating to a physical system undergoing change of state are of the general form

$$\left.\begin{aligned}\frac{dX_1}{dt} &= F_1(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j)\\ \frac{dX_2}{dt} &= F_2(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \frac{dX_n}{dt} &= F_n(X_1, X_2, \ldots X_n;\ P_1, P_2, \ldots P_j)\end{aligned}\right\} \qquad (1)$$

where $X_1, X_2, \ldots X_n$ are the masses of the components of the system and the quantities P measure certain parameters defining the state of the system (e. g. pressure, temperature).

The functions F commonly considered in discussions of this subject are those defined by the law of mass action. Some other special cases have also been considered. The most general discussion of the subject is that given by Jüttner, but is restricted to a broad treatment of a somewhat general form of the law of mass action.

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[18] In the thermodynamical sense.

[19] Nature, Oct. 5, 1911, p. 466; A. J. Lotka, ibid. Feb. 8, 1912, p. 497.

In the present paper a very general treatment of the kinetics of material transformations is developed on the sole assumption that the functions F are analytic in the neighborhood of the equilibrium values of the variables X.

A general solution is given of the system of differential equations (1) for the case that the parameters P are held constant during the transformation. This solution is oscillatory or aperiodic according to the nature (complex or real) of the roots λ of a certain determinantal equation $\Delta\ (\lambda) = 0$

The effect of equations of constraint is discussed.

Convergence of the series solution is shown to be related to the stability of the system at the equilibrium point.

Several special cases are considered.

1. Single transformation.

2. Consecutive transformations. It is well known that in chains of consecutive radio-active transformations the products are present in constant ratio. This property is now shown to be quite general for the terminal stages of any series of purely consecutive reactions taking place according to any law whatever, provided only that the functions F are analytic as previously assumed.

3. A similar conclusion applies to all systems in which the process of transformation is aperiodic.

4. In general, in a complicated system, the process is oscillatory.

The entire development set forth presents an analogy to the theory of small oscillations according to Lagrange. However, instead of $2n$ arbitrary constants, corresponding to n co-ordinates, as in Lagrange's theory, we have here only n such constants. This fact stands in close relation to the circumstance that the systems here considered are essentially "inertia-free" or "completely damped," and the transformations accordingly are typically irreversible processes, typical cases of evolution (see for example Chwolson, Textbooks of Physics, German Edition 1905, v. 3, p. 499; J. Perrin, Traité de Chimie Physique, 1903, Chapter V). It is on this account, with a view to preparing for the treatment of the general problem of evolution that the reflections here set forth were conceived along broad and comprehensive lines.

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ROTATIONS IN SPACE OF EVEN DIMENSIONS.

BY H. B. PHILLIPS AND C. L. E. MOORE.

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Introduction. Motions in three-space have been studied from three points of view: (a) the geometric in which the reasoning is primarily synthetic; (b) the algebraic expressed in terms of quaternions or Gibbs' dyadics; (c) that of groups of motions which was started by Jordan [1] (1884) and was continued by Lie and others. The literature of motions in three-space is quite extensive. The study of motions in n-space has not been nearly so extensive but has followed the general lines of development of motions in three-space. One of the first papers on the subject was by Cole [2] (1889). He discussed rotations in four-space by means of a sort of generalized Plücker coördinates. The principal result of the paper is that a rotation in four-space leaves two and only two completely perpendicular planes invariant. P. H. Schoute [3] (1891) by what might be called geometrical argument discussed the general motion in n-space. The algebraic method has not been used much so far as the authors of the present paper know. Clifford [4] and Lipschitz [5] generalized quaternions and thus established a transformation leaving the sum of squares invariant. The papers on this subject have dealt mostly with the algebra and very little with the geometry of motions.

The Gibbs' dyadic [6] is readily extended to n-space but its use for studying motions has not been very extensive. C. L. E. Moore [7] (1918) discussed rotations in hyperspace by treating first the resolution of a complex two-vector into the sum of completely perpendicular simple two-vectors or planes. Gibbs' vector analysis and dyadics form a foundation of this work. E. B. Wilson [8] (1919) published a note outlining a discussion of rotations founded on the properties of the characteristic equation.

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[1] Annali di mathematica, 1884.

[2] On rotations in space of four dimensions. American Journal, 12, 1889, page 191.

[3] Le deplacement le plus général dans l'espace à n dimensions. Annales de l'Ecole Polytechnique de Delft, 7, 1891.

[4] Applications of Grassmann's Extensive Algebra, American Journal 1, 350, 1878. Papers, London, 1882.

[5] Untersuchungen ueber Summen von Quadraten. Bonn, Max Cohen & Sohn, 1886.

[6] E. B. Wilson. On the theory of double products and strains in hyperspace. Trans. Conn. Acad. Arts Sci. 14, 1–57, 1908.

[7] Rotations in Hyperspace. These Proceedings. 53, 1918.

The group side of the subject has received more attention than the other two. Bemporad [9] (1904) returning to the original method of Jordan applied the general theory of Lie to determine the different types of groups of motions in three and four dimensions. E. E. Levi [10] (1905) considered the $\frac{1}{2}\, n(n+1)$ parametered group of motions in n-space. He treated the two and three parametered subgroups of the general group. The paper is intimately connected with a paper by Bianchi on generalized motion. C. L. E. Moore [11] (1918) discussed motions in n-space from the point of view of the infinitesimal transformation.

In this paper we use the Gibbs dyadic to discuss rotations in a space of $2n$ dimensions, particularly in four dimensions. It is shown first that any such rotation is the product of rotations in n fixed planes. The dyadic representing the general rotation is the sum of dyadics representing the rotations in these fixed planes. If we associate with each plane the two-vector in that plane equal in magnitude to the angle of rotation, the sum of such two-vectors for all the fixed planes is a complex two-vector M which plays a role analogous to that of the one-vector, equal to the angle of rotation, along the axis of a three dimensional rotation. A rotation in $2n$ dimensions may thus be described as a rotation through a two-vector angle M. Just as a rotation in the plane can be expressed in the form

$$\Xi' = e^{i\theta}\cdot\Xi$$

where Ξ and Ξ' are corresponding complex variables, so the rotation in $2n$ dimensions can be expressed in the form

$$r' = e^{I\cdot M} r$$

where r' and r are corresponding one-vectors. Furthermore, just as

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[8] Note on Rotations in Hyperspace. Journal, Washington Acad. Sci. 9, 1919.

[9] Sui gruppe dei movimenti. Annali della r. scuola normale sup. di Pisa, 8, 1904.

[10] Sui gruppi di movimenti. Atti dei Lincei, series 5, 14, part 1, 1905.

[11] Motions in Hyperspace. Annals of Math. second series, 19, 1918.

$i\theta$ represents a rotation through 90° and stretching of modulus θ, so $I \cdot M$ represents a rotation through 90° in each of the fixed planes and stretching in each of modulus equal to the angle of rotation.

In the study of commutative rotations it is found useful to define a new product of two-vectors which we call the star product. The star product of M_1 and M_2 is defined as the vector of the dyadic

$$(I \cdot M_1) \cdot (I \cdot M_2).$$

The cross product of two one-vectors in three dimensions is again a one-vector. No such product of one-vectors in higher space can, however, be defined. The star product combines two-vectors in three dimensions in the same way that the cross product combines one-vectors. It is however definable in space of any number of dimensions, the product being always a two-vector. It may therefore be considered a generalization in higher space of the Gibbs cross product. In four dimensions, if M_1 and M_2 are unit simple two-vectors and π_1 and π_2 perpendicular to and cutting both the star product is

$$M_1 * M_2 = \cos\theta_2 \, \pi_1 + \cos\theta_1 \, \pi_2,$$

θ_1 and θ_2 being the angles intercepted in π_1 and π_2 between M_1 and M_2.

If the rotations in the fixed planes are all through the same angle θ there is a fixed plane through each vector of space and all vectors are rotated through the same angle. These equiangular rotations have characteristic equations of order two. They are analogous to the rotations produced by quaternion multiplication [12] and are of two types corresponding to quaternions referred to right and left hand axes. In four dimensions equiangular rotations of opposite types are commutative and every rotation is the product of equiangular rotations of opposite types.

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[12] H. B. Phillips, Application of quaternions to four dimensions. Johns Hopkins University Circular. January 1905, page 8.

In four dimensions a quaternion may be considered as a vector referred to system of four mutually perpendicular axes, one being the axis of reals, the other three the axes on which i, j, k lie. The product of two such vectors α and β is a generalization of the product of two complex numbers, the result being a vector which makes with each of the given vectors the same angle that the other makes with the axis of reals. Two vectors are thus determined one being $\alpha\beta$ the other $\beta\alpha$.

I. ROTATIONS IN $2n$ DIMENSIONS.

1. **Characteristic equation.** In a space of $2n$ dimensions a rotation about the origin, considered as an operation on a one-vector, is represented by a matrix or dyadic ϕ of order $2n$. If r and r' are corresponding vectors the transformation is represented by

$$r' = \phi \cdot r$$

If this is a rotation r and r' will have the same length and therefore

$$r' \cdot r' = (\phi \cdot r) \cdot (\varphi \cdot r) = r \cdot \phi_c \, \phi \cdot r = r \cdot r.$$

Hence

$$\phi_c \, \phi = I. \qquad (1)$$

(Here ϕ_c is used to represent the conjugate of ϕ, that is $\phi \cdot r = r \cdot \phi_c$ and I represents the identical transformation).

The dyadic ϕ satisfies a characteristic or Hamilton-Cayley equation

$$\phi^{2n} + a_1 \, \phi^{2n-1} + a_2 \, \phi^{2n-2} + \ldots . + a_{2n-1} \, \phi + a_{2n} = 0. \qquad (2)$$

The conjugate of (2) is

$$\phi_c{}^{2n} + a_1 \, \phi_c{}^{2n-1} + a_2 \, \phi_c{}^{2n-2} + \ldots . + a_{2n-1} \, \phi_c + a_{2n} = 0.$$

Multiplying by ϕ^{2n} and reducing by use of (1)

$$1 + a_1 \, \phi + a_2 \, \phi^2 + \ldots . + a_{2n-1} \, \phi^{2n-1} + a_{2n} \, \phi^{2n} = 0. \qquad (3)$$

Since this must be identical with (2)

$$a_{2n} = \pm 1. \qquad (4)$$

It is well known that

$$(-1)^{2n} \, a_{2n} = a_{2n}$$

is the determinant of the transformation. If this is positive the rotation is called a proper rotation, if negative an improper one. In case of a proper motion the characteristic equation is then reciprocal. If λ is a root $\frac{1}{\lambda}$ is also a root. In case of an improper motion if λ is a root $-\frac{1}{\lambda}$ is also.

2. **Invariant planes.** If (2) has a real root λ, there must be a real vector r such that

$$r' = \phi \cdot r = \lambda r_1$$

Since r' and r have the same length

$$\lambda = \pm 1.$$

In case of a proper motion, there is a second real root

$$\frac{1}{\lambda} = \pm 1.$$

In case of an improper motion the second root is

$$-\frac{1}{\lambda} = \mp 1.$$

In either case there are two real roots (equal for a proper motion). There is then an invariant plane. That there is an invariant plane also follows from the fact that the line along which a lies is left invariant point for point and consequently the space of $2n - 1$ dimensions perpendicular to this is left invariant. In this space the characteristic equation is of odd order and so has a real root. Two vectors are thus left invariant and so their plane is invariant. In this plane each point is left fixed.

If (2) has no real root it must have two conjugate imaginary roots. Two conjugate imaginary vectors are therefore left invariant. The real plane containing these two vectors is then left invariant. Hence in any case a rotation in $2n$ dimensions has an invariant real plane. This plane is left invariant as a plane but the individual points are not left invariant. Since the $2n - 2$ dimensional space perpendicular to this invariant plane is also left invariant, by an application of the same argument there is an invariant plane in this perpendicular space. By continuing we see that *the rotation must leave n completely perpendicular real planes invariant.*[13]

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[13] It follows from this fact that positive rotations can be defined. For take two perpendicular unit vectors in each invariant plane arranged so that the rotation in each plane is from the first to the second. Then a positive rotation can be defined as one such that the outer product of these unit vectors (keeping the order indicated in each plane) is positive.

If the characteristic equation has a pair of roots equal to $\lambda + i\mu$ it will have a second pair equal to $\lambda - i\mu$. There will then be a pencil of imaginary lines corresponding to $\lambda + i\mu$ and another pencil corresponding to $\lambda - i\mu$. The imaginary planes in which these pencils lie are left invariant. Any plane through the origin cutting each of these invariant planes in a line will contain two invariant lines and so will be invariant. In particular the ∞^2 real planes cutting the planes of the invariant pencils in lines will be invariant. Thus repeated imaginary roots characterize the motions which have ∞^2 real invariant planes discussed by Moore.

3. **Canonical form of the dyadic.** Let k_1 and k_2 be perpendicular unit vectors. The dyadic

$$\Psi = (k_1 k_1 + k_2 k_2) \cos q + (k_2 k_1 - k_1 k_2) \sin q. \tag{5}$$

represents a rotation of the plane k_{12} though the angle q measured from k_1 toward k_2. That is if the vector r is rotated into r'

$$r' = \Psi \cdot r.$$

This is the most general form of a proper rotation of the plane into itself. Any improper rotation can be expressed in the form

$$x = (k_1 k_1 - k_2 k_2) \cos q + (k_2 k_1 + k_1 k_2) \sin q. \tag{6}$$

By properly choosing k_1 and k_2 this can always be reduced to a simpler form. Let

$$k_1' = k_1 \cos \tfrac{1}{2} q + k_2 \sin \tfrac{1}{2} q,$$
$$k_2' = k_1 \sin \tfrac{1}{2} q - k_2 \cos \tfrac{1}{2} q.$$

By direct expansion it is seen that

$$k_1' k_1' - k_2' k_2' = (k_1 k_1 - k_2 k_2) \cos q + (k_2 k_1 + k_1 k_2) \sin q.$$

If then we use k_1', k_2', instead of k_1, k_2 any improper rotation of the plane into itself can be expressed in the form

$$\chi = k_1 k_1 - k_2 k_2. \tag{7}$$

Let

$$k_{ij},\ i = 1, 3, \ldots 2n-1,\ j = i + 1, \tag{8}$$

be a set of n completely perpendicular planes and let k_i, k_j be perpendicular unit vectors in the plane k_{ij}. The dyadic

$$\phi = \Sigma\, (k_i k_i \pm k_j k_j) \cos q_{ij} + (k_j k_i \mp k_i k_j) \sin q_{ij}, \tag{9}$$

the summation being for all values of i and j in (8), represents a motion in which each plane k_{ij} is rotated through an angle q_{ij}. If the rotation in k_{ij} is proper the upper signs in formula (9) are used and if improper the lower signs are used. Any rotation Ψ in $2n$ dimensions can be expressed in this form. To do so let k_{ij} be the n invariant planes of Ψ and q_{ij} the angles of rotation in those planes. Then $\phi^{-1}\,\Psi$ is a dyadic leaving all the vectors k_i invariant. Hence

$$\phi^{-1}\,\Psi = I$$

and so $\phi = \Psi$.

If the rotations in two of the planes k_{ij} are improper, these can be replaced by two new planes in which the rotations are proper. Thus suppose k_{12} and k_{34} are the planes in question. By proper choice of k_1, k_2, k_3, k_4 the part of (9) belonging to these planes can be reduced, by (7), to the form

$$k_1\,k_1 - k_2\,k_2 + k_3\,k_3 - k_4\,k_4.$$

This is equivalent to

$$k_1\,k_1 + k_3\,k_3 - (k_2\,k_2 + k_4\,k_4)$$

which represents a proper motion leaving all the vectors in the plane k_{13} invariant and rotating the vectors in the plane k_{24} through 180°. The corresponding rotations will be proper. By a continuation of the process we can ultimately reduce (9) to a form in which not more than one of the rotations in the invariant planes will be improper. If the motion in only one of the planes is improper the $2n$ dimensional rotation represented by Ψ is improper. Hence *any proper rotation can be represented by a dyadic of the form*

$$\phi = \Sigma\,(k_i\,k_i + k_j\,k_j)\cos q_{ij} + (k_j\,k_i - k_i\,k_j)\sin q_{ij}. \qquad (10)$$

We shall now obtain the characteristic equation[14] satisfied by ϕ. In case of a rotation Ψ in a plane k_{12} we have

$$\Psi = (k_1\,k_1 + k_2\,k_2)\cos q_{12} + (k_2\,k_1 - k_1\,k_2)\sin q_{12},$$
$$\Psi^2 = (k_1\,k_1 + k_2\,k_2)(\cos^2 q_{12} - \sin^2 q_{12}) + (k_2\,k_1 - k_1\,k_2)\,2\sin q_{12}\cos q_{12},$$
$$I = k_1\,k_1 + k_2\,k_2.$$

Hence

$$\Psi^2 - 2\,\Psi\cos q_{12} + I = 0.$$

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[14] See E. B. Wilson, note 8.

An equation satisfied by ϕ is then

$$\pi\,(\phi^2 - 2\,\phi\cos q_{ij} + I) = 0 \tag{11}$$

π indicating the product of all the factors obtained by letting i, j have the values in (8). For if x is any vector in the plane k_{ij}, we have

$$(\phi^2 - 2\,\phi\cos q_{12} + I)\cdot x = 0$$

and so

$$\pi\,(\phi^2 - 2\,\phi\cos q_{12} + I)\cdot x = 0 \tag{12}$$

Hence the left side of (11) is such that its product with any vector in any one of the fixed planes of ϕ is zero. Any vector in space can be represented as a linear function of vectors in these fixed planes. Consequently (12) is satisfied by any vector in space which proves relation (11).

If some of the angles q_{ij} are equal, (10) can be replaced by the equation of lower degree obtained by using only one factor for each distinct value of q_{ij}. For the above argument requires only one factor of each type. If one of the angles is zero the resulting factor

$$\phi^2 - 2\,\phi + I = (\phi - I)^2$$

can be replaced by $\phi - I$. For any vector x in the corresponding plane is left invariant and so

$$(\phi - I)\cdot x = x - x = 0.$$

If one of the roots is π, the factor

$$\phi^2 + 2\,\phi + I = (\phi + I)^2$$

can be replaced by $\phi + I$.

The resulting equation containing only one quadratic factor for each distinct value of q_{ij}, *not more than one factor* $\phi - I$ *and not more than one factor* $\phi + I$ *is the equation of lowest degree satisfied by* ϕ. For the roots of ϕ are

$$\cos q_{ij} + \sqrt{\cos^2 q_{ij} - 1} = e^{q_{ij}\sqrt{-1}}$$

and any equation satisfied by ϕ must be satisfied by all these roots.

4. **Exponential form of dyadic.** If Ψ is any dyadic, we shall write

$$e^{\Psi} = I + \Psi + \frac{\Psi^2}{\left|\underline{2}\right.} + \ldots\ldots + \frac{\Psi^n}{\left|\underline{n}\right.} + \ldots \tag{13}$$

where I is the identical dyadic. The sum of n terms in (13) is a dyadic. That this sum approaches a definite limit, when n increases indefinitely has been shown by various writers.[15] By direct expansion it is seen that

$$e^{\Sigma q_{ij}\,(k_i k_j - k_j k_i)} = \Sigma\,(k_i k_i + k_j k_j) \cos q_{ij} + (k_j k_j - k_i k_j) \sin q_{ij}.$$

Hence the dyadic representing any proper rotation can be written in the form

$$\phi = e^{\Sigma q_{ij}\,(k_j k_i - k_i k_j)} \tag{14}$$

According to the inner multiplication of Lewis

$$\begin{aligned} k_i \cdot k_{ij} &= -\,k_j, \\ k_j \cdot k_{ij} &= \,k_i, \\ k_l \cdot k_{ij} &= \,0, \quad l \neq i, j. \end{aligned}$$

The identical dyadic is

$$I = k_1 k_1 + k_2 k_2 + \ldots . k_{2n} k_{2n}.$$

Hence

$$I \cdot k_{ij} = k_j k_i - k_i k_j. \tag{15}$$

The product being obtained by multiplying the consequents of I with k_{ij}. If then we write

$$M = \Sigma q_{ij} k_{ij}, \tag{16}$$

equation (14) takes the form

$$\phi = e^{I \cdot M}. \tag{17}$$

This is a form in which the dyadic representing any proper rotation in $2n$ — dimensional space can be written.

In a space of $2n + 1$ dimensions the dyadic ϕ representing a given rotation has a characteristic equation of the form

$$\phi^{2n+1} + a_1 \phi^{2n} + \ldots . a_{2n} \phi + a_{2n+1} = 0.$$

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[15] Cf. H. B. Phillips. Functions of Matrices, Amer. J. of Math. Vol. XLI.

In case of a proper rotation

$$a_{2n+1} = -1.$$

Since the equation is reciprocal, it has a root $+1$. Hence there is a vector i such that

$$\phi \cdot i = i.$$

The space perpendicular to i will be left invariant and in that space ϕ will determine a rotation with dyadic Ψ. Now

$$\phi = i\, i + \Psi.$$

For $i\, i + \Psi$ converts i into i and transforms the perpendicular space in the same way as Ψ. Also

$$\Psi = e^{I \cdot M}$$

where

$$I = k_1 k_1 + k_2 k_2 + \ldots . + k_{2n} k_{2n}$$

is the identical transformation in the space perpendicular to i. If however we let

$$I = k_1 k_1 + k_2 k_2 + \ldots . + k_{2n} k_{2n} + i\, i.$$

be the identical dyadic in the complete space of $2n + 1$ dimensions, it is easily seen that

$$e^{I \cdot M} = i\, i + \Psi = \phi.$$

For M is a linear combination of the invariant planes which lie in the $2n -$ dimensional space perpendicular to i and $I \cdot M$ will be the same whether I is the $2n$ or the $2n + 1 -$ dimensional form of the identical dyadic. Then

$$e^{I \cdot M} = i\, i + I + I \cdot M + \frac{(I \cdot M)^2}{2} + \ldots .$$

where I on the right signifies the $2n -$ dimensional form. The above formula then follows at once. Hence, *whether the space rotated is of* *dimensions* $2n$ *or* $2n+1$, *any proper rotation leaving the origin fixed can be determined by a dyadic of the form* (17).

Conversely any dyadic of the form

$$\phi = e^{I \cdot M}$$

represents a rotation. To show this, it is sufficient to express M as a linear function of completely perpendicular planes k_{ij}. Then

$$e^{I \cdot M} = e^{\Sigma q_{ij} (k_j k_i - k_i k_j)}$$

which is equivalent to (9) with all upper signs used.

The dyadic

$$I \cdot k_{ij} = k_j k_i - k_i k_j$$

determines a rotation through 90° in the plane k_{ij}. This is analogous to $i = \sqrt{-1}$ which produces a rotation of 90° in the complex domain. Similarly

$$\phi = e^{qI \cdot k_{ij}}$$

represents a rotation through the angle q in the plane k_{ij} just as $e^{i\varphi}$ produces a rotation through the angle φ in the complex domain. We can then consider $q\, k_{ij}$ as the vector angle of rotation. For this reason M might be called the *vector angle of rotation* of the dyadic

$$\phi = e^{I \cdot M}.$$

5. **Uniqueness of exponent.** The angles q_{ij} may be of any size. However, *if they are restricted to the region*

$$-\pi < q_{ij} < \pi$$

we shall show that *for each rotation there is a unique value of* M, that is, if

$$\phi = e^{I \cdot M_1} = e^{I \cdot M_2} \tag{18}$$

then

$$M_1 = M_2.$$

To show this we first consider the dyadic

$$I \cdot M = \Sigma q_{ij} (k_j k_i - k_i k_j).$$

Direct multiplication gives

$$(I \cdot M)^2 = \Sigma\, q_{ij}^2\,(-\,k_i\,k_i - k_j\,k_j).$$

Also

$$I = \Sigma\,(k_i\,k_i + k_j\,k_j).$$

The characteristic equation of $I \cdot M$ is then

$$\pi[(I \cdot M)^2 + q_{ij}^2\, I] = 0 \tag{19}$$

π representing the product for all values of q_{ij}. To obtain the equation of lowest degree satisfied by $I \cdot M$ we need only use one factor for each distinct value of q_{ij} and one factor $I \cdot M$ for each angle $q_{ij} = 0$. This is shown as in the case of the equation of lowest degree satisfied by ϕ by taking the product with any vector in an invariant plane. Hence the equation of lowest degree is of the same order as that of ϕ. Furthermore the values of q_{ij} (between $-\,\pi$ and $+\,\pi$) are determined by the roots

$$e^{q_{ij}\sqrt{-1}}$$

of ϕ. Consequently if (18) is satisfied, M_1 and M_2, satisfy the same equation of lowest degree (19) and the degree m of that equation is the same as that of ϕ. By using (19), terms of degree higher than $m - 1$ in

$$\begin{aligned}\phi &= I + I \cdot M_1 + \frac{(I \cdot M_1)^2}{4} + \frac{(I \cdot M_1)^3}{\underline{|3}} + \ldots. \\ &= I + I \cdot M_2 + \frac{(I \cdot M_2)^2}{\underline{|2}} + \frac{(I \cdot M_2)^3}{\underline{|3}} + \ldots.\end{aligned}$$

can be replaced. Let the result be

$$\begin{aligned}\phi &= a_{11}\, I + a_{12}\,(I \cdot M_1) + a_{13}(I \cdot M)^2 + \ldots. + a_{1m}(I \cdot M_1)^{m-1} \\ &= a_{11}\, I + a_{12}\,(I \cdot M_2) + a_{13}(I \cdot M)^2 + \ldots. + a_{1m}(I \cdot M_2)^{m-1}.\end{aligned} \tag{20}$$

The coefficients a_{11}, a_{12}, etc., are the same in both cases because we reduce the same analytical expression by the use of the same identity

(19). Squaring both sides of (20) and using (19) as before

$$\begin{aligned}\phi^2 &= a_{21}\, I + a_{22}(I\cdot M_1) + \ldots + a_{2m}(I\cdot M_1)^{m-1} \\ &= a_{21}\, I + a_{22}(I\cdot M_2) + \ldots + a_{2m}(I\cdot M_2)^{m-1}.\end{aligned} \tag{21}$$

Continuing this process we have finally

$$\begin{aligned}\phi^{m-1} &= a_{m-11}\, I + a_{m-12}(I\cdot M_1) + \ldots + a_{m-1m}(I\cdot M_1)^{m-1} \\ &= a_{m-11}\, I + a_{m-12}(I\cdot M_2) + \ldots + a_{m-1m}(I\cdot M_2)^{m-1}.\end{aligned} \tag{22}$$

Since $\phi, \phi^2, \phi^3 \ldots . \phi^{m-1}, I$ are linearly independent the determinant

$$\begin{vmatrix} a_{12} & a_{13} \ldots\ldots & a_{1m} \\ a_{22} & a_{23} \ldots\ldots & a_{2m} \\ \ldots & \ldots\ldots & \ldots \\ \ldots & \ldots\ldots & \ldots \\ a_{m-12} & a_{m-13} \ldots & a_{m-1m} \end{vmatrix} \neq 0. \tag{23}$$

From (20), (21), (22) we get

$$\begin{aligned} a_{12}[I\cdot M_1 - I\cdot M_2] + \ldots . + a_{1m}[(I\cdot M_1)^{m-1} - (I\cdot M_2)^{m-1}] &= 0, \\ a_{22}[I\cdot M_1 - I\cdot M_2] + \ldots . \quad a_{2m}[(I\cdot M_1)^{m-1} - (I\cdot M_2)^{m-1}] &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m-12}[I\cdot M_1 - I\cdot M_2] + \ldots . + a_{m-1m}[(I\cdot M_1)^{m-1} - (I\cdot M_2)^{m-1}] &= 0. \end{aligned}$$

There are $m - 1$ homogeneous linear equations in $m - 1$ unknowns $(I\cdot M_1 - I\cdot M_2)$, $[(I\cdot M_1)^2 - (I\cdot M_2)^2]$ etc. and since the determinant does not vanish all the unknowns must be zero. In particular

$$I\cdot M_1 - I\cdot M_2 = I\cdot(M_1 - M_2) = 0.$$

Hence

$$M_1 - M_2 = 0$$

which was to be shown

The case where one or more of the angles is equal to π is exceptional. The argument just given does not apply because the characteristic equation of $I\cdot M$ has the quadratic factor

$$(I\cdot M)^2 + \pi^2 I$$

corresponding to the linear factor

$$\phi + I$$

in the characteristic equation of ϕ. For the discussion of this case we may assume that there are no angles $q_{ij} = 0$. For such an angle corresponds to an absolutely fixed plane. If there are several such angles, the corresponding planes determine an absolutely fixed space of ϕ. Neither M_1 nor M_2 will have any planes lying in or cutting this space. The discussion is then identical with that of motions in the lower space perpendicular to this fixed space.

If there is only one angle equal to π, we have

$$M_1 = \pi\alpha_1 + q_2\alpha_2 + \ldots . q_n\alpha_n$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ are a set of completely perpendicular planes. This can be written

$$M_1 = \pi(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n) + (q_2 - \pi)\alpha_2 + \ldots . + (q_n - \pi)\alpha_n.$$

It is easily seen that

$$e^{I \cdot \pi(\alpha_1 + \alpha_2 \ldots + \alpha_n)} = -I$$

since it represents a rotation of 180° in each of the planes $\alpha_1, \alpha_2 \ldots \alpha_n$. Hence

$$e^{I \cdot M} = -e^{I \cdot [(q_2 - \pi)\alpha_2 + \ldots . + (q \ - \pi)\alpha_n]}.$$

Similarly, we can write

$$M_2 = \pi\beta_1 + q_2\beta_2 + \ldots . + q_n\beta_n$$

and

$$e^{I \cdot M_2} = -e^{I \cdot [(q_2 - \pi)\beta_2 + \ldots . + (q_n - \pi)\beta_n]}.$$

If now

$$e^{I \cdot M_1} = e^{I \cdot M_2}$$

we can apply the previous argument and so find

$$(q_2 - \pi)\alpha_2 + \ldots . + (q_n - \pi)\alpha_n = (q_2 - \pi)\beta_2 + \ldots . + (q_n - \pi)\beta_n,$$

whence

$$M_1 - \pi(\alpha_1 + \alpha_2 + \ldots + \alpha_n) = M_2 - \pi(\beta_1 + \beta_2 + \ldots . + \beta_n).$$

Instead of $\alpha_1 + \alpha_2 + \ldots + \alpha_n$ we could have used the function

$$\alpha_1 \pm \alpha_2 \pm \ldots \pm \alpha_n,$$

the positive and negative signs being choosen independently providing we used the same signs for corresponding values of β_i.
Hence

$$M_1 - \pi(\alpha_1 \pm \alpha_2 \pm \ldots \pm \alpha_n) = M_2 - \pi(\beta_1 \pm \beta_2 \pm \ldots \pm \beta_n)$$

and so

$$\alpha_2 = \beta_2, \alpha_3 = \beta_3, \ldots \alpha_n = \beta_n.$$

Since α_1 is the plane perpendicular to $\alpha_2, \alpha_3 \ldots \alpha_n$ and forming with them a right hand system, and β_1 has the same relation to $\beta_2, \beta_3, \ldots \beta_n$, we have finally $\alpha_1 = \beta_1$ and so

$$M_1 = M_2.$$

If two angles are equal to π the theorem may not be true. Thus in four dimensions

$$e^{\pi I \cdot (\alpha_1 + \alpha_2)} = e^{\pi I \cdot (\beta_1 + \beta_2)} = -I,$$

whether $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ or not. In higher dimensions the part of M_1 and M_2 in the four dimensional space of two planes rotated through 180° would be subject to the same indetermination.

6. **Vector of the dyadic.** If M is expressed as a linear function

$$M = \Sigma q_{ij} k_{ij}$$

of completely perpendicular planes k_{ij}, it is clear from the expression of ϕ in the form (9) that these planes are all left invariant and that the motion in each of them is proper.

The planes k_{ij} are invariant in the sense that if k_i and k_j go by the rotation into k_i' and k_j', since these new vectors are perpendicular and lie in the plane k_{ij},

$$k_i' \times k_j' = k_i \times k_j$$

For the same reason M is invariant. Also any linear function of the planes k_{ij} is invariant. A particular linear function of considerable interest is the vector of the dyadic.

$$\phi_x = \Sigma\, 2 \sin q_{ij}\, k_{ij} \tag{24}$$

obtained by replacing each dyad of ϕ by the vector product of its two factors. The rotation cannot be determined uniquely when the vector is known, however, for the angles q_{ij} might be replaced by their supplements without changing the vector of the dyadic.

7. **Commutative rotations and star product.** Consider two rotations

$$\phi_1 = e^{I\cdot M_1},\ \phi_2 = e^{I\cdot M_2}.$$

Since ϕ_1 and ϕ_2 are sums of powers of $I\cdot M_1$ and $I\cdot M_2$ they will be commutative if $I\cdot M_1$ and $I\cdot M_2$ are commutative, that is if

$$\Psi = (I\cdot M_1)\cdot(I\cdot M_2) - (I\cdot M_2 - I\cdot M_1) = 0. \tag{25}$$

Represent M_1 and M_2 in the symbolic form

$$M_1 = AB,\ M_2 = CD.$$

Then

$$I\cdot M_1 = BA - AB,\ I\cdot M_2 = DC - CD,$$

and

$$\begin{aligned}\Psi &= (A\cdot D)\,(BC - CB) + (B\cdot C)\,(AD - DA) - (A\cdot C)\,(BD - DB) - \\ &\qquad\qquad (B\cdot D)\,(AC - CA)\\ &= I\cdot\,[(A\cdot D)\,(C\times B) + (B\cdot C)\,(D\times A) - (A\cdot C)\,(D\times B) - \\ &\qquad\qquad (B\cdot D)\,(C\times A)]\\ &= -\,I\cdot[(I\cdot M_1)\cdot(I\cdot M_2)]_\times \end{aligned} \tag{26}$$

where $[\ \ ]_\times$ signifies the vector of the dyadic contained in the brackets. Let

$$\begin{aligned}M_1 * M_2 &= -\,[(I\cdot M_1)\cdot(I\cdot M_2)]_\times = [(I\cdot M_2)\cdot(I\cdot M_1)]_\times\\ &= (A\cdot C)\,(B\times D) + (B\cdot D)\,(A\times C) - (A\cdot D)\,(B\times C) - \\ &\qquad\qquad (B\cdot C)\,(A\times D).\end{aligned} \tag{27}$$

We will call this the *star product* of M_1 and M_2. It is evidently distributive since the operations in terms of which it is expressed are distributive. Equation (26) shows that ϕ_1 *and* ϕ_2 *are commutative if*

$$M_1 * M_2 = 0.$$

Conversely, if ϕ_1 and ϕ_2 are commutative and none of the angles equal to π.

$$M_1 * M_2 = 0.$$

For we can write ϕ_1 in the form

$$\phi_1 = a_{11}\, I + a_{12}\,(I\cdot M_1) + \dots + a_{1\cdot m}\,(I\cdot M_1)^{m-1}$$

as in (20). If then ϕ_2 is commutative with ϕ_1

$$\phi_2\,\phi_1 - \phi_1\,\phi_2 = a_{12}[\phi_2\,(I\cdot M_1) - (I\cdot M_1)\phi_2] + \dots + a_{1m}[\phi_2(I\cdot M_1)^{m-1} - (I\cdot M)^{m-1}\phi_2] = 0.$$

Since ϕ_2 will also be commutative with ϕ_1^2, from (21)

$$a_{22}\,[\phi_2\,(I\cdot M_1) - (I\cdot M_1)\,\phi_2 + \dots + a_{2m}[\phi_2(I\cdot M_1)^{m-1} - (I\cdot M_1)^{m-1}\phi_2] = 0.$$

Similarly for other powers of ϕ_1. Finally

$$a_{mm-1}\,[\phi_2(I\cdot M_1) - (I\cdot M_1)\phi_2] + \dots + a_{m-1m}[\phi_2(I\cdot M_1)^{m-1} - (I\cdot M_1)^{m-1}\phi_2] = 0.$$

There are $m-1$ equations linear and homogeneous in $m-1$ unknowns $[\phi_2\,(I\cdot M_1) - (I\cdot M_1)\,\phi_2]$, *etc.* Since the determinant (23) is not zero each of the unknowns must be zero. Hence

$$\phi_2\,(I\cdot M_1) - (I\cdot M_1)\,\phi_2 = 0. \tag{28}$$

This shows that ϕ_2 is commutative with $I\cdot M_1$. Expanding ϕ_2 in the form (20) and repeating the argument, we then get,

$$(I\cdot M_1)\,(I\cdot M_2) - (I\cdot M_2)\,(I\cdot M_1) = 0,$$

that is $I \cdot M_1$ and $I \cdot M_2$ are commutative. Equation (25) is then satisfied and so

$$M_1 * M_2 = 0,$$

since this equation was shown to be a consequence of (25).

The star product is defined by (27) for the product of two planes or complex two vectors. We can obtain $M_2 * M_1$ by interchanging A with C and B with D. It follows from (27) that

$$M_2 * M_1 = - M_1 * M_2. \tag{29}$$

In particular if $M_1 = M_2 = M$, we get

$$M * M = 0. \tag{30}$$

Representing M_3 in the symbolic form EF, by direct expansion we get

$$M_1 \cdot (M_2 * M_3) = M_3 \cdot (M_2 * M_1). \tag{31}$$

8. **Star product of two planes.** Suppose the two planes M_1 and M_2 lie in a 3-space. They can be written

$$M_1 = A \times B, \quad M_2 = A \times C$$

where A is a unit vector and B and C are perpendicular to A. Then

$$M_1 * M_2 = B \times C.$$

This is a plane vector perpendicular to M_1 and M_2 and having a magnitude equal to the product of the magnitudes of M_1 and M_2 and the sine of the angle between them. In particular the product vanishes when and only when M_1 and M_2 lie in the same plane.

If M_1 and M_2 are not contained in three dimensions they always lie in a 4-space. In that space we can always find two planes $A \times C$ and $B \times D$ perpendicular to and cutting both. We can then choose the vectors A, B, C, D such that

$$M_1 = A \times B, \quad M_2 = C \times D.$$

Then since

$$A \cdot B = A \cdot D = C \cdot B = C \cdot D = 0,$$
$$M_1 * M_2 = (A \cdot C)\, B \times D + (B \cdot D)\, A \times C. \tag{32}$$

The product is a linear function of the two vectors perpendicular to both. In particular the product is zero only when

$$A \cdot C = B \cdot D = 0.$$

that is, when M_1 and M_2 are completely perpendicular. Thus *the star product of two plane vectors vanishes when and only when they lie in the same plane or are completely perpendicular.*

An interesting case occurs when

$$A \cdot C = \pm B \cdot D \neq 0.$$

For simplicity let A, B, C, D be unit vectors. Any unit vector in M_1 can be written

$$X = A \cos \alpha + B \sin \alpha$$

and similarly a vector in M_2 can be written

$$Y = C \cos \beta + D \sin \beta.$$

The angle θ between these vectors satisfies the relation

$$\begin{aligned} \cos \theta &= A \cdot C \cos \alpha \cos \beta + B \cdot D \sin \alpha \sin \beta \\ &= A \cdot C \, (\cos \alpha \cos \beta \pm \sin \alpha \sin \beta) \\ &= A \cdot C \cos (\alpha \pm \beta). \end{aligned}$$

The minimum value of θ is given by

$$\cos \theta = \pm A \cdot C.$$

This is the angle between X and M_2. Hence all vectors in M_1 make the same angle with M_2 and vice versa. Since $A \times C$ and $B \times D$ are completely perpendicular unit planes, we could choose the vectors A, B, C, D such that $A \times C$ is the complement of $B \times D$ in their four dimensional space. If then $A \cdot C = B \cdot D$, $M_1 * M_2$ is equal to its complement, that is, is self-complementary. If $A \cdot C = -B \cdot D$, $M_1 * M_2$ is the negative of its complement that is, it is anti-self-complementary. For all vectors in either plane to make the same angle with the other plane it is then necessary and sufficient that $M_1 * M_2$ be self-complementary or anti-self-complementary.

In four dimensions all vectors X through the origin making a given angle with M_2 belong to the quadric cone

$$(M_2 \cdot X) \cdot (M_2 \cdot X) = 0.$$

This cone has two sets of generating planes. Any plane M of the one set is such that $M * M_2$ is self-complementary and any plane of the other set such that $M * M_2$ is anti-self-complementary.

II. ROTATIONS IN FOUR DIMENSIONS.

9. **Identities.** In terms of unit two-vectors k_{ij} the laws of star multiplication are

$$\begin{aligned} k_{12} * k_{12} &= k_{12} * k_{34} = 0, \\ k_{12} * k_{13} &= k_{23}. \end{aligned} \tag{33}$$

The case of four dimensions is the one in which we shall be most interested. Let

$$M = \Sigma \lambda_{ij} k_{ij}$$

be any two-vector in four dimensions. Then

$$\begin{aligned} k_{12} * M &= \lambda_{13} k_{23} + \lambda_{14} k_{24} - \lambda_{23} k_{13} - \lambda_{24} k_{14}, \\ k_{34} * M &= - \lambda_{13} k_{41} + \lambda_{4} k_{31} - \lambda_{23} k_{42} + \lambda_{24} k_{32}. \end{aligned} \tag{34}$$

Denoting the complement by the subscript c we then have

$$(k_{12} * M)_c = (k_{34} * \mathrm{M}) = (k_{12})_c * M.$$

Since any unit plane could be taken as k_{12}

$$(M_1 * M_2)_c = (M_1)_c * M_2 = M_1 * (M_2)_c. \tag{35}$$

The operations in (35) being distributive the equations are valid whether M_1 and M_2 are planes or complex two-vectors in four dimensions.

Since the complement of $(M_1 * M_2)_c$ is $M_1 * M_2$ we also have

$$M_1 * M_2 = (M_1)_c * (M_2)_c. \tag{36}$$

10. **Geometrical significance of $M_1 * M_2 = 0$.** Let M_1 and M_2 be complex two-vectors. They can be expressed in the form

$$M_1 = \lambda_1 \pi_1 + \mu_1 (\pi_1)_c$$
$$M_2 = \lambda_2 \pi_2 + \mu_2 (\pi_2)_c.$$

Expanding and using (35) and (36) we have

$$M_1 * M_2 = (\lambda_1 \lambda_2 + \mu_1 \mu_2)\, \pi_1 * \pi_2 + (\lambda_1 \mu_2 + \lambda_2 \mu_1)\, \pi_1 * (\pi_2)_c, \qquad (37)$$

Now $\pi_1 * \pi_2$ is a linear function of two completely perpendicular planes perpendicular to and cutting both π_1 and π_2. Since

$$\pi_1 * (\pi_2)_c = (\pi_1 * \pi_2)_c$$

it is a linear function of the same planes. Therefore *if M_1 and M_2 are resolved in any way into linear functions of pairs of completely perpendicular planes $M_1 * M_2$ is a linear function of two completely perpendicular planes, perpendicular to and cutting the planes of each pair.*

Suppose now

$$M_1 * M_2 = 0.$$

The complement of this is

$$(M_1)_c * M_2 = 0.$$

Let

$$M_1 = \alpha\, k_{12} + \beta\, k_{34}; \quad M_2 = \Sigma\, \lambda_{ij}\, k_{ij}.$$

Then $(M_1)_c = \alpha\, k_{34} + \beta\, k_{12}$ and

$$0 = [\alpha M_1 - \beta (M_1)_c] * M_2 = (\alpha^2 - \beta^2)\, k_{12} * M_2.$$

If M_1 is not self-complementary or anti-self-complementary

$$\alpha^2 - \beta^2 \neq 0, \quad k_{12} * M_2 = 0.$$

From (34) we then have

$$\lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = 0.$$

Hence

$$M_2 = \lambda_{12}\, k_{12} + \lambda_{34}\, k_{34}.$$

Consequently M_1 and M_2 are linear functions of the same pair of completely perpendicular planes k_{12} and k_{34}. We could obtain a similar conclusion if M_2 was not self-complementary or anti-self-complementary. If both belong to these special types, let

$$M_1 = \alpha\,(k_{12} + k_{34}).$$

From (34) we then have

$$M_1 * M_2 = \alpha\,[(\lambda_{13} - \lambda_{24})\,k_{23} + (\lambda_{14} + \lambda_{23})\,k_{24} + (\lambda_{23} + \lambda_{14})\,k_{31} + (\lambda_{24} - \lambda_{13})\,k_{41}.$$

If this is zero

$$\lambda_{13} = \lambda_{24}, \quad \lambda_{14} = -\lambda_{23}.$$

Hence M_2 is anti-self-complementary and so

$$M_2 = \lambda_{12}\,(k_{12} - k_{34}) + \lambda_{13}\,(k_{13} + k_{24}) + \lambda_{14}\,(k_{14} - k_{23}).$$

By direct multiplication we get

$$\begin{aligned} M_1 \cdot M_2 &= M_1 \times M_2 = M_1 * M_2 = 0, \\ M_1 \times M_1 &= M_1 \cdot M_1 = 2\,\alpha^2 > 0, \\ M_2 \times M_2 &= -M_2 \cdot M_2 = -2\,(\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{14}^2) < 0. \end{aligned}$$

Hence

$$(M_1 + \lambda\,M_2) \times (M_1 + \lambda\,M_2) = M_1 \times M_1 + \lambda^2\,M_2 \times M_2 = 0$$

has two real roots λ. If then

$$\pi_1 = M_1 + \lambda\,M_2, \quad \pi_2 = M_1 - \lambda\,M_2,$$

π_1 and π_2 are perpendicular planes and M_1 and M_2 linear functions of them. Hence in any case *if the star product of two two-vectors in four dimensions is zero, they are expressible as linear functions of the same pair of completely perpendicular planes.* Incidentally we have shown that the star product, cross product and dot product of a self-complementary and an anti-self-complementary two-vector are all zero.

Let π_1', π_2', π_3' be the complements of π_1, π_2, π_3. We shall now prove the identity

$$\pi_1 * (\pi_2 * \pi_3) = (\pi_1 \cdot \pi_3)\pi_2 - (\pi_1 \cdot \pi_2)\pi_3 + (\pi_1 \cdot \pi_3')\pi_2' - (\pi_1 \cdot \pi_2')\pi_3'. \quad (38)$$

Since both sides are distributive functions of π_1, π_2, π_3 it is sufficient to prove the identity when they are products of units. If $\pi_2 = \pi_3$ or $\pi_2 = \pi_3'$ both sides vanish. In any other case with a proper choice of axes we could write

$$\pi_2 = k_{12}, \quad \pi_3 = k_{13}, \quad \pi_1 = k_{ij}.$$

Since the subscripts 2 and 3 enter in a similar manner, we need consider only the four cases $\pi_1 = k_{12}$, k_{14}, k_{23} or k_{24}. In these cases the identity becomes

$$-k_{13} = -k_{13}, \quad 0 = 0, \quad 0 = 0, \quad k_{43} = -k_{34}$$

respectively.

By applying (38) to each term and adding, since $\pi_1 \cdot \pi_2' = \pi_1' \cdot \pi_2$ ecc., we get

$$\pi_1 * (\pi_2 * \pi_3) + \pi_2 * (\pi_3 * \pi_1) + \pi_3 * (\pi_1 * \pi_2) = 0. \tag{39}$$

Another identity is

$$\begin{aligned}(\pi_1 * \pi_2) * (\pi_3 * \pi_4) = {} & (\pi_1 \cdot \pi_3)\, \pi_2 * \pi_4 - (\pi_1 \cdot \pi_4)\, \pi_2 * \pi_3 \\ & + (\pi_2 \cdot \pi_4)\, \pi_1 * \pi_3 - (\pi_2 \cdot \pi_3)\, \pi_1 * \pi_4 \\ & + (\pi_1 \cdot \pi_3')\, \pi_2 * \pi_4' - (\pi_1 \cdot \pi_4')\, \pi_2 * \pi_3' \\ & + (\pi_2 \cdot \pi_4')\, \pi_1 * \pi_3' - (\pi_2 \cdot \pi_3')\, \pi_1 * \pi_4'.\end{aligned} \tag{40}$$

Both sides vanish when $\pi_1 = \pi_2$, $\pi_1 = \pi_2'$, $\pi_3 = \pi_4$ or $\pi_3 = \pi_4'$. Since there are only three units and their complements, we must have in any other case π_1 or π_2 equal to π_3, π_3', π_4 or π_4'. These cases are all handled in practically the same way. We consider then only the case $\pi_1 = \pi_3$. The identity then reduces to

$$(\pi_1 * \pi_2) * (\pi_1 * \pi_4) = (\pi_1 \cdot \pi_1)\, \pi_2 * \pi_4. \tag{41}$$

Both sides vanish when $\pi_2 = \pi_4$ or $\pi_2 = \pi_4'$. We could then take

$$\pi_1 = k_{12}, \quad \pi_2 = k_{13}, \quad \pi_4 = k_{14} \text{ or } k_{23}.$$

In these cases (41) becomes

$$k_{23} * k_{24} = k_{34}$$

and

$$k_{23} * k_{31} = k_{12}$$

respectively, both of which are correct. Formula (40) is then correct for all unit two-vectors and since it is distributive it is therefore true when π_1, π_2, π_3, π_4 are complex two vectors.

11. **Rotation of a vector.** Let

$$\phi = e^{I \cdot M} = e^{I \cdot (q_1 \pi_1 + q_2 \pi_2)}$$

where π_1 and π_2 are two completely perpendicular unit planes in terms of which M is linearly expressible. Suppose ϕ rotates a unit vector r through an angle θ into a unit vector

$$r' = \phi \cdot r.$$

Then

$$\cos\theta = r \cdot r' = r \cdot \phi \cdot r. \tag{42}$$

Let α be the angle between r and the invariant plane π_1. We can then write

$$r = r_1 + r_2$$

where r_1 and r_2 are the projections of r on the invariant planes π_1 and π_2. Since $\phi \cdot r_1$ and $\phi \cdot r_2$ are in the planes π_1, and π_2 respectively

$$r_1 \cdot \phi \cdot r_2 = 0,$$

and so

$$\begin{aligned} \cos\theta &= r \cdot \phi \cdot r = r_1 \cdot \phi \cdot r_1 + r_2 \cdot \phi \cdot r_2 \\ &= r^2 \cos^2\alpha \cos q_1 + r^2 \sin^2\alpha \cos q_2. \end{aligned}$$

Hence

$$\cos\theta = \cos^2\alpha \cos q_1 + \sin^2\alpha \cos q_2. \tag{43}$$

The angle θ then depends only on α, q_1 and q_2. Since π_1 is invariant r rotates into a vector r' making the same angle α with π_1. Consequently the motion rotates r' through the same angle θ that it rotates r through.

If q_1 and q_2 are not equal it is clear from (43) that the maximum and minimum valves of θ are determined by $\alpha = 0, \frac{\pi}{2}$, that is, when r lies in π_1 or π_2. *Vectors in one of the invariant planes are rotated through the maximum angle and vectors in the other are rotated through the minimum angle and those in neither plane are rotated through an intermediate angle.* If $q_1 = \pm q_2$

$$\cos\theta = \cos q_1 = \cos q_2$$

and it would appear that all vectors are rotated through the same angle. In this case M is self-complementary or anti-self-complementary.

12. **Equiangular rotations.** If M is any two-vector and k_1 a unit one-vector, M can be expressed in terms of four perpendicular vectors including k_1. The terms of M containing k_1 will have for sum a plane since the sum of three plane vectors with a vector in common, is a plane vector. Let k_2 be perpendicular to k_1 in that plane and let k_3, k_4, form with k_1 and k_2 a system of orthogonal unit vectors. Then

$$M = \lambda_{12}\, k_{12} + \lambda_{23}\, k_{23} + \lambda_{24}\, k_{24} + \lambda_{34}\, k_{34}$$

and

$$M_c = \lambda_{12}\, k_{34} + \lambda_{23}\, k_{14} + \lambda_{24}\, k_{31} + \lambda_{34}\, k_{12}.$$

If $M = M_c$

$$\lambda_{12} = \lambda_{34}, \quad \lambda_{23} = \lambda_{24} = 0$$

and

$$M = \lambda_{12}\,(k_{12} + k_{34}).$$

If $M = -M_c$

$$\lambda_{12} = -\lambda_{34}, \quad \lambda_{23} = \lambda_{24} = 0$$

and

$$M = \lambda_{12}\,(k_{12} - k_{34}).$$

In either case M is a linear function of two completely perpendicular planes one of which may be taken through the arbitrary vector k_1. By properly choosing k_2 we can always make λ_{12} positive. Then λ_{12} will have the same value for all choices of k_1. For if k'_{12} and k'_{34} are two other completely perpendicular planes and

$$M = \lambda\,(k_{12} \pm k_{34}) = \mu\,(k'_{12} \pm k'_{34}),$$
$$M \cdot M = 2\,\lambda^2 = 2\,\mu^2.$$

If now

$$M = \pm M_c, \quad \phi = e^{I \cdot M} \tag{44}$$

we can write ϕ in the form

$$\phi = e^{I \cdot M} = e^{qI \cdot (k_{12} \pm k_{34})}$$

where k_1 is an arbitrarily chosen vector and q is a constant of the motion. The motion rotates k_1 through the angle q in the fixed plane k_{12}. Hence *every vector moves in a fixed plane and all are rotated through the same angle q.* For this reason we shall call the rotation equiangular when

$$M = \pm M_c. \tag{45}$$

Conversely, if all vectors are rotated through the same angle q, in particular the rotation in the fixed plane must be through the angle

$$q_1 = q_2 = q$$

and so (44) is satisfied.

In the case just discussed, if x is any vector, since $x, \phi x, \phi^2 x$ lie in a plane and the angle from x to ϕx is q and that from ϕx to $\phi^2 x$ is also q,

$$x + \phi^2 x = 2 \cos q \, \phi \, x.$$

Since this is true for all values of x

$$\phi^2 = 2 \phi \cos q - I. \tag{46}$$

This is the characteristic equation of ϕ. Conversely if any rotation has this for characteristic equation, x, ϕx, $\phi^2 x$, lie in the same plane. This plane will then be invariant and the rotation in the plane will be through the angle q. The rotation will therefore be equiangular.

These rotations resemble very much the rotations that result from quaternion multiplication. If the angle of rotation is q, by (44)

$$M \cdot M = 2 q^2. \tag{47}$$

In terms of a given set of orthogonal vectors k_1, k_2, k_3, k_4, any self-complementary two-vector of this kind be written,

$$M = q \left[\lambda (k_{12} + k_{34}) + \mu (k_{13} - k_{24}) + \nu (k_{14} + k_{23})\right]$$

where

$$\lambda^2 + \mu^2 + \nu^2 = 1. \tag{48}$$

Expanding and using (48) we get

$$\phi = e^{I \cdot M} = I \cos q + \lambda \sin q \, I \cdot (k_{12} + k_{34}) + \mu \sin q \, I \cdot (k_{13} - k_{24}) + \nu \sin q \, I \cdot (k_{14} + k_{23}).$$

If we take

$$\begin{aligned} i &= I \cdot (k_{12} + k_{34}), \\ j &= I \cdot (k_{13} - k_{24}), \\ k &= I \cdot (k_{14} + k_{23}), \end{aligned}$$

then

$$\phi = I \cos q + \sin q \, (\lambda i + \mu j + \nu k).$$

Also by direct multiplication it is seen that

$$\begin{gathered} i^2 = j^2 = k^2 = -I, \\ ij = k, \quad jk = i, \quad ki = j, \end{gathered}$$

as in quaternions. Two dyadics of this kind can then be multiplied like quaternions the principal difference being that 1 in quaternions is replaced by I. If M is anti-self-complementary we get similar results with

$$\begin{aligned} i &= I \cdot (k_{12} - k_{34}), \\ j &= I \cdot (k_{13} + k_{24}), \\ k &= I \cdot (k_{23} - k_{14}). \end{aligned}$$

Rotations with self-complementary M differ from those with anti-self-complementary M in the same way that quaternions referred to a right hand system of axes differ from quaternions referred to a left hand system.

13. **Invariant complexes.** If M is self-complementary the rotation

$$\phi = e^{I \cdot M}$$

leaves all anti-self-complementary two-vectors invariant. For if M' is such a two-vector M and M' can be expressed in the form

$$M = q \, (\pi_1 + \pi_2), \quad M' = \lambda \, (\pi_1 - \pi_2)$$

where π_1 and π_2 are completely perpendicular unit planes. Now π_1 and π_2 being fixed planes of ϕ, are left invariant. Hence ϕ leaves M' invariant. Similarly the rotation

$$\phi = e^{I \cdot M'}$$

leaves any self-complementary two-vector M invariant.

Let

$$\phi = e^{I \cdot M} = e^{I \cdot} k_{12}{}^{(q_1 k_{12} + q_2 k_{34})}.$$

The rotation evidently leaves the self-complementary-complex

$$k_{12} + k_{34}$$

invariant. Any self-complementary complex can be written

$$\lambda\,(k_{12} + k_{34}) + \mu\,(k_{13} - k_{24}) + \nu\,(\lambda_{14} + k_{23}).$$

If ϕ leaves this invariant, it must leave

$$\mu\,(k_{13} - k_{24}) + \nu\,(k_{14} + k_{23})$$

invariant. Since k_1 transforms into

$$k_1 \cos q_1 + k_2 \sin q_1, \text{ etc.}$$

we must then have

$$\begin{aligned}
&\mu(k_{13} - k_{24}) + \nu(k_{14} + k_{23}) = \\
&\quad = \mu\,[(k_1 \cos q_1 + k_2 \sin q_1) \times (k_3 \cos q_2 + k_4 \sin q_2) - (k_2 \cos q_1 \\
&\qquad\qquad - k_1 \sin q_1) \times (k_4 \cos q_2 - k_3 \sin q_2)] \\
&\quad + \nu\,[(k_1 \cos q_1 + k_2 \sin q_1) \times (k_4 \cos q_2 - k_3 \sin q_2) + (k_2 \cos q_1 \\
&\qquad\qquad - k_1 \sin q_1) \times (k_3 \cos q_2 + k_4 \sin q_2).
\end{aligned}$$

Equating coefficients of k_{13}, we get

$$\mu \cos (q_1 + q_2) - \nu \sin (q_1 + q_2) = \mu.$$

Similarly from the coefficients of k_{14} we get

$$\mu \sin (q_1 + q_2) + \nu \cos (q_1 + q_2) = \nu.$$

If these equations are satisfied simultaneously

$$\begin{vmatrix} \cos(q_1 + q_2) - 1, & -\sin(q_1 + q_2) \\ \sin(q_1 + q_2), & \cos(q_1 + q_2) - 1 \end{vmatrix} = 0,$$

that is,

$$\cos(q_1 + q_2) = 1.$$

Hence

$$q_1 = -q_2.$$

Consequently M is anti-self-complementary. If then ϕ leaves any self-complementary complex invariant other than $k_{12} + k_{34}$, M must be anti-self-complementary. In particular *if M is self-complementary, the rotation cannot leave any self complementary complex invariant except a multiple of M.* Similar theorems hold for anti-self-complementary rotations.

The self-complementary complexes are analogous to one system of the generators on a quadric surface, the anti-self-complementary complexes constituting the generators of the other system, each generator of one system cutting each generator of the other in the sense that the star product is zero. The equiangular rotations, are the rotations that leave this quadric invariant. Any rotation of the self-complementary type leave all generators of the opposite type invariant and one of the self-complementary type. Similarly for the anti-self-complementary rotations.

14. **Analogy to Gibbs' vector semi-tangent.**[16] Any self-complementary complex can be written

$$M = q\,(k_{12} + k_{34}).$$

Hence

$$M \cdot M = M \times M = 2q^2, \tag{49}$$

$$I \cdot M = q(k_2\, k_1 - k_1\, k_2 + k_4\, k_3 - k_3\, k_4), \tag{50}$$

$$(I \cdot M) \cdot (I \cdot M) = -\,q^2 I = -\tfrac{1}{2}\,(M \cdot M)\,I. \tag{51}$$

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[16] Gibbs-Wilson Vector Analysis.

If M_1 and M_2 are self-complementary $M_1 + M_2$ is also and so

$$I\cdot(M_1 + M_2)\cdot I\cdot(M_1 + M_2) = -\tfrac{1}{2}\,(M_1 + M_2)\cdot(M_1 + M_2)\,I.$$

Since

$$\begin{aligned}(I\cdot M_1)\cdot(I\cdot M_1) &= -\tfrac{1}{2}\,(M_1\cdot M_1)\,I,\\ (I\cdot M_2)\cdot(I\cdot M_2) &= -\tfrac{1}{2}\,(M_2\cdot M_2)\,I.\end{aligned}$$

by subtraction we get

$$(I\cdot M_1)\cdot(I\cdot M_2) + (I\cdot M_2)\cdot(I\cdot M_1) = -\,(M_1\cdot M_2)\,I. \tag{52}$$

also

$$(I\cdot M_1)\cdot(I\cdot M_2) - (I\cdot M_2)\cdot(I\cdot M_1) = I\cdot(M_1 * M_2),$$

whence by addition we get

$$(I\cdot M_1)\cdot(I\cdot M_2) = \tfrac{1}{2}\,[I\cdot(M_1\ *\ M_2 - (M_1\cdot M_2)\,I]. \tag{53}$$

Let

$$M_1\cdot M_1 = M_2\cdot M_2 = 2.$$

Then

$$\begin{aligned}\phi_1 &= e^{q_1 I\cdot M_1} = I\cos q_1 + I\cdot M_1 \sin q_1,\\ \phi_2 &= e^{q_2 I\cdot M_2} = I\cos q_2 + I\cdot M_2 \sin q_2.\end{aligned}$$

Multiplying and using (53) we get

$$\begin{aligned}\phi_1\phi_2 = {} & I\,[\cos q_1 \cos q_2 - \tfrac{1}{2}\,(M_1\cdot M_2)\sin q_1 \sin q_2] \qquad (54)\\ & + I\cdot[M_1 \sin q_1 \cos q_2 + M_2 \cos q_1 \sin q_2 + \tfrac{1}{2}\,M_1 * M_2 \sin q_1 \sin q_2.\end{aligned}$$

Let

$$Q_1 = M_1 \tan q_1, \qquad Q_2 = M_2 \tan q_2.$$

The corresponding quantity for $\phi_1\,\phi_2$ is

$$\begin{aligned}Q &= \frac{M_1 \sin q_1 \cos q_2 + M_2 \cos q_1 \sin q_2 + \tfrac{1}{2}(M_1 * M_2)\sin q_1 \sin q_2}{\cos q_1 \cos q_2 - \tfrac{1}{2}\,M_1\cdot M_2 \sin q_1 \sin q_2}.\\ &= \frac{Q_1 + Q_2 + \tfrac{1}{2}\,Q_1 * Q_2}{1 - \tfrac{1}{2}\,Q_1\cdot Q_2}. \qquad (55)\end{aligned}$$

The quantities Q_1, Q_2, Q are analogous to Gibbs' vector semi-tangent except that the whole angles q_1, q_2, q are used instead of the half angles.

We may call Q_1 the tangent vector of ϕ_1, Q_2 the tangent vector of Q_2 etc. Then (55) has the same general form as Gibbs' relation between the vector semi-tangents except for the $\frac{1}{2}$ and that the star product is used instead of the cross product. Similar results are obtained if M_1 and M_2 are anti-self-complementary.

If M_1 is self-complementary and M_2 anti-self-complementary

$$M_1 * M_2 = 0.$$

Hence if

$$\phi_1 = e^{q_1 I \cdot M_1}, \qquad \phi = e^{q_2 I \cdot M_2}$$

ϕ_1 and ϕ_2 are commutative. That is *equiangular motions of opposite types are always commutative.* If, however, M_1 and M_2 are of the same type equation (54) shows that

$$\phi_1 \phi_2 = \phi_2 \phi_1$$

when and only when

$$\sin q_1 \sin q_2 \, M_1 * M_2 = 0.$$

If $\sin q_1 = 0, \quad q_1 = 0$ or π and

$$\phi_1 = \pm I.$$

Hence *if neither dyadic is a multiple of the idemfactor, two equiangular rotations of the same type are commutative only when*

$$M_1 * M_2 = 0.$$

In this case M_1 and M_2 are expressible as multiples of the same completely perpendicular planes and so M_1 is a multiple of M_2.

15. Expression of a rotation as a product of equiangular rotations.

Any rotation in four dimensions can be expressed as the product of two equiangular rotations of opposite type. For, we have

$$\phi = e^{I \cdot M} = e^{I \cdot \frac{1}{2}(M + \qquad + I \cdot \frac{1}{2}(M - M_c)}.$$

Now the dyadics

$$I \cdot \tfrac{1}{2}(M + M_c), \qquad I \cdot \tfrac{1}{2}(M - M_c)$$

are commutative. Hence

$$\phi = e^{I \cdot \frac{1}{2}(M + M_c)} \cdot e^{I \cdot \frac{1}{2}(M - M_c)}$$

which is the resolution required. Conversely, this can be done, except for sign, in only one way. For suppose

$$\phi = \phi_1 \phi_2 = \phi_3 \phi_4,$$

ϕ_1 and ϕ_3 being equiangular rotations of one type, ϕ_2 and ϕ_4 of the opposite type. Then

$$\phi_3^{-1} \phi_1 = \phi_4 \phi_2^{-1}$$

Now $\phi_3^{-1} \phi_1$ and $\phi_4 \phi_2^{-1}$ are equiangular motions of opposite type. Since they are equal they must leave all self-complementary complexes invariant and also all anti-self-complementary ones. They then leave all complexes invariant and so

$$\phi_3^{-1} \phi_1 = \phi_4 \phi_2^{-1} = \pm I,$$

whence

$$\phi_1 = \pm \phi_3, \qquad \phi_2 = \pm \phi_4.$$

We shall call ϕ_1 and ϕ_2 the components of ϕ.

Suppose

$$\phi = \phi_1 \phi_2, \qquad \phi' = \phi_1' \phi_2',$$

ϕ_1 and ϕ_1' being equiangular rotations of one type and ϕ_2 and ϕ of the opposite type. Since ϕ_1 and ϕ_1' are commutative with ϕ_2 and ϕ

$$\phi \phi' = \phi_1 \phi_1' \cdot \phi_2 \phi_2'.$$

The product of two rotations is thus obtained by multiplying together corresponding components. If ϕ and ϕ' are commutative, either

$$\phi_1 \phi_1' = \phi_1' \phi_1, \qquad \phi_2 \phi_2' = \phi_2' \phi_2$$

or

$$\phi_1 \phi_1' = - \phi_1' \phi_1, \qquad \phi_2 \phi_2' = - \phi_2' \phi_2.$$

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By Arthur Searle.

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THE present investigation attempts to solve some problems which were interesting to me, and which did not seem to be satisfactorily treated in such works as I could conveniently consult. It is possible that additional research among mathematical treatises would have proved more successful; but in any case, it is not probable that the precise method which I have here employed would have been adopted by previous writers. The plan of inquiry, therefore, may be new, even if its conclusions only confirm older discoveries.

Under the laws of inertia and gravitation, a particle of negligible mass circulates in one of the conic sections about a central point from which emanates an attraction proportional to the inverse square of its distance. Denoting this variable distance by R and its inverse square by R^{-2}, the question may be asked what effects will follow if we substitute for -2 an exponent n to which we may assign any value. Various cases of this kind are considered in ordinary text-books, especially that in which $n = +1$, but some which seem particularly interesting are evaded. In attempting the solution of these, I have been led to the general principles to be stated below.

I.

Transverse motion and the force derived from it.

When the moving particle proceeds directly toward or directly from the central fixed point, its course is controlled only by the velocity attributed to it and by a single force. In other cases, its motion may be resolved into radial and transverse components perpendicular to each other. Its transverse motion produces a second force, called centrifugal by established usage only when the orbit described is a circle. If not, a distinctive name for this second force is required, and if such a name is to be derived from an ancient language, the words apocentric or divellent might be suggested. But, in the present discussion, it will be sufficient to employ the words outward and inward for forces tending to increase or to diminish the distance between the moving particle and the central point.

The outward force is left without a name in the usual treatises upon our present subject. They make it clear, however, that such a force exists, and that it varies inversely to the cube of the radius vector. For instance, we find in Price's Treatise on Infinitesimal Calculus, Vol. III, p. 462, the equation $d^2r/dt^2 - h^2/r^3 = -P$, in which P denotes a central force, and h denotes the constant $r^2d\theta/dt$ resulting from the law of inertia. This equation, in the form $d^2r/dt^2 = h^2/r^3 - P$, denotes the excess of a force tending to elongate the radius vector r, and inversely proportional to r^3, over any force tending to decrease r. If $h^2/r^3 < P$, the inward force is the stronger.

Since it will also be convenient to have an expression for the transverse motion in terms of the outward force, a geometrical proof of the principle just stated will here be added.

In Fig. 1, let C represent the fixed point about which the moving particle describes its orbit, and AB the straight line through which that particle moves under the law of inertia during one of the intervals between successive applications of force ultimately to be supposed continuous and consequently infinitesimal in the second degree. From B let fall the perpendicular BD on CA; and at B erect a perpendicular on CB, intersecting CA, produced if necessary, at F. Since CB>CD, while CF>CB, there is a point E between D and F such that CE = CB. When AB becomes infinitesimal with respect to CA or CB, DE = EF, as is easily shown, so that DF = 2DE. The effect of outward force in elongating CD is represented by DE.

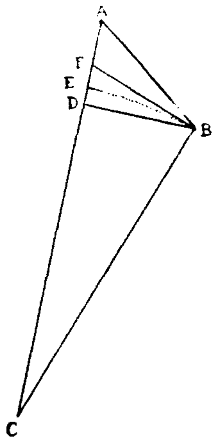

Fig. 1.

The transverse motion represented by BD is inversely proportional to CD, according to the fundamental law of equality of areas, which need not here be repeated. The squared transverse motion is therefore inversely proportional to the squared radius vector, which is equal to the product of CD and DF. Accordingly, DF, and consequently DE, is proportional to the inverse cube of CD or CA, which are ultimately equal; and the transverse motion is expressed by the square root of the product of the radius vector and twice the outward force. Any reader familiar with the elements of orbital motion will readily fill up the outline of this demonstration by adding details which it seems needless to state here.

II.

The relation of outward and inward force, when $n > -3$.

The present discussion will be confined to those cases in which the exponent n, defining the law of inward force, as above stated, is at least not less than -3. Under this restriction, there may be, and if $n > -3$, there must be, with the exceptions noticed at the end of section IX, some radius vector, to be denoted by P, at which the outward and inward forces are equal. Under the law of gravitation, this radius vector is the semiparameter of the orbit, as appears from the ordinary treatment of the subject. For values of n other than -2, it is to be found by methods hereafter to be described. For the present, it is required only to show that it will occur.

Suppose that in a particular case, and at a given moment, the inward force is the stronger, while at the same time the radius vector is diminishing by the operation of the law of inertia. The momentary effects of the forces are both infinitesimals of the second order, so that a definite ratio may exist between them. Let this ratio be denoted by a, so that the momentary effect of the inward force may be a times as great as that of the outward force. Let the inward force vary directly as that power of the distance R denoted by the exponent n, and let $n > -3$. At a later time, the ratio of the forces will be expressed by $aR^n/R^{-3} = aR^{n+3}$, in which $n + 3 > 0$. Let R decrease without limit, and before it becomes equal to 0, R^{n+3} will become equal to $1/a$, since a is finite. The forces will then be equal.

At this time, the inward velocity will reach its maximum, since it has previously been increasing from an excess of inward force, and must afterwards decrease. If this velocity is finite, it must be extinguished, before $R = 0$, by the unlimited decrease of R^{n+3}.

If $n = -3$, the inward and outward forces are always equal or never equal. If $a > 1$, the inward force is always the stronger; if $a < 1$, it is always the weaker. The various results to be expected from such conditions have been discussed in previous treatises, for example, by Price (Treatise on Infinitesimal Calculus, Vol. III., p. 487), and will not be examined in the present article.

III.

Definitions of terms and symbols here to be employed.

The words periastron and apastron will be used in their ordinary senses of the situations in which the radius vector has its least or its greatest value. We have just seen that periastron will always occur in the cases here to be examined, but that apastron may never be attained. The straight lines drawn through the fixed point in the directions of periastron and apastron will be called the axes of periastron and apastron.

The geometrical quantities having special designations in the present discussion are R, radius vector; Q, least value of R; L, greatest value of R if such a value occurs; P, value of R at which the inward and outward forces are equal, and the inward or outward velocity is at a maximum; Y, variable distance measured from the fixed point, toward periastron, to the foot of the perpendicular U, let fall upon the axis of periastron from any position of the moving particle; Z, variable distance measured in the same manner and direction as Y, having the value 0 not at the fixed point, but at the foot of that value of U which occurs when $R = L$; K, greatest value of Z, occurring when $R = Q$; X, variable distance measured from the fixed point, toward apastron, to the foot of the perpendicular let fall upon the axis of apastron from any position of the moving particle; S, variable distance measured in the same manner and direction as X, having the value 0 when $R = Q$; H, greatest value of S, occurring when $R = L$; r, angle between the direction of R and the axis of periastron, having the value 0 when $R = Q$; r_0, the value of r when $R = L$. Hence $Y = Z + L \cos r_0$, and $X = S + Q \cos r_0$.

Designations of algebraic quantities are n, that exponent of R which shows the law of inward force under consideration, so that for the law of gravitation $n = -2$; m, infinitesimal of the second order, expressing the immediate effect of inward force in orbits of the same system; m', similar infinitesimal, showing the immediate effect of transverse motion in increasing R; c, a constant ratio between functions of R and Z, these functions to be determined for different values of n, and the value of c to be determined accordingly; w, an abbreviation for $\sqrt{(n+3)}$; c', the constant cw.

Since P denotes that value of R at which the inward and outward forces are equal, $mP^n = m'P^{-3}$, whence $m' = mP^{n+3}$. Hence for any value of R, the outward force is expressed by mP^{n+3}/R^3, and the inward force by mR^{n+3}/R^3, so that $m(P^{n+3} - R^{n+3})/R^3$ expresses the excess of outward force if positive, and of inward force if negative. This expression, then, denotes the actual force momentarily existing at any value of R.

Fig. 2 illustrates the meaning of the geometrical terms above defined. It represents a curve of the form which will result from an inward force inversely proportional to the first power of the distance, so that $n = -1$. The fixed point, from which the inward force is assumed to emanate, is represented by C, the point of periastron by Q, and the point of apastron by L. The point R is the assumed position of the moving particle. The line CR represents the radius vector R; CQ represents Q, and CL represents L. The position of the radius vector P has not been computed for this particular figure. The lines CY, RY, KY, KQ, CX, HX, HL, respectively represent the quantities Y, U, Z, K, X, S, H. The angle RCQ represents v, and LCQ v_0.

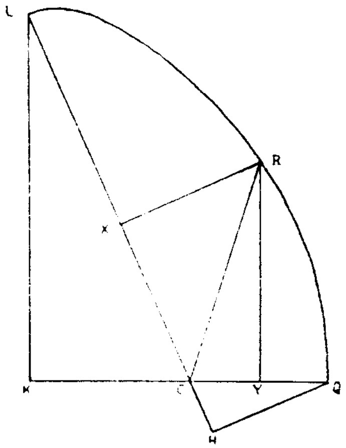

Fig. 2.

IV.

Law of variation of actual force.

The actual force, for any value of R assumed to be momentarily constant, will vary with respect to R^{n+3}, as has just appeared. When $n = -2$, $R^{n+3} = R$, and we know that in this case dR and dY

vary proportionally, while $w^2 = n + 3 = 1$. If $w > 1$, dR is affected by a force greater than that affecting dY. While R is decreasing from the value P, Y is increasing toward the value Q, so that the ratio dR/dY is negative. The variation tending to change it is $n + 3$ times as great as that which exists when $w = 1$, so that this variation is proportional to $(n + 3)\, dR/dY$. It may be desirable, however, to examine this result more minutely, since it is to be the basis of subsequent inquiry.

In discussing the effect of forces, we are obliged to consider them as applied, not continuously, but at intervals which we may ultimately regard as infinitesimal. During each of these intervals, the rates of variation, both of R and of Y, are assumed to be uniform; that is, t denoting time, dR/dt and dY/dt are constants. In the case of R, if no actual force exists, the outward and inward forces are equal. In the absence of force, the variation of Y is uniform by the law of inertia. In differentiating any function of R or of Y, we accordingly regard these quantities, for the moment, as independent variables.

Let any particular value of R be denoted by R_0, and regard it as constant for the moment. In the succeeding interval of time, we will suppose R_0 to be reduced to a smaller value R, so near R_0 that the difference is infinitesimal. To determine the variation of the force acting on dR when $R = R_0$, we differentiate the expression $m\ (R_0^{n+3} - R^{n+3})/R_0^3$ with respect to R, regarding R_0 as constant. The resulting differential coefficient is $-\ m(n + 3)R^{n+2}/R_0^3$, and disregarding the difference between R_0 and R, it becomes $-\ m(n + 3)\ R^{n-1}$. If we regard the denominator as R^3 instead of $R^3{}_0$, we have for the differential itself $-\ m[R^3(n + 3)R^{n+2} + 3R^2(R_0^{n+3} - R^{n+3})]\ dR/R^6$. The second term of the numerator is of a higher order of infinitesimals than the first, $(R_0^{n+3} - R^{n+3})$ being infinitesimal. The differential coefficient becomes as before $-\ m(n + 3)R^{n-1}$.

The variation of Y is not affected by any outward force, but is affected by the inward force on R in the ratio Y/R, as appears from consideration of the right triangle shown in Fig. 2, of which R is the hypotenuse while U and Y are the legs. The effect of inward force on R being $-\ mR^n$, its effect on Y is $-\ mR^n Y/R_0 = -\ mR_0^{n-1}\ Y$ for any particular value of R denoted by R_0, which is to be regarded as constant for the moment. The differential coefficient is $-\ mR_0^{n-1}$, and we have $(n + 3)\ dR/dY$ for the ratio of the variations of force at any moment affecting dR and dY. By the definition of Z, $dY = dZ$, and the action of force upon Z is the same as its action on Y. The ratio of corresponding variations of force upon dR and dZ is therefore

$(n + 3)\, dR/dZ = w^2 dR/dZ$, the ratio dR/dZ being negative, since R decreases as Z increases.

We have now to determine the corresponding ratio of forces.

The action of a constant force upon a finite distance is proportional to the square of the time allowed for such action; and a force depending on the exponent of that distance is to be regarded as constant while we regard the distance as momentarily constant. This requires the time of action to be infinitesimal, and the squared time to be infinitesimal in the second order.

Similarly, the action of a constant variation of force upon the first differential of a distance is proportional to the squared time allowed for its action; and such a variation of force is to be regarded as constant while we regard that first differential as momentarily constant. This requires the time of action to be infinitesimal with respect to that first differential, and the squared time to be similarly infinitesimal in the second order. If we consider the time of action to be in the first instance infinitesimal in the second order, its square is infinitesimal in the fourth order. But the relative order of the two infinitesimals of time is in either case that of a quantity and its square.

In the absence of force, dR and dZ vary uniformly with time; and while the time allowed for their variation is the same for each, their ratio remains unaffected by the order of infinitesimals to which that time belongs.

In the expression for the ratio of variations of force, $w^2 dR/dZ$, the factor w^2 is equivalent to a ratio between the squared times allowed for the variations of dR and dZ, to whatever order of infinitesimals these times belong. For if the original ratio dR/dZ is multiplied by such a ratio between those squared times, the product will be w^2 times as great as before.

The square root of that ratio of squared times will be the ratio of the times by which dR and dZ are multiplied in the corresponding expression for the ratio of the forces acting upon them. This process is an integration merely in the sense that the integral of an infinite series of squared differentials of time is the first power of such a differential. The ratio dR/dZ is unaffected, as shown above, by the change thus indicated in the order of infinitesimals of time.

If, therefore, $w^2 dR/dZ$ represents a ratio of variations of force, the corresponding ratio of forces is $w dR/dZ$.

V.

Equations of curves derived from the preceding laws.

Let us first consider cases in which $w > 1$. In these, an apastron will ultimately occur. For, when $R > P$ and is increasing, the inward force is the stronger, and tends to retard the increase of R and the decrease of Z. But if $w > 1$, and the forces have the ratio wdR/dZ, the tendency to check the increase of R exceeds that checking the decrease of Z. Hence dR will ultimately become less than $-dZ$, however much it may have exceeded $-dZ$ when R had the value P; so that R will subsequently increase more slowly than Z decreases, Z will reach the value 0 before R reaches the value ∞, and, by the definition of Z, R cannot further increase, and must attain its greatest value L. The quantities K and L must then be regarded as finite, although we cannot set a definite limit to their increase.

When $w < 1$, the chance of an apastron diminishes. When $w = 0$, it has already appeared that R always or never has the value P.

We will now attempt to find functions of R and Z, the variations of which, when corrected for the corresponding forces, shall have a constant ratio.

The ratio wdR/dZ is a ratio of forces acting directly on dR and dZ, and indirectly on R and Z themselves. The corresponding ratio of forces acting directly on $R^{w-1}\,dR$ and $Z^{w-1}dZ$ is R^{w-1}/Z^{w-1} times as great, that is, $wR^{w-1}dR/Z^{w-1}dZ$. This is also the ratio of variations of wR^w and of Z^w, which is $w^2R^{w-1}dR/wZ^{w-1}dZ$, when we regard R and Z as varying uniformly, undisturbed by any force. The ratio of forces acting upon the variations is accordingly the same as the ratio of the variations themselves when undisturbed by force.

In all such cases, the existing ratio of the variations is not changed by the application of the forces, not merely at a single instant, when the forces are negligible as compared with the variations on which they act, but after they have developed accelerating or retarding effects upon the variations. Let A and B denote two velocities, a and b forces acting upon them. If $A/B = a/b = k$, so that $A = kB$ and $a = kb$, $(A + a)/(B + b) = k(B + b)/(B + b) = k$.

We conclude, therefore, that the variations of wR^w and of Z^w have a constant ratio c' when the forces acting upon them have the ratio wdR/dZ. Hence the variations of R^w and of Z^w have the constant ratio $c'/w = c$.

At periastron, the value of R is Q and that of Z is K. At other points, therefore, $(R^w - Q^w) = e(K^w - Z^w)$, which will be one equation of the curve described by the moving particle. When apastron occurs, $R = L$ and $Z = 0$, so that $e = (L^w - Q^w)/K^w$.

Unless e' is infinite, e tends to approach 0 as w increases, and the orbit ultimately becomes a circle. This is also evident from the consideration that, with w indefinitely large, any radial movement corresponding to a given transverse movement will instantly be checked by the unlimited increase of inward force if the radial movement is outward, or of outward force if that movement is inward.

A second form of the equation of the curve, when apastron occurs, is obviously $(L^w - R^w) = eZ^w$, in which, when $R = Q$, $Z = K$, and $e = (L^w - Q^w)/K^w$ as before.

Equations representing the curve may also be derived from the quantities X, S, and H, corresponding, when measures are made along the axis of apastron, to Y, Z, and K in the system above employed. The values of X and S, like those of Y and Z, vary uniformly in the intervals between successive applications of force, so that $dR/dX = dR/dS$, and is proportional to dR/dZ, but not equal to it except in a circular orbit.

By the method already explained, we find the ratio of the corresponding variations of force on R and on S to be $(n + 3)\,dR/dS$; the ratio of forces $w\,dR/dS$; and the resulting equations $(L^w - R^w) = e\,(H^w - S^w)$ and $R^w - Q^w = e\,S^w$. The value of e now becomes $(L^w - Q^w)/H^w$. Denoting the former value of e by e_1, and the new value by e_2, we find $e_2 = e_1K^w/H^w$.

We found above $(L^w - R^w) = e_1Z^w$, and we now find $(L^w - R^w) = e_2\,(H^w - S^w)$, whence $(K^w/H^w)\,(H^w - S^w) = Z^w$, and $H^w Z^w + K^w S^w = H^w K^w$. When $w = 2$, $H = A$, the semi-axis major; $K = B$, the semi-axis minor; $Z = Y$, and $S = X$. We then have the familiar equation $A^2\,Y^2 + B^2\,X^2 = A^2\,B^2$. When $w = 1$, and apastron occurs, as is here assumed to be the fact, $H = L + Q$ and $K = Q + L$, the value of each being the major axis $2A$; $S = 0$ at periastron, and $Z = 0$ at apastron; the equation becomes $S + Z = 2A$.

By the substitution of $R \cos r$ for Y, we convert the equation $(L^w - R^w) = e\,Z^w$ into $L^w = R^w + e\,(R \cos r - L \cos r_0)^w$, but unless $w = 1$ or 2, this form is less convenient than those above stated.

VI.

Angles formed by the axes of periastron and apastron.

The substitution of $R \cos v$ for Y is advantageous, however, in finding the value of v_0, when $R \cos v$ is taken as the independent variable, since $dZ = dY$. Hence the corresponding changes in R and in $\cos v$ are reciprocal.

At periastron, $dR = dZ = 0$, and the ratio dR/dZ is indeterminate for the moment. The subsequent values of dR and dZ result from forces which have the ratio w, as was found above, in section IV. This ratio is independent of the order to which the infinitesimals of time belong, since the time allowed for the action of the forces is the same for dR as for dZ. The variations produced by these forces, and the consequent changes in R and Z themselves, retain this ratio.

While R varies from Q to L, Z varies from K to 0, and $(K - Z)$ from 0 to K. The change in R is w times the change in Z. Hence the corresponding change in $\cos v$ is $1/w$ as great as that in Y, which is equal to that in Z.

When $w = 1$, dR/dZ is constant, and the total variation of $\cos v$, from periastron to apastron, is 2, from $+ 1$ to $- 1$. In other cases, the total variation of $\cos v$ is $2/w$. When $w = 2$, $2/w = 1$; hence $\cos v$ varies from $+ 1$ at periastron to 0 at apastron, and $v_0 = 90°$, as we know independently.

For values of w between 1 and 2, v_0 is obtuse, and $\cos v_0 = - (2/w - 1)$. For example, when the inward force is inversely proportional to the first power of the distance, $w = \sqrt{2}$, $\cos v_0 = - (\sqrt{2} - 1)$, and the approximate value of v_0 is $180° - 65°.5 = 114°.5$. When the inward force is constant, $w = \sqrt{3}$; $\cos v_0 = - (2/\sqrt{3} - 1)$, and the approximate value of v_0 is $180° - 81°.1 = 98°.9$.

When $w > 2$, v_0 becomes acute, and if w increases without limit, v_0 approaches 0. We have already seen that in this case the orbit is circular.

When $w < 1$, v_0 will exceed 180°, approaching 360° as w approaches 0. These cases will not here be minutely examined.

The value of K results from that of $\cos v_0$ by means of the equation $K = Q - L \cos v_0$, in which $\cos v_0$ is negative if v_0 is obtuse. The value of H is $L - Q \cos v_0$. The curve can accordingly be drawn when values of Q, L, and w are assumed.

When $w > 1$, $H > K$. For $H - K = (L - Q \cos v_0) - (Q - L \cos v_0) =$

$(L - Q)$ $(1 + \cos r_0)$, and $\cos r_0$ cannot be less than -1. If $w = 1$, $H = K$, the value of either being the major axis of the ellipse if apastron occurs. If not, $L = \infty$, and $(L - Q)$ $(1 + \cos r_0)$ becomes indeterminate. In either case, $e = (L - Q)/K = 2C/2A = C/A$, in which C and A have their usual meanings in the equations of a conic section, and e denotes the eccentricity.

When $w = 2$, $\cos r_0 = 0$, while H and K are respectively equal to the major and minor semi-axes of the ellipse. If the curve is referred to the axis of periastron, $e = (A^2 - B^2)/B^2$, in which A and B have their usual meanings of the two semi-axes, and e may have any value. If the curve is referred to the axis of apastron, $e = (A^2 - B^2)/A^2$, and cannot exceed 1, being the squared eccentricity of the ellipse.

VII.

Relative distances of points on the curve from the axis of periastron.

Since $R^w = (L^w - eZ^w)$ and $Y = Z + L \cos r_0$, while $R^2 = U^2 + Y^2$, $U^2 = (L^w - eZ^w)^{2/w} - (Z + L \cos r_0)^2$. By differentiation with respect to Z, $d(U^2) = (2/w)\,(L^w - eZ^w)^{2/w-1}\,(-e\,w\,Z^{w-1})\,dZ - 2\,(Z + L \cos r_0)dZ$, whence $d(U^2)/2 = -\,[(L^w - eZ^w)^{2/w-1}\,eZ^{w-1} + (Z + L \cos r_0)]\,dZ$. When $R = Q$, $dZ = 0$, and a minimum of U^2 and hence of U occurs, as is obviously the fact.

When r_0 is obtuse, a maximum of U^2 and hence of U occurs if $(Z + L \cos r_0) = -\,(L^w - eZ^w)^{2/w-1}\,eZ^{w-1}$. Since $(L^w - eZ^w) = R^w$, it is positive; Z and e are also positive. The maximum occurs, then, when $L \cos r_0$, which is negative, numerically exceeds Z.

In the ellipse, when $w = 1$, $L = A + C$, $e = C/A$, and $r_0 = 180°$. The condition for a maximum of U becomes $Z - A - C = -\,[(A + C) - CZ/A]\,C/A$. Multiplying by A^2 and reducing, $(A^2 - C^2)\,Z = A^2\,(A + C) - AC(A + C)$, whence $Z = A$, and the maximum of U is B, as is otherwise known.

When $w = 2$, $r_0 = 90°$ and $L \cos r_0 = 0$; the condition for a maximum of U is $Z = -\,eZ$; hence $Z = 0$, and the maximum of U is A, as is otherwise known.

When $w > 2$, $(Z + L \cos r_0)$ is positive, and no maximum of U, in the strict sense of that word, will occur. The greatest value of U will be $L \sin r_0$.

VIII.

Area included by the curve and its axes.

The value of U, as above, is $\sqrt{[(L^w - cZ^w)^{2/w} - (Z + L \cos v_0)^2]}$. The integration of UdZ between the limits $Z = K$ and $Z = 0$ should give twice the area enclosed by the curve and its axes, with the addition of the rectangle $L^2 \sin v_0 \cos v_0$. When $w = 1$ or $w = 2$, either $\sin v_0$ or $\cos v_0$ has the value 0, and this rectangle disappears. The integration, in other cases, if not directly practicable may be effected by the customary indirect methods. It is not here discussed.

When $w = 1$, $U^2 = [(A + C) - CZ/A]^2 - [Z - (A + C)]^2$, that is $(A+C)^2 - 2(A+C)CZ/A + C^2Z^2/A^2 - Z^2 + 2(A+C)Z - (A+C)^2$, and after reduction $[(C^2 - A^2)Z^2 - 2A(A + C)(C - A)Z]/A^2 = (2AB^2Z - B^2Z^2)/A^2$.

In the circle with radius A, the squared perpendicular which corresponds to U is expressed by $A^2 - (A - Z)^2 = 2AZ - Z^2$; so that U^2 has the ratio B^2/A^2 to this squared perpendicular, and U has the ratio B/A to the perpendicular itself, whence we find the elliptic area in the usual way, when apastron occurs.

When $w = 2$, $U^2 = A^2 - (A^2 - B^2)Z^2/B^2 + Z^2 = [A^2B^2 - A^2Z^2 + B^2Z^2 - B^2Z^2]/B^2$, which becomes $A^2(B^2 - Z^2)/B^2$. In the circle with radius B, the squared perpendicular which corresponds to U is $B^2 - Z^2$, so that U has the ratio A/B to that perpendicular, and the elliptic area is found as before.

IX.

Value of the radius vector at which the forces are equal.

It is now desired to find the value of the radius vector P. When $R = P$, the variation of R with respect to time is at a maximum. Denoting time by t, we now require a maximum value for $dR/2dt$, which may be expressed either as $dR/U\,dZ$ or as dR/R^2dv. In the first case, the independent variable will be Z; in the second, $\cos v$ may be more convenient than v itself.

A maximum of dR^2/U^2dZ^2 will occur with that of dR/UdZ. In the differentiation required to determine this maximum, it must be noticed that U^2 increases while Z diminishes, and hence that $d(U^2)/dZ$ is negative.

We have first to express dR^2/dZ^2 in terms of Z by means of the equation $R^w = L^w - eZ^w$, from which $wR^{w-1}\,dR = -weZ^{w-1}\,dZ$, and $-dR/dZ = eZ^{w-1}/R^{w-1} = eZ^{w-1}/(L^w - eZ^w)^{(w-1)/w}$, whence $dR^2/dZ^2 = e^2Z^{2(w-1)}/(L^w - eZ^w)^{2(w-1)/w}$.
Since $U^2 = (L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2$,
$dR^2/U^2dZ^2 = e^2Z^{2(w-1)}/[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2](L^w - eZ^w)^{2(w-1)/w}$.

We may omit the factor e^2, which will not affect the result.

The denominator of the expression obtained for dR^2/U^2dZ^2 is $U^2R^{2(w-1)}$. It is always positive, and $U = 0$ only at periastron. We have therefore only to consider the numerator of the required differential expression. It will consist of three terms, the differential of dR^2 multiplied by U^2dZ^2, the differential of U^2 multiplied by dR^2 and dZ^2, and the differential of dZ^2 multiplied by dR^2 and U^2. The second and third of these terms will have their original signs reversed.
The first term is $2(w-1)Z^{2(w-1)-1}\,dZ[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]$ $(L^w - eZ^w)^{2(w-1)/w}$. The second is $-Z^{2(w-1)}\,dZ[(L^w - eZ^w)^{2(w-1)/w}]$ $[2/w)(L^w - eZ^w)^{2/w-1}\,weZ^{w-1} + 2(Z + L\cos v_0)]$. The third is $Z^{2(w-1)}$ $[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]$ $[(2(w-1)/w)(L^w - eZ^w)^{2(w-1)/w-1}$ $weZ^{w-1}dZ]$.

We may remove the factor $2Z^{2(w-1)-1}\,dZ$, which indicates minima at periastron and apastron. The remaining factors may be reduced to the form

$$(w-1)\,[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]\,(L^w - eZ^w)^{2(w-1)/w} \qquad (1)$$
$$-Z\,[(L^w - eZ^w)^{2(w-1)/w}]\,[(L^w - eZ^w)^{2/w-1}\,eZ^{w-1} + (Z + L\cos v_0)] \qquad (2)$$
$$(w-1)Z\,[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]\,[(L^w - eZ^w)^{2(w-1)/w-1}\,eZ^{w-1}] \qquad (3)$$

the sum of these three terms being 0 when the maximum is reached. For any given value of w, the corresponding value of Z at the maximum may be found, indirectly if not directly. From this value of Z, the value of P is found by means of the equation of the curve.

When $w = 1$, the expression to be differentiated becomes $1/U^2$, and the proposed operation is not directly practicable. But in this case dR/dZ is constant, so that we may regard R as the independent variable. Since $U = R\sin v$, $dU < dR$ except when $U = R$, that is, when R is perpendicular to the axis of periastron, and an extreme value of dU will then occur. The semiparameter of the conic section is otherwise known to be the radius vector P, at which the outward and inward forces are equal.

When $w = 2$, the three terms found above, the sum of which is 0 when the maximum is reached, become

$$(A^2-eY^2-Y^2)\,(A^2-eY^2) = A^4-(1+e)A^2Y^2-eA^2Y^2+e(1+e)Y^4 \quad (1)$$

$$-(A^2-eY^2)\,(eY^2+Y^2) = -(1+e)A^2Y^2+e(1+e)Y^4 \quad (2)$$

$$(A^2-eY^2-Y^2)eY^2 = +eA^2Y^2-e(1+e)Y^4 \quad (3)$$

since then $L = A$, the semi-axis major, and $Z = Y$. The value of e in this case was found above to be $(A^2 - B^2)/B^2$, in which B denotes the semi-axis minor.

The equation determining the value of Y when $R = P$ is

$$A^4-2\,(1+e)A^2Y^2+e\,(1+e)Y^4 = 0.$$

The corresponding results from the use of R^2dv as an equivalent for $2dt$ will next be deduced from the equation $L^w = R^w + eZ^w = R^w + eR^w \cos^w v$, whence $R^w = L^w/(1 + e \cos^w v)$. To find dR, we have $wR^{w-1}\,dR = -L^w ew \cos^{w-1} v\, d \cos v/(1 + e \cos^w v)^2$, and $R^{w-1} = L^{w-1}/(1 + e \cos^w v)^{(w-1)/w}$, so that

$$dR = -d \cos v[eL \cos^{w-1} v(1+e \cos^w v)^{(w-1)/w-2}].$$

Also $1/R^2 = (1 + e \cos^w v)^{2/w}/L^2$, and $1/dv = -\sin v/d \cos v$.

Hence $dR/R^2dv = e \sin v\, L^{-1}\cos^{w-1} v(1+e \cos^w v)^{(w-1)/w-2}(1+e \cos^w v)^{2/w}$, in which we may omit the factor eL^{-1}, and differentiate the remaining expression with respect to $\cos v$ to obtain the condition for the maximum.

It will here be considered sufficient to examine the results thus obtained for the two cases $w = 1$ and $w = 2$.

When $w = 1$, $dR/R^2dv = e \sin v/L$, which obviously has a maximum when $v = 90°$, confirming the result less directly found above.

When $w = 2$, $dR/R^2dv = e \sin v \cos v/L\sqrt{(1 + e \cos^2 v)}$. Omitting e/L, and squaring, we require a maximum for $(1 - \cos^2 v) \cos^2 v/(1 + e \cos^2 v)$. The squared denominator is always positive, and we need to consider only the numerator of the differentiated expression. It will be the sum of three terms

$$-2 \cos^3 v\,(1 + e \cos^2 v) \quad (1)$$

$$2 \cos v\,(1 + e \cos^2 v)\,(1 - \cos^2 v) \quad (2)$$

$$-2e \cos^3 v\,(1 - \cos^2 v) \quad (3)$$

The minimum occurs when $\cos v = 0$, and in finding the maximum the factor $2 \cos v$ may be omitted.

$$\begin{array}{rcll} -\cos^2 v(1 + e\cos^2 v) & = & -\cos^2 v \qquad\qquad - e\cos^4 v & (1) \\ (1 - \cos^2 v)(1 + e\cos^2 v) & = & 1 - \cos^2 v + e\cos^2 v - e\cos^4 v & (2) \\ -e\cos^2 v(1 - \cos^2 v) & = & \qquad\qquad -e\cos^2 v + e\cos^4 v & (3) \end{array}$$

and the required equation is $1 - 2\cos^2 v - e\cos^4 v = 0$, which may also be expressed as $\sin^2 v - \cos^2 v - e\cos^4 v = 0$, the value of e being $(A^2 - B^2)/B^2$ as before. It appears from this equation that as e approaches 0, so that the orbit becomes nearly circular, the value of v, when $R = P$, approaches 45°; and that it increases toward 90° as the ellipse elongates.

We may also refer the axial measurements to the major axis of the ellipse by means of the equation $R^2 - B^2 = e^2X^2$, in which e denotes the eccentricity of the ellipse, so that $e^2 = (A^2 - B^2)/A^2$. Since $X = R \sin v$, $R^2 = B^2/(1 - e^2 \sin^2 v)$. The value of R^2dv is now $R^2d \sin v/\cos v = B^2d \sin v/\cos v\,(1 - e^2 \sin^2 v)$. From the value of R^2 just found, $2RdR = 2B^2e^2 \sin v\, d \sin v/(1 - e^2 \sin^2 v)^2 = 2BdR/\sqrt{(1 - e^2 \sin^2 v)}$, and $dR = Be^2 \sin v\, d \sin v/(1 - e^2 \sin^2 v)^{3/2}$, whence $dR/R^2dv = e^2 \sin v \cos v/B\sqrt{(1 - e^2 \sin^2 v)}$. Omitting e^2/B, a maximum is required for $\sin v \cos v/\sqrt{(1 - e^2 \sin^2 v)}$ or for its square $\sin^2 v \cos^2 v/(1 - e^2 \sin^2 v)$. The numerator of the differential coefficient is the sum of three terms.

$$\begin{array}{rl} 2 \sin v(1 - \sin^2 v)\,(1 - e^2 \sin^2 v) & \qquad (1) \\ -\,2 \sin^3 v(1 - e^2 \sin^2 v) & \qquad (2) \\ 2\, e^2 \sin^3 v(1 - \sin^2 v) & \qquad (3) \end{array}$$

The minimum occurs when $\sin v = 0$. Omitting the factor $2 \sin v$

$$\begin{array}{rcll} (1 - \sin^2 v)\,(1 - e^2 \sin^2 v) & = & 1 - \sin^2 v - e^2 \sin^2 v + e^2 \sin^4 v & (1) \\ -\sin^2 v\,(1 - e^2 \sin^2 v) & = & -\sin^2 v \qquad\qquad + e^2 \sin^4 v & (2) \\ e^2 \sin^2 v\,(1 - \sin^2 v) & = & \qquad\qquad + e^2 \sin^2 v - e^2 \sin^4 v & (3) \end{array}$$

and the required equation is $\cos^2 v - \sin^2 v + e^2 \sin^4 v = 0$, in which $e^2 = (A^2 - B^2)/A^2$.

We previously found $\sin^2 v - \cos^2 v - (A^2 - B^2) \cos^4 v/B^2 = 0$. The sum of the two equations is $(A^2 - B^2) \sin^4 v/A^2 - (A^2 - B^2) \cos^4 v/B^2 = 0$, whence $\sin^2 v : \cos^2 v = A : B$, and since $\sin^2 v = X^2/P^2$ and $\cos^2 v = Y^2/P^2$, $X^2 : Y^2 = A : B$ when $R = P$. This equation may also be derived from the employment of UdZ for the equivalent of $2dt$. It is needless to give the details of this process.

When $R = P$, $X = P \sin v$ and $Y = P \cos v$. It has just appeared that $A^2 = B^2 \tan^4 v$. Hence, from the equation $B^2X^2 + A^2Y^2 = A^2B^2$, after division by B^2, $P^2 (\sin^2 v + \tan^4 v \cos^2 v) = A^2$. Since $\tan^4 v \cos^2 v = \sin^4 v/\cos^2 v$, $\sin^2 v + \tan^4 v \cos^2 v = \sin^2 v(1 + \tan^2 v) = \sin^2 v/\cos^2 v = \tan^2 v$, and $P^2 \tan^2 v = A^2$. Accordingly, while the value of v increases as the ellipse elongates, as appeared above, the ratio A/P also increases.

Since $\tan^2 v = X^2/Y^2 = A/B$, when $R = P$, $AP^2/B = A^2$, and $P^2 = AB$. Also $\sin^2 v + \cos^2 v : \cos^2 v = (A + B) : B$, and $\cos^2 v = B/(A + B)$. Similarly, $\sin^2 v = A/(A + B)$. The position and length of the radius vector P, when $w = 2$, are thus determined.

As stated in section VI, cases in which $w < 1$ are not minutely examined in the present discussion. But it may be remarked that since, when $w = 1$, the radius vector P is perpendicular to the axis of periastron, we may expect P to make with that axis an angle v, greater than 90°, when $w < 1$. If no apastron occurs, the curve will not reach this direction of P at all, and the outward force will always be in excess; if R ever assumes the value P, it will subsequently attain the value L, for which value the angle v_0 will exceed 180°.

X.

Relative times of revolution in closed orbits of the same system.

Since the momentary tendency to transverse motion is expressed by the square root of the product of the radius vector and twice the outward force, as appeared above in section I, and since the outward force is expressed by mP^{n+3}/R^3, as in section III, the momentary transverse motion is expressed by $\sqrt{(2mP^{n+3}/R^2)}$. The corresponding increase of area is the product of this quantity by half the radius vector, hence $\sqrt{(mP^{n+3}/2)}$. The ratio of momentary increments of area in two orbits belonging to the same system is therefore the ratio of their respective values of $\sqrt{P^{n+3}}$. Let these values be denoted by $\sqrt{(P_1{}^{n+3})}$ and $\sqrt{(P_2{}^{n+3})}$, and let the corresponding areas traversed by the radii vectores from periastron to apastron be denoted by a_1 and a_2. The corresponding times of these angular movements are directly proportional to the areas and inversely proportional to the momentary increments of area, so that their ratio is $a_1\sqrt{(P_2{}^{n+3})}/a_2(\sqrt{P_1{}^{n+3}})$.

The possibility of determining the total areas is discussed above in section VIII.

When $w = 1$, $n + 3 = 1$, and the ratio of the areas in a closed orbit is that of half the products of the major and minor semi-axes. The value of P in this case is B^2/A, the proof of which need not here be repeated. The conclusion follows that the times of revolution are proportional to the square roots of the cubes of the major semi-axes.

When $w = 2$, $n + 3 = 4$, and $\sqrt{(P^{n+3})} = P^2$. The area described by the radius vector from periastron to apastron is one quarter of the area of the ellipse, and apastron always occurs. Hence the ratio of the two areas a_1 and a_2 is that of $A_1 B_1$ and $A_2 B_2$, in which A_1 and A_2, B_1 and B_2, respectively denote the major and minor semi-axes. We have found the value of P^2 in this case to be AB. Hence $a_1\sqrt{(P_2^{n+3})}/a_2\sqrt{(P_1^{n+3})}$ becomes $A_1B_1A_2B_2/A_2B_2A_1B_1 = 1$; that is, when $w = 2$, the same time is required for all orbits of that system. This result is obvious in the case of circular orbits, as Newton states in the first book of the Principia.

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OF HARVARD UNIVERSITY. No. LXXXVII.

NEW DIMORPHOMYCETEAE.

By Roland Thaxter.

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THE Dimorphomyceteae, or unisexual Peyritschiellaceae, include those unisexual forms among the Laboulbeniales in which the male individual possesses one or more compound antheridia, and include the three genera *Dimorphomyces*, *Dimeromyces* and *Streblomyces*, to which a third, *Polyandromyces*, is now added. In the present Contribution I have given a list of all forms of this type which have been published hitherto, and have added a large number from various parts of the world which are new and in many cases very peculiar and interesting. The greater number of the latter belong to *Dimeromyces*, a genus which, like *Rickia*, has proved to be unexpectedly important numerically, and to occur on a considerable variety of unrelated hosts belonging to various families of the Coleoptera, Diptera, Orthoptera and Acarini.

A study of this large series of species, and a comparison of more abundant material of *Dimorphomyces* than has been hitherto available, has served to make clear a number of doubtful points in regard to homologies and development in these two genera, and has emphasized the fact that the two are much more closely related than I had at first supposed.

Typical forms of *Dimorphomyces* are clearly and easily distinguished from normally developed species of *Dimeromyces* by the cell-relations in the mature female, which, in the last mentioned genus, consists of a primary axis of superposed cells, variable in number and terminated by a more or less clearly distinguished terminal primary appendage. In very young individuals this axis consists of a two-celled primary receptacle, terminated by the appendage. The basal and subbasal cell of the primary receptacle, which are formed by the first division of the basal spore-segment, are thus in contact one above the other;

but owing to the early activity of the former, from the upper portion of which new cells are rapidly separated, the relation of the two is changed, and the primary subbasal cell becomes the terminal cell of a series which is thus intercalated between the two, and usually forms the fertile portion of a variably developed axis. This simple condition may be complicated by further secondary cell-activities in special cases, but the fundamental cell-relations are the same in all.

In *Dimorphomyces*, on the other hand, although essentially the same process of development takes place and the series of intercalated cells, which has been called the secondary receptacle and from which the perithecia and secondary appendages arise, is produced in a similar fashion, the primary subbasal cell, with its primary appendage, retain their original relations to the basal cell. The apparent difference between the two is due to the fact that an outgrowth from the basal cell pushes up beside the subbasal cell, and cutting off a variable number of successive cells from its tip, gives rise to the so-called secondary axis. Although the base of this axis appears to have been derived from the subbasal cell, it is necessarily associated throughout its length with the progressive protrusion of the basal cell which forms a continuous margin, the lumen of which extends from its apex to the foot; although in older individuals it may be somewhat obliterated by secondary thickening of the wall. The secondary fertile axis is thus also a secondary development intercalated between the primary basal and subbasal cells, and although the latter remains in contact with the former, it represents the terminal cell of the axis in *Dimeromyces*, the subbasal cell of which would correspond to the distal cell of the secondary axis in *Dimorphomyces*. In certain species of the genus a further complication arises from the production of a second fertile axis, similar to the first in structure and development, and often symmetrical in form and position, on the opposite side; but which originates from the primary subbasal cell, an extension of the latter forming a continuous margin on its under side exactly like that of the first, but differing from the fact that its lumen is continuous with that of the subbasal cell. In certain aberrant species of *Dimeromyces*, like *D. Necrotalis* herewith described, a somewhat similar condition is present; the fertile axis being more or less horizontal, and two or three of its lower cells being margined by a slight extension of the basal cell. As in all other species of the genus, however, the primary subbasal cell has become the terminal cell, far removed from contact with the basal and bearing the primary appendage. On the other hand certain species of *Dimorphomyces* like *D. Pachytelis* and certain

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Polyandromyces nov. gen.

Male individual consisting of a two-celled receptacle bearing a single terminal antheridium. Antheridium consisting of a well defined stalk-cell, a venter containing very numerous antheridial cells discharging its bacillus-like antherozoids into a large cavity and out through a well developed neck.

Female individual. Receptacle consisting of five superposed cells, terminated by a two-celled undifferentiated appendage, a single perithecium arising from the third cell. Stalk- and basal cells of the perithecium obliterated at maturity, the cavity continuous below the ascogenous cell to the insertion.

Except for the greater number of cells in its receptacle, the female of this type closely resembles that of *Nycteromyces* to which it is most nearly allied. The male, however, differs widely in possessing a single terminal antheridium, which is more highly developed than that of any other genus, with the exception of *Eumonoicomyces.*

Polyandromyces Coptosomalis nov. sp.

Male individual. Basal cell of receptacle relatively large and long, hyaline or yellowish, straight or often strongly curved, obliquely adjusted to the small triangular brownish yellow subbasal cell. Antheridium straight, erect, reddish brown below, the base of the venter darker; the stalk-cell squarish, or slightly broader than long; the venter relatively large, symmetrical, somewhat inflated, often rather abruptly broader; the efferent chamber large, almost dome-shaped, separated by a slight constriction from the characteristic enlargement which forms the base of the efferent neck. Antheridium 64 × 25 μ; stalk-cell 16 × 16–21 μ; venter 30 × 26 μ; efferent chamber 125 × 16 μ; neck 22 μ. Total length, average, 135 μ.

Female individual yellowish, becoming more or less deeply tinged with reddish brown. Receptacle consisting of five obliquely superposed cells; the basal larger, the subbasal separating a small cell which subtends, externally, the short stalk of the perithecium which arises from the third cell: the fourth cell smaller than the rest, narrowly triangular, oblique: appendage erect or suberect, the distal cell slightly longer, somewhat tapering and bluntly rounded. Perithecium erect, nearly symmetrical above the very short curved stalk, the margins slightly and symmetrically convex: the tip clearly distinguished by a more prominent basal and less distinct distal elevation and intervening depression: apex very short, abruptly defined, distally flattened or slightly rounded. Perithecia 24-280–50–55 μ. Appendage 30 × 20 μ. Receptacle 105 × 34 μ. Total length to tip of perithecium 300–345 μ.

On the legs of *Coptosoma maculatum* Westw. No. 2800 (Type), Madagascar (M. C. Z., Wulsin). No. 2960 on *Coptosoma* sp. Suene, Fiji (M. C. Z., Mann), also 2960, Wai Ai, Solomon Islands (M. C. Z., Mann).

The material from Fiji and the Solomons is somewhat smaller than the type and grows for the most part on the margin of the abdomen. The differentiation of the tip is even more pronounced, the appendage slightly narrower and more reddish: but there are no essential differences. The antheridia seem to contain several dozen antheridial cells, and the sperm-cells are discharged in·a mass which fills the chamber and neck with bacillus-like elements.

A form has been observed on a *Coptosoma* from Kamerun which is perhaps sufficiently distinct for varietal separation and may be distinguished as

POLYANDROMYCES COPTOSOMALIS, var. **minor** nov. var.

The general structure similar to that of the type, smaller and stouter, the perithecium asymmetrical, externally straight or concave, the tip hardly distinguished, the apex relatively broader. The male similar but somewhat smaller. Perithecia 135 × 38–150 × 42 μ. Receptacle 40 × 34 μ. Appendage 20 × 12 μ. Total length 160–190 μ. The male 100–105 μ.

On *Coptosoma* sp. 3103. Kamerun, West Africa.

Although this form is separable at a glance from any of the abundant specimens of the type, owing to its smaller size and somewhat different perithecium, the variation does not seem to be more than might readily be included in a single species. The variety was found only at the tip of the abdomen, and is seen with difficulty beneath the projecting segment.

Dimorphomyces Bledii nov. sp.

Male individual relatively large and stout, faintly tinged with brownish yellow, receptacle composed of two nearly equal cells obliquely separated. Appendage short stout two celled, the wall of the upper of the two nearly equal cells distally thickened. Appendage and antheridium diverging slightly, the latter sessile on a broad base, the stalk-cell very small, the venter inflated, broad, the neck well distinguished, bent upward. Antheridium 23 × 9.5 μ. Appendage 9.5 × 5 μ. Receptacle 13 × 8 μ. Total length to tip of antheridium 42 μ.

Female individual pale yellowish with brown tinges. Receptacle developing two fertile axes; a primary, larger and margined by the basal cell; a secondary shorter, asymmetrical in relation to the primary, developed from and margined by an outgrowth of the subbasal cell. The axes of the usual type, the cells bearing either long flexed appendages, of from seven to fourteen cells, more slender toward the base, the terminal cell also often distinctly more slender; or here and there short bent appendages of two cells separated by a dark septum. Perithecia two or several, the stalk well developed, not abruptly distinguished, the ascigerous portion straight or somewhat bent distally and darker brown, subfusiform, tapering rather gradually to the blunt termination; the tip and apex not distinguished. Primary appendage subcylindrical, two-celled (? always), the cells about equal, the distal thick-walled and bluntly rounded, the basal somewhat darker. Perithecia 80–135 × 21–22 μ. Primary appendage 25 × 8 μ. Longest appendages 250 × 8 μ, short appendage 20 μ.

On the head and prothorax of *Bledius emarginatus* Say. Lawrence, Kansas.

I am indebted to Professor Alban Stewart for the host of this species, which he kindly collected for me at electric light. The form described was accompanied by a smaller one on the abdomen which may prove a different species; but owing to the scantiness of the material it has seemed best to defer its separation. It is allied to *D. vulgatissimus*, but differs in its long stout secondary appendages which bear no resemblance to the slender tapering distally attenuated appendages of the last mentioned species.

Dimorphomyces furcatus nov. sp.

Male individual pale brownish, the subbasal cell externally nearly opaque. Basal cell slightly longer than the subbasal, separated from the base of the antheridium by an abrupt indentation. Appendage slender, tapering, its three cells subequal in length, the tip of the distal slightly enlarged. Antheridium stout, sessile, asymmetrical, bulging externally, the stalk-cell well defined; the neck clearly distinguished rather short. Antheridium 28.5 × 11.5 μ. Appendage 29 μ. Receptacle, without foot, 32 μ. Total length to tip of appendage 56 μ.

Female individual becoming rather deeply tinged with brown. Secondary axes two; one, usually slightly larger, developed directly from the small basal cell; the other from the subbasal cell; the general habit irregularly furcate, asymmetrical, the plane of one axis being usually turned at right angles to that of the other: the first axis diverging at an angle of about 45°, or less; the eight to sixteen cells obliquely superposed. Primary appendage short, two-celled, becoming blackened about the septum and distally: secondary appendages of two forms: short blackened of one or two cells; or much longer, four to six celled, the distal cell distinguished by a slight broad blackened somewhat constricted area. Perithecia relatively large, much longer than the secondary appendages; the stalk portion stout, rather clearly but not abruptly distinguished; the ascigerous region subsymmetrical, the margins convex, progressively darker smoky brown to its junction with the abruptly paler dirty yellowish tip and apex, which are distinguished below by a broad and often very distinct depression; the margins of the tip slightly convex, the short apex slightly distinguished, somewhat asymmetrical as a rule, blunt, the lips often slightly and variably prominent. Perithecia 140–180 × 22–27 μ. Longer secondary appendages 45-95 × 8.5 μ. Secondary receptacles 60–95 × 20 μ.

On *Apocellus* sp., No. 1581, and 1582, Agua Caliente, Guatemala. (Kellerman).

Although the material of the female is sufficiently abundant, one male, only, has been examined. The former seems well distinguished from *D. verticalis* and *D. Myrmedoniae* by its furcate habit, long stout multicellular secondary appendage, and more specialized perithecia. The position of the second secondary appendage, which arises from the subbasal cell and which developes sidewise and is viewed edgewise, makes it almost impossible to determine the exact character of the small dark primary appendage which it subtends. It is also nearly allied to *D. vulgatissimus* of Spegazzini, from which it differs in the character of its secondary appendages and in the relative position and character of its secondary receptacles.

Dimorphomyces brevirostris nov. sp.

Male individual yellowish, becoming tinged with brownish, a deep blackish brown suffusion involving a part, or almost the whole, of the basal cell, and more or less of the posterior margin of the rest of the receptacle; which consists of two or three cells, according as one or two antheridia are developed: the basal cell rather long and narrow, forming a rather well developed stalk, its narrow termination extending beneath the stalk-cell of the lowest antheridium. Appendage of two stout subequal cells, the upper much thickened distally, and rounded or flattened and somewhat blackish. Antheridia one or often two, straight, erect or slightly divergent, sessile or nearly so, the stalk-cell clearly defined, the venter relatively large, stout and abruptly distinguished from the broad efferent tube region; the neck very short and stout, almost obsolete. Antheridia 23–26 × 9.5–11 μ. Appendage 13 μ. Receptacle 25–27 × 10 μ. Total length to tip of antheridia 45–50 μ.

Female individual becoming tinged with brownish yellow, a clearly defined and usually contrasting blackish brown area involving the upper half or more of the basal cell proper and the adjacent lower half of the subbasal cell, sometimes also the lower cell of the secondary axis: the basal cell proper forming a well defined rather abruptly distinguished short stalk. Primary appendage usually two-celled, the cells nearly equal, stout with blackish shades about the septa, distally rounded, the walls thickened. Secondary receptacle of six to twelve cells, the series diverging at an angle of about 45°, usually straight and rigid, more rarely slightly curved, slightly tapering, distally bluntly rounded; producing from one to four perithecia, alternating with much shorter two-celled stout secondary appendages; their distal cells tapering very slightly, with rather broadly rounded terminations. Perithecia long fusiform the stalk hardly distinguished; the tip reddish brown, often distinguished by a slight depression when fully mature; the apex slightly broader, flattened or rounded, the inner margin slightly more prominent. Perithecia 75–106 × 16–20 μ. Primary appendage 23–28 × 10–11.5 μ. Secondary 33–42 × 11.5 μ. Secondary receptacle 30–55 × 12–13 μ. Primary basal cell 9 μ.

On the abdomen of Erchomus (*Coproporus*) sp.? No. 1662, Mandeville, Jamaica. No. 1652, Los Amates, Guatemala. No. 2228, Amazon (Mann).

This species is most nearly related to *D. Myrmedoniae*, which differs in its brown subcylindrical perithecia, longer secondary appendages, in the absence of a conspicuous blackish suffusion of the basal cell-region, so conspicuous in the present species; but especially in the male, which in *D. brevirostris* is clearly defined by its blackish suffusion, and the almost obsolete neck of its antheridium.

Dimorphomyces Eleusinus nov. sp.

Male individual tinged with pale brownish yellow, the neck of the antheridium reddish brown. Receptacle two-celled; the appendage two- (or ? three-) celled, the distal cell distinguished by a dark septum, flask-shaped, its distal third forming a narrow terminal point. Antheridium relatively large, the stalk well defined, the rather short neck slightly curved outward or sidewise. Antheridia 35 × 9 μ. Appendage 20 μ. Receptacle 22 μ. Total length to tip of antheridium 75 μ.

Female individual yellowish, becoming tinged with brownish. Secondary axis of from four or five to ten cells obliquely superposed, clearly margined below by the basal cell, the series diverging at an angle of somewhat less than 45°. Primary appendage similar to that of the male, three-celled. Secondary a pendages two or three to seven, mostly dark red-brown, with one or two septa, the lower sometimes dark; the extremity abruptly narrower paler and bent. Perithecium falcate, curved strongly outward over the secondary appendages; the stalk short and not distinguished from the ascigerous part; which is broadest in the middle; the tip not distinguished except by a deep red-brown suffusion along its concave exterior margin; the apex undifferentiated except for a slight indentation below the slightly prominent inner lip. Perithecia 95–105 × 22–23 μ. Secondary receptacle 40–66 × 13 μ. Primary appendage 20 × 9 μ at base: secondary 50–58 × 8 μ. Total length 130–144 μ to tip of perithecium.

On the margin of the abdomen near the tip of *Eleusis* (*Isomalus*) sp. No. 3128, Los Baños, Luzon, Philippine Is.

The male of this species has been seen only in association with the female and viewed edgewise, so that it is uncertain whether the appendage consists of two or three cells, or what the form of the receptacle would be in lateral view. The form is well distinguished by the peculiar compressed terminations of all the appendages in both sexes, and by the falcate habit and external distal red-brown suffusion of the perithecium, which is single in each of the four specimens examined, arising close to the primary appendage.

Dimorphomyces Grenadinus nov. sp.

Male individual nearly hyaline, rather slender, the basal cell very small the subbasal long and narrow, about equal to the basal cell of the three-celled appendage, the distal cell of which is subulate. Antheridium relatively large, the stalk-cell long and narrow, free only externally, the venter well distinguished from the very long slender neck. Antheridia 38 × 8 μ. Receptacle 13 μ. Appendage 20 μ. Total length 51 μ.

Female individual pale yellowish. The basal cell of the primary appendage similar to the subbasal cell of the receptacle, two small cells separated distally, the upper abruptly attenuated. Secondary appendages stout, mostly curved away from the perithecium, somewhat irregular in outline, six to nine-celled. Perithecium single, next the primary appendage and lateral, the stalk not distinguished, the distal third, or half, tapering very slightly to the broad termination, the margin of which is slightly convex, angular on the outer, and subtended by a variably developed short nearly cylindrical projection on the inner side. Perithecia 64–75 × 17 μ. Primary appendage 25–35 × 8 μ. Secondary appendages, longer, 105 × 10 μ at base.

At the tip of the abdomen of a small aleocharid. Grand Etang, Granada, B. W. I.

The host of this species, which was captured by sweeping in the forest about the Grand Etang, has been unfortunately lost or mislaid. The characters of the species are quite distinctive, the stout curved undifferentiated secondary appendages, the compressed terminal subulate cell of the primary appendage in both sexes, the very long slender neck of the antheridium, and the small well defined projection from the inner angle of the apex of the perithecium, serve clearly to distinguish it. The conformation is such that the individuals do not lie flat, and a complete side view of the female cannot be obtained.

Dimorphomyces simplex nov. sp.

Male individual short and stout, uniform pale yellowish brown. Receptacle consisting of two or three cells, and bearing one or two antheridia, the basal short and protruding externally beneath the base of the cell or base of the antheridium above it. Appendage rather short, erect, two-celled. Antheridia stout, erect, or but slightly divergent, the stalk-cell clearly defined, the efferent portion of the neck bent abruptly sidewise. Receptacle, without foot, 19 μ. Antheridia, to curvature of neck, 20–23 × 8 μ. Appendage 12 μ. Total length to curvature of neck 38 μ.

Female individual uniform pale yellowish brown, the body of the perithecium and distal cell of the appendages darker. Receptacle short and broad; the basal cell short, its extension clearly defined, developed on one side only, and curved upward abruptly beneath the usually single secondary appendage; the, usually three, remaining cells bearing the primary appendage, perithecium and secondary appendage respectively. Primary appendage erect, consisting of two nearly equal cells, or sometimes three, the distal terminally somewhat disorganized; the cell of the receptacle which bears it, rarely producing a second perithecium; the secondary appendage similar to the primary, two-celled, slightly shorter and more slender. Perithecium usually single, the stalk often rather abruptly broader, distally, below the ascigerous portion; which is bent abruptly sidewise at an angle of about 45°; the inner margin strongly convex, the outer concave; the tip distinguished by a very slight elevation externally; tapering to the bluntly pointed apex. Perithecium 65–80 × 12–15 μ. Receptacle, exclusive of foot, 16 × 16–20 × 20 μ. Primary appendage 21 × 6 μ, secondary 19 × 5.5–6 μ. Total length to tip of perithecium 95–105 μ. Foot large.

On the prothorax and elytra of a small tenebrionid (?), No. 2388, Mindanao, P. I. (Weber).

The male and female of this species are so firmly united that I have not been able to isolate the former in good condition. The antheridial neck seems well developed, but is turned abruptly at right angles so that it is viewed end on. The receptacle of the female is relatively very small, with the extension of the basal cell conspicuous and externally upcurved. The goose-neck habit of the perithecium and its stalk are characteristic.

Dimorphomyces clavuliferus nov. sp.

Male individual consisting of two cells, or with a small third cell not distinguished from the base of the erect, two- to three-celled appendage; the terminal cell of which is brown, the subterminal sometimes distinguished by dark septa above as well as below and becoming brownish. Basal cell of the receptacle extending out below the base of the single antheridium, and obliquely separated from the smaller subbasal cell. Antheridium sessile, its stalk-cell small and flat, the venter bulging externally, the neck curved or bent sidewise. Antheridia 16 × 6 μ. Appendage 8–10 × 3.5 μ. Receptacle 15 × 9 μ. Total length to tip of appendage 32 μ.

Female individual subflabelliform, the basal cell forming a slender short stalk and extending sidewise below the four to nine cells of the compact receptacle. Primary appendage of two or three cells, the distal brown and rounded, the middle, if present, distinguished by dark septa. The cell next the subbasal producing the single perithecium, the rest secondary appendages; which consist of a short basal cell, distinguished above from a long clavate terminal cell by a broad dark septum; the termination conspicuously swollen, darker brown than the more slender portion below. Perithecium with a very short abruptly narrow stalk, or almost sessile; long ovoid, bent to one side, with brown, blunt termination. Perithecia about 30 × 15 μ. Primary appendage 9–12 × 5.5 μ; secondary 26–28 × 7.5 μ at tip. Receptacle about 20 × 20 μ more or less.

On the inferior surface of a species of *Uroscius*. Baguio Mt., Luzon, P. I.

The mite bearing this very peculiar species was found in the sediment of a small lot of miscellaneous beetles kindly sent me by Mr. T. V. Reed. The material is not abundant, but the form is very clearly distinguished, by the character of its relatively short broadly clavate secondary appendages, from any other with which it might be confused. It is most nearly related to *D. triangularis*.

Dimorphomyces triangularis nov. sp.

Male individual. Receptacle two-celled, the basal cell long and obliquely flattened, extending upward and outward beneath the antheridium, which appears to be seated on its extremity: subbasal cell oblique below; the appendage three-celled; the basal cell much larger, as long as the second and third combined, the second distinguished above and below by dark septa, and tinged with brownish. Antheridium sessile, the stalk-cell minute and triangular, nearly straight on the inner side, the outer margin bulging prominently, the neck relatively long, erect, strongly curved outward distally, and faintly purplish. Antheridia 17 × 5.7 μ. Appendage 7.6 × 3.5 μ. Receptacle 11.5 × 7.5 μ. Total length to tip of perithecium 32 μ.

Female individual becoming unevenly tinged with brownish, usually bent sidewise so that it lies flat on the substratum; the receptacle small and compact, consisting of five or six cells, including the basal which extends out below the secondary portion; the subbasal cell larger than the others, bearing the stout usually three-celled primary appendage, its basal cell broad and undistinguished, the two others subequal, abruptly much smaller, separated by a narrow black septum; the terminal cell dark brown, a single perithecium arising from the cell next the subbasal, the rest producing secondary appendages consisting usually of three cells, the basal and subbasal subequal and hyaline, the rest of the appendage more or less elongate often slightly narrower above the subbasal cell, purplish brown, tapering more or less to the blunt, or usually rather abruptly enlarged, clavate termination; the successive appendages tending to lie close together and parallel. Perithecium triangular or subtriangular, wholly ascigerous, except for the very short narrow stalk which attaches it by its greater angle; becoming reddish brown, the short apex, only, slightly distinguished, and symmetrically rounded. Perithecia 28–30 × 15–17 μ. Primary appendage 10–12 × 7.5 μ, at base: Secondary 40–125 μ. Receptacle 22–28 × 11.5 μ.

On the legs of species of *Celaenopsis*, No. 2069 (Type) Wainoni Bay, Solomon Islands, and No. 2970, Fulakora.

This species is most nearly allied to *D. clavuliferus* and is chiefly remarkable for its triangular perithecium which is shaped somewhat like the ascigerous portion in *Dioicomyces malleolaris*, and is similarly inserted on its stalk. In the specimens from Fulakora the appendages are longer and more attenuated, none of them ending in the bulbous termination which usually occurs in the material from Wainoni Bay. The general appearance of the form, apart from the specific peculiarities of its perithecium and receptacle suggests that of *Rickia Formicicola* Speg., and it seems not impossible that Spegazzini has not accurately interpreted the characters of the latter, which may possibly prove to be, after all, a species of *Dimorphomyces* or of *Dimeromyces*.

Dimorphomyces obliqueseptatus nov. sp.

Male individual pale yellowish. Receptacle consisting of three cells, the basal and subbasal similar, the latter larger, both distally prominent, about twice as long as broad, subovoid, distinguished by a slight constriction: appendage usually four-celled, the three basal broad and short, the two middle distinguished by rather deep constrictions and distinct black septa. Antheridia usually two, from the subbasal and terminal cells, nearly sessile and erect, the regions clearly distinguished; the neck strongly curved outward or even recurved. Receptacle 38–42 × 16 μ. Appendage 30 × 5 μ. Antheridia 30 × 8.5 μ. Total length to tip of antheridia 68 μ, of appendage 70 μ.

Female individual pale yellowish or brownish yellow. Receptacle rather stout and compact, the basal and subbasal cells nearly equal or the latter usually larger and reaching nearly to the foot: the three to four cells of the secondary portion of the receptacle triangular, more or less divergent from the angle between them, and producing a perithecium and two to three secondary appendages. The former arising either from the first secondary cell or from the second, in which case it is associated with the first secondary appendage which arises externally from the same cell: the second and third secondary cells becoming so displaced, as a rule, that they lose contact with the basal cell. Primary appendage consisting of a large basal cell bearing the appendage terminally and a branch laterally, the basal cell of which occupies its whole outer face: the appendage five celled above its base, the second third, and to a less degree the fourth, roundish, separated by distinct constrictions and horizontal black septa; the terminal cell elongate and tapering; the external branch variably developed, the four subterminal cells short; the two middle ones distinguished by conspicuously oblique and blackened septa: the lower, secondary, appendages variable, of ten cells or less, the four subterminal cells similarly modified. Perithecia erect, usually slightly curved, the regions hardly indicated, rather short, the termination blunt and unmodified. Perithecia 55–75 × 13–14 μ. Receptacle 30 × 16 μ. Primary appendage 38–50 × 4.5 μ: its branch and the secondary appendages, longer 120 μ. Total length to tip of perithecium 105–120 μ.

On the legs of *Pachyteles* sp. Verdant Vale, Arima, Trinidad, B. W. I., No. 2821.

This species, which is closely allied to *D. Pachytelis*, appears to be distinguished from it by the characters already mentioned; the curvature of the antheridial necks which resemble those of *D. ramosus*, and the conspicuously blackened and oblique septa near the extremities of the secondary appendages and the basal branch of the primary, being the most evident differences.

Owing to the displacement of the second and third of the secondary cells of the receptacle, they lose their contact with the basal cell, from which they were derived, more or less completely; so that the cell relations in the mature individual are quite confusing, and constitute a deviation from the type which serves still further to diminish the fundamental differences between this genus and *Dimeromyces*.

Dimorphomyces Pachytelis nov. sp.

Male individual tinged with brownish yellow. Receptacle consisting of three cells, the basal long and narrow, extending up to the base of the lower antheridium, below which it is slightly prominent; nearly twice as long as the subbasal, the distal end of which is much broader than the base of the terminal cell which is much smaller. Appendage erect, curved toward the antheridia, consisting of usually five cells, the three middle ones distinguished by very slight constrictions and somewhat darker septa, the terminal cell long and tapering. Primary antheridium arising from the subbasal cell; one, or sometimes two, usually developing later from the terminal cell. Antheridia almost sessile, the venters relatively stout, the necks rather long and clearly distinguished, nearly straight, but bent outwards at the base. Receptacle 65–85 × 18–28 μ, including foot. Appendage, longest, 68 × 6 μ. Antheridia 35–42 × 9.5–11 μ.

Female individual dull brownish yellow. Basal and subbasal cells of the receptacle, including their extensions, subequal, obliquely associated, the former reaching nearly to the base of the latter. The secondary portion of the axis consisting of usually two cells obliquely margined below by the extension upward of the basal cell; the second, outer, giving rise to the usually single secondary appendage externally; the first, inner, bearing the normally single perithecium. Secondary appendage consisting of about a dozen cells, rather stout, rigid, slightly tapering, one to three of its subterminal cells short, small and distinguished by darker septa. Primary appendage consisting of a relatively large slightly tapering basal cell, about twice as long as broad, which bears terminally the short, more slender, abruptly distinguished terminal portion which tapers distally and consists of four or five cells, the three lower slightly longer than broad, the second and third distinguished by darker horizontal septa: while its outer margin is occupied by the bases of usually two lateral branches, less often one, which project upward, diverging very slightly, and similar in general to the *secondary* appendage, the cells somewhat less numerous. Perithecium straight or curved outwards, becoming yellowish, distally gradually broader, the short blunt tip and apex rather clearly distinguished, 100–185 × 18–34 μ. Longer appendages 100–210 μ: primary appendage 100–140 × 6.5 μ, its basal cell 25 × 13 μ. Receptacle 75–125 × 25–40 μ.

On the legs of *Pachyteles* sp., Verdant Vale, Arima, Trinidad, B. W. I. No. 2821.

Although a considerable number of specimens of this form have been examined, few of the females are in perfect condition and I have had some hesitation in separating it from the smaller *D. obliquesceptatus;* although the differences which appear to distinguish the two seem constant. In the present species the dark septa of the appendages are always horizontal, the secondary receptacle consists of but two cells bearing a single secondary appendage, while in the male the antheridia lack the abrupt distinction between its successive regions which characterize *D. obliquesceptatus*, as well as the shorter recurved neck which gives the latter its greater individuality.

The species belongs to a small group of forms on the same host, characterized by the presence of adventitious branches from the basal cell of the primary appendage, which are most highly developed, through successive branching, in *D. ramosus*. The perithecium normally develops from the first cell of the secondary receptacle, next the subbasal cell; but if it is unfertilized or its growth for any reason arrested, one may arise from the second cell, which also gives rise to the secondary appendage and is subtended by the latter. A fourth species of this type is known to me from Jamaica, but the material is not sufficient to warrant its description. In general habit all these forms resemble species of *Dimeromyces* more closely than the ordinary types of *Dimorphomyces*, and although the relations of the basal and primary subbasal cells are those characteristic of the latter genus, they form, with *D. Necrotalis* and some of the acarine types, a series of connecting links between the two which suggests that sooner or later they may have to be united.

Dimorphomyces ramosus nov. sp.

Male individual pale somewhat dirty yellowish. Receptacle consisting of three cells; the basal narrow and extending up obliquely beside the base of the subbasal, forming an external prominence which subtends the base of the lowest antheridium: basal septum of the terminal cell transverse; the primary appendage uniformly slender, straight, erect or divergent from the antheridia, its three lower cells small and similar, the second and third distinguished above and below by black septa, the terminal one as long or longer than the rest of the appendage. Antheridia two or three, the upper two arising directly from the terminal cell, without septation, when three are present; sessile, rather stout, the necks well distinguished, strongly curved outward or even recurved. Receptacle including foot, 55–62 × 17–20 μ. Appendage 38 × 5.5 μ. Antheridia 30 × 8 μ. Total length to tip of antheridia 85–90 μ, to tip of appendage 100 μ.

Female individual dirty yellowish with a slight tinge of brown. Receptacle consisting of a rather large basal cell, the distal surface of which is obliquely adjusted to the bases of three other cells lying transversely; the anterior bearing the basal of the lower secondary appendage, from which a small cell is usually laterally separated; a middle cell which bears the perithecium, and a posterior cell, the primary subbasal, which extends upward above the insertion of the perithecium and is followed by the large basal cell of the primary appendage, the rest of which is short, much narrower similar to the appendage of the male, brownish, consisting of four, or rarely five cells, the second and third distinguished by dark septa, the terminal one much longer; the whole inserted in the angle made by two highly developed branches which arise on either side of its base from the distal end of the basal cell: these branches greatly elongated, curved, forming a tuft which usually curves outward, downward and backward, at first with much regularity; both similar, two or three times branched, rather closely septate throughout, the ultimate branchlets of the two combined ten or twelve in number; one of these in each case straight, rigid, three- or four-celled, with slightly suffused middle septa. Secondary appendage simple, curved toward and with the branches of the primary appendage, a short two- to three-celled terminal portion with dark septa rather clearly distinguished; or sometimes variably branched, like the branches of the primary appendage, though shorter and less highly developed. Perithecium very large, usually slightly curved throughout the base abruptly bent so that the perithecial axis is turned nearly at right angles to that of the receptacle: the stalk-portion elongate, not distinguished, the ascigerous portion somewhat shorter and slightly broader distally, the tip clearly and abruptly distinguished by subtending elevations, subconical; the apex bluntly rounded, small, not distinguished. Perithecia 350–425 × 20–22 μ. Receptacle including rather large foot 70–85 × 22–25 μ. Primary appendage 64 × 5 μ, upper secondary, 500 to over 1000 μ: lower secondary to 150 μ.

On the inferior tip of abdomen of *Pachyteles* sp. Nos. 2835 and 2821: Maraval Valley and Verdant Vale, Arima, Trinidad, B. W. I.

Although allied to the other American species on *Pachyteles* this species is well distinguished by its highly developed secondary branches, the primary terminations of which are usually recognizable, if unbroken, by their resemblance to the termination of the secondary appendage, when the latter is simple. The structure of the receptacle, as in *D. Pachytelis*, conforms strictly to the type of the genus, the basal cell being in direct contact with all the rest.

Dimeromyces Galumnae nov. sp.

Male individual slightly curved throughout, faintly tinged with brown, consisting of a three-celled receptacle, terminating in a simple terminal three-celled appendage, not abruptly distinguished, the two distal cells somewhat shorter, narrower and paler than the basal. The antheridium partly lateral, incorporated in the axis, arising from the subbasal cell its very short neck lateral. Total length to tip of appendage 50–58 × 10 μ.

Female individual. Receptacle reddish brown, consisting of usually seven cells, the basal nearly hyaline, the second to sixth flattened and very obliquely superposed, the third, fourth and sixth producing secondary appendages; straight, rigid short subhorizontal, or bent upward, that from the second and third usually rudimentary or obsolete; the fifth producing the usually single perithecium, which is recurved when young, later divergent and straight or slightly curved inward, the long slender stalk-portion rather clearly distinguished from the subsymmetrically fusiform body; the inner lip slightly higher and flattened. Basal cell of primary appendage concolorous with the receptacle, more than twice as long as broad, tapering slightly, followed by one or two distinct cells which pass into a very slender and greatly elongated flagellum, stouter, curved or almost clavate at the tip. Perithecia including stalk 115–190μ; the body × 15–19 μ, the stalk × 7.5–11.5 μ. Primary appendage, longest 325 μ, its smaller diameter 3.8 μ, its basal cell 20–24 × 10–12 at base.

Prostrate on the upper surface of *Galumna* sp. No. 2976, Lasema, Fiji Is. Mann Coll. M. C. Z.

This species is chiefly peculiar for its enormously elongated primary appendage. The male is unlike that of any other species from the position and very short lateral neck and discharge tube of its antheridium. The receptacle of the female finally lies prostrate on the surface of the host, the long-stalked perithecium projecting upward at right angles.

Dimeromyces Parasiti nov. sp.

Male individual hyaline, relatively long and slender; the foot large, the basal and subbasal cells subequal; the former usually obsolete, or its lumen almost obliterated; the terminal cell longer and narrow, terminated by a very short two-celled subconical appendage, distinguished by a broad blackish ring, the terminal cell minute and more or less obliterated. Venter of the antheridium closely united to the upper cell of the receptacle, slightly inflated, the region of the efferent tubes concolorous with the long slender translucent brown neck, which is bent over the terminal appendage and slightly curved upward. Total length, including foot, (12 μ) about 40 μ, the appendage 5.5 × 4.5 μ. Venter of antheridium, including efferent tubes, 12 × 6.5 μ: the neck 24 × 2.5 μ.

Female individual. Axis of the receptacle normally consisting of three cells, the basal sometimes short and subtriangular, more often longer, narrower and nearly uniform; the two remaining cells hardly longer than broad, obliquely superposed, the lower larger; the upper cutting off a small cell next the perithecium by a septum which lies opposite the oblique insertion of the free divergent basal cell of the primary appendage; which is nearly uniform, colorless, and separated by a constriction and broad blackened septum from the abruptly distinguished three celled terminal portion; the lower cell of which is squarish and hyaline, the next slightly longer and broader and concolorous with the much longer olive brown terminal cell: the latter slightly narrowed in the middle, and terminating in an abruptly hyaline glandular subspherical tip, edged below with contrasting blackish brown. Secondary appendage usually single, rarely two subtended by small cells lying anterior to the subbasal cell; the first, and usually the only one, having a considerably elongated basal cell, united on its inner side to the lower third or more of the perithecium; its extremity, only, free and somewhat divergent, hyaline or slightly involved by the olive brown suffusion of the rest of the appendage; which is distinguished by a deep blackish septum and is similar to the distal portion of the primary appendage, except that its base is three-celled; the two lower cells squarish, both suffused, the third nearly twice as long, the fourth usually bent slightly outward. Perithecium hyaline, or becoming slightly suffused near the tip, nearly straight, subsymmetrical, slightly enlarged below the rather abruptly distinguished hyaline relatively large apex, the broadly rounded slightly sulcate extremity of which is subtended on either side by a distinct prominence, the anterior somewhat lower. Spores, female, 30× 3.8 μ, the apical segment very short. Perithecia 58–68 × 22 μ. Primary appendage 75–80 × 8 μ, its basal cell 15–21 μ: free portion of first secondary appendage 54–56 × 8.5 μ. Total length to tip of perithecium 100–125 μ.

On *Parasitus* sp. and *Macrocheles* sp. No. 2493, M. C. Z. (Mann. Collection). Orizaba, Mexico.

This well marked species is more nearly allied to Paoli's *D. mucronatus* and *D. muticus*, but is so clearly distinguished by its three celled primary appendage, as well as by numerous other characters that further comparison is unnecessary. The two appendages form the arms of a more or less symmetrical Y, in the angle of which the perithecium rests.

Dimeromyces subuliferus nov. sp.

Male individual hyaline; the receptacle consisting of two very obliquely superposed cells, the upper slightly larger and corresponding to the subbasal cell, as well as to the subbasal cell of the terminal appendage, which is separated from it distally by a slight constriction and dark septum, and consists of two cells; the upper subulate, its extremity straight, recurved, or bent. Venter and small stalk-cell of the antheridium separated from the subbasal cell by a curved oblique septum, erect; the brown neck clearly defined, rather long and usually bent across the distal portion of the appendage. Total length to tip of antheridium, including foot, 50–55 × 10 μ.

Female individual becoming suffused with blackish brown above, the basal cell of the receptacle hyaline, forming a more or less well defined stalk on which is borne the compact and somewhat rounded abruptly broader upper portion, which is composed of usually three-smaller somewhat obliquely superposed nearly median cells lying below the base of the single perithecium, which separate a corresponding number of obliquely superposed, closely united, externally prominent cells anteriorly, each bearing a single simple terminal portion; which is distinguished by a dark septum and constriction, is three celled, the two lower cells subequal stout and about as broad as long, the terminal one forming a slender prolongation which usually is recurved inward and lies against the surface of the receptacle; the remainder of the receptacle consisting of three very obliquely superposed larger cells lying on the opposite side of the perithecium, the *lower* of which is smaller and gives rise to the latter: the *middle* one longer and narrow, its upper two thirds or less united to the perithecium, its upper third or more separated by a septum, and terminated by a rigid subulate appendage which is two-celled, its basal cell partly hyaline or translucent, strongly inflated, distinguished below by a blackened septum and constriction, the upper portion tapering to a slender blunt hyaline apex; below red brown, the suffusion becoming deeper with age and involving the basal cell also; its distal half lying above the apex of the perithecium; the *upper* cell subtriangular, terminating below the apex of the middle one, and bearing a single terminal primary appendage similar to those on the anterior side. Perithecium stout, subsymmetrical, erect or slightly tilted toward the anterior side, ovoid or broad-fusiform; the apex subtruncate, hyaline, subtended by a reddish brown suffusion which also involves the venter and eventually the whole perithecium and the adjacent portions of the receptacle. Perithecia 55–64 × 25–30 μ. Spores about 30 × 4 μ. Basal cell of receptacle 25–42 μ. Subulate appendage 38–42 μ. Total length to tip of perithecium 92–125× 42–50 μ.

On *Uropoda* sp. sensu latiore. No. 2692. Verdant Vale, Arima, Trinidad, B. W. I.

This species is at once distinguished from other known forms by its rigid, suffused, subulate subterminal appendage, as well as by the character of the other appendages and the position of their slender recurved terminations. In general appearance it closely resembles species of *Rickia* of compact form which occur on similar hosts.

Dimeromyces adventitiosus nov. sp.

Male individual pale yellowish, the receptacle consisting of five or more cells; the basal larger, subtriangular, the rest flattened and somewhat oblique, producing single slightly divergent antheridia, or occasionally short scattered simple sterile branches. The appendage, an undifferentiated continuation of the axis, often bearing numerous antheridia on one or both sides; a large secondary appendage sometimes developed from the subbasal cell, of indefinite growth, and bearing numerous antheridia from the upper margin. Antheridia straight, or the necks slightly curved; the stalk-cell well distinguished, venter slightly inflated, neck moderately well distinguished. Length of receptacle proper about 25 μ. Primary appendage 40 μ or more, sometimes over 200 μ, the large secondary appendages somewhat shorter. Antheridia about 22 × 5 μ.

Female individual pale yellowish, the receptacle consisting of about six to eight flattened oblique cells; the primary appendage often greatly elongated, undifferentiated. Three or four of the cells of the receptacle usually producing simple, undifferentiated secondary appendages; while from one or more of the remainder, perithecia may develop; adventitious perithecia may also arise from any cell of the primary or from those of the lower secondary appendages. Perithecia straight, somewhat asymmetrical, the stalk-portion short and not clearly defined, one margin more strongly convex; tapering distally to the short, nearly symmetrical, rather clearly differentiated, small, blunt apex. Perithecia 80–110 × 20–18 μ. Receptacle and foot 40 μ or less. Appendages variably elongated up to 425 × 9 μ.

On the elytra and prothorax of *Tenebrio quadrihamatus* Fairm. No. 2801B, M. C. Z., Madagascar, (Wulsin).

This species is remarkable from its habit of producing adventitious perithecia and antheridia from primary and secondary appendages, as many as thirty-five of the latter being sometimes formed by a single individual, which may take on an irregularly furcate habit owing to the development of a large antheridiiferous branch from the subbasal cell. A majority of individuals, however, are more nearly of the normal type for the genus, although the primary appendage usually bears antheridia, like the undifferentiated axis below it. Both the primary and secondary appendages of the female are often greatly elongated and adventitious perithecia may appear even near their extremities. The species is most nearly allied to *D. anomalus*, but is very clearly distinct.

Dimeromyces anomalus nov. sp.

Male individual pale yellowish; the axis rather slender, consisting of an indeterminate number of cells, slightly tapering, without differentiation of an appendage; the subbasal cell usually developing a secondary appendage which is similar to the main axis, and gives the individual a furcate habit; any cells of the axis, or the branch, producing antheridia of which there may be six more or less on the inner upper side. Antheridia rather stout, with short well defined stalk-cells, the necks rather short and stout, well distinguished, somewhat curved. Antheridia 20 × 6 μ, longest axis 66 μ.

Female individual pale yellowish, the subbasal cell giving rise to a secondary appendage which forms an axis similar to the main axis, and like it diverges at right angles to that of the basal cell; the two axes hardly distinguishable from one another, producing from the upper side near the base one or several perithecia and sterile branches: the cells of the axes and branches slightly longer than broad, the latter hardly tapering. Perithecia usually not more than one to two or three in all, the stalk rather slender and well developed, rather clearly distinguished from the ascigerous portion; which is somewhat asymmetrical, slightly more convex on one side, the rather broad apex with one or both margins concave, subsymmetrical, or oblique when viewed sidewise. Perithecia 95 × 15–16 μ; ascigerous part 47–52 × 16 μ. The divergent axes 85–95 × 6 μ.

On the left elytron of a small Tenebrionid. No. 3005, Florida, Solomon Islands (Mann).

I have hesitated to describe this very peculiar species owing to the scanty material, which is not in perfect condition, consisting of four males and five females, only two of which have mature perithecia. Its characters are so well marked, however, owing to the Y-shaped habit of the male and the T-shaped habit of the female, that it will probably be easily recognized. It is most nearly allied to *D. adventitiosus*, also parasitic on a tenebrionid; but while in the males of the latter species the large secondary axis from the subbasal cell is apparently rare, all of the four individuals of the present species are thus developed. The production of adventitious perithecia and antheridia is far more restricted, the largest male examined bearing only six antheridia from both axes, the others less. The female of *D. adventitiosus* has differently shaped, short-stalked perithecia, and no one of the several secondary appendages is specially developed as in *D. anomalus*, although it is only from one or two of the lowest that any formation of adventitious perithecia seems to take place.

Dimeromyces Peltoidis nov. sp.

Male individual hyaline, or becoming tinged with brown. Receptacle four- or five-celled, the basal larger, sometimes more or less involved by the blackening of the foot; terminal appendage two- or three-celled, rarely furcate, undifferentiated and not distinguished from the receptacle. Antheridia slightly divergent, usually two, some times four, rather short and stout, the necks short, blunt, not abruptly distinguished, slightly curved. Total length 40–46 μ. Antheridia 15–20 × 4.5–5.5 μ.

Female individual pale yellowish, or becoming tinged with brown. Receptacle short and stout consisting of usually seven cells; the basal rather small, but longer; the rest flattened, subtriangular, in a curved series, each producing one, sometimes two, appendages or perithecia;

the terminal primary appendage, like the secondary, undistinguished, nearly uniform, rather closely septate, tapering slightly at the apex, often furcate above the basal cell which may also produce an adventitious perithecium or appendage: the successive cells of the appendages usually separated by slight constrictions at the septa, or somewhat broader at the ends. Perithecia usually elongate, subsymmetrical, the medium, blunt apex subtended by a slight general enlargement of the tip-region; the stalk undifferentiated, or sometimes rather abruptly distinguished, slender and curved. Spores 26 × 2.5 μ, female. Perithecia including stalk 85–180 × 14–22 μ. Longest appendages 100–190 × 6–9 μ. Receptacle 38–46 × 4.5–5.5 μ.

On *Peltoides pustulatus* Fairm., Kamerun, West Africa, No. 3082.

This species appears to vary considerably according to its position and luxuriance of growth. Individuals from the legs and tip of the abdomen are darker, more fasciculate with short-stalked perithecia. Those on the elytra and prothorax are simpler and smaller, the appendages usually simple and single. A specimen growing near the base of the posterior legs is much larger, with branched appendages and supernumerary perithecia. In the more compact and fasciculate types the structure of the receptacle is more or less completely concealed and the appendages form a dense tuft which appears to arise from a more or less common base.

Dimeromyces Strongylii nov. sp.

Male individual pale yellowish with a faint brownish tinge. Receptacle compact, slightly curved, erect; consisting of from five to ten flattened cells obliquely superposed; the basal cell subtriangular, usually flattened like the rest, which bear either antheridia or simple sterile tapering appendages of which seldom more than two, sometimes none, are present; the antheridia varying in number according to the number of cells, arising in a unilateral series, the successive members of which diverge slightly right and left; their axes, in general, coincident with that of the cell which bears them. Primary and secondary appendages undifferentiated, similar, tapering. Antheridia rather stout, straight, the necks relatively short. Receptacle 25–42 × 16 μ. Antheridia 25–30 × 9 μ. Appendages 40–80 μ.

Female individual pale yellowish, faintly tinged with brown. Receptacle, similar to that of the male, consisting of from eight to ten cells, bearing secondary appendages and perithecia, seldom more than three of the latter. The primary appendage similar to the secondary, undifferentiated, tapering; the distal portion proliferous through a collar above the fifth or sixth cell. Axes of the appendages and perithecia coincident in general with those of the cells which bear them. Perithecia subfusiform, straight, subsymmetrical, the stalk not abruptly distinguished and about as long as the ascigerous portion; the termination blunt, rather broad, unmodified. Perithecia 75–120 × 20–25 μ. Receptacle 40–60 × 16–25 μ. Appendages longer 100–150 μ.

On the elytra and inferior surface of *Strongylium* sp. No. 2353, Kamerun, West Africa.

The species is more nearly allied to *D. Peltoidis* and *D. adventitiosus*. from both of which it is, however, abundantly distinct. The appendages are rigid and often distally attenuated, the distal portion being usually distinguished by a collar formed through the disorganization of the outer walls, usually above the fifth or sixth cell, the lumen of this portion being very small.

Dimeromyces Amarygmi nov. sp.

Male individual hyaline or faintly yellowish, erect, rather slender; the receptacle consisting of about ten flattened cells, the upper three or four somewhat oblique; the terminal cell, and one to six or seven of the cells below it, giving rise to antheridia. Antheridia suberect, or but slightly divergent, the stalk-cells small and well defined; the neck nearly as long as the venter and rather strongly bent outward. Primary appendage simple, the distal portion slender, usually longer than the receptacle. Antheridia 25–30 × 7–8 μ; receptacle 45–55 × 7–12 μ; primary appendage 55–85 μ.

Female individual pale yellowish. Receptacle erect, or bent but slightly distally; consisting of from twelve to eighteen cells; the upper three or four oblique, the rest more flattened, all except the lowest usually giving rise to simple secondary appendages or perithecia. Appendages rather closely septate below, the upper longer and distally attenuated; one or more of the lower usually bent characteristically downward to the substratum: the primary appendage similar, but shorter than those below it. Perithecia one or several, straight, nearly symmetrical, increasing in diameter slightly and continuously from the insertion of the stalk to the tip-region; which is clearly distinguished by slight elevations on either side, whence it tapers rather abruptly to the slightly inflated apex. Perithecia 125–235 × 20–24 μ, the subconical extremity about 20 μ. Longest appendages 150–275 μ. Receptacle 65–100 × 20–25 μ.

On elytra of *Amarygmus* sp. No. 2333 and 2567, Kamerun, West Africa.

This species is very closely allied to *D. Strongylii* differing in the form of its perithecium and antheridia, its attenuated upper and down curved lower appendages, as well as in minor characters.

Dimeromyces Trycheri nov. sp.

Male individual pale yellowish. Receptacle rather stout, consisting of from five to eight cells, all except the basal and distal much flattened and more or less oblique. A single terminal appendage, straight, somewhat divergent, usually two-celled, about as long as the antheridia. Antheridia two to six; the venter stout, a slight enlargement below the neck, which is short and rather well distinguished: the stalk-cell short. Receptacle about 30 × 12 μ. Appendage 30 μ. Antheridia 25 × 8 μ.

Female individual pale yellowish. Receptacle consisting of more often sixteen cells, the basal subtriangular, the rest usually much flattened, the distal ones slightly oblique, all bearing perithecia or appendages, except the lowest. Primary appendage short, two- or three-celled, the more slender terminal portion subtended by a more or less distinct brownish suffusion; the secondary ones tapering from a stout base, longer, the third cell brown, more or less clearly distinguishing the basal from the terminal part; the lower ones not thus modified, and often curved downward, or even across the foot. Perithecia straight, subsymmetrical, subfusiform; the stalk shorter than the ascigerous part, which is hardly distinguished from it; the slightly inflated tip-region distinguished below by a slight depression; the apex very slightly inflated, subsymmetrical, pointed. Perithecia 100–125 × 25 μ. Receptacle 80–100 × 25–30 μ. Primary appendage 30–40 μ; secondary appendages, longer 85–100 × 9 μ.

On the elytra and legs of *Trycherus bimaculatus* Guer., and *Trycherus* sp. Metet, Kamerun, W. Africa, Nos. 2445, 3087 and 3088.

This species is closely allied to *D. Amarygmi*, but is at once distinguished from this and other allied forms by the brown modification of the third cell of a majority of the secondary appendages. The specimens occurring on the legs of the host are stouter and more compact, with shorter appendages.

Dimeromyces Cryptici nov. sp.

Male individual uniform pale yellowish, becoming faintly tinged with brown. Receptacle and appendage forming an undifferentiated axis somewhat divergent and tapering distally; the walls thick, the transverse septa horizontal, the lower cells but slightly flattened. Antheridia one to four, slightly divergent, the well developed stalk-cell bent abruptly upward, the venter short and stout, the hyaline neck straight, rather short, abruptly distinguished. Total length to tip of appendage 70 × 9 μ. Antheridia 25 × 8 μ.

Female individual like the male in general color and structure, slightly darker. Subbasal cell of receptacle giving rise to a stout slightly tapering secondary appendage, curved upward against the perithecium and reaching about to its middle. Perithecia one or several, the stalk short and bent upward, the rest subsymmetrically fusiform, the tip hardly distinguished, the apex small, short, hyaline, abruptly distinguished by a slight constriction on both sides, that on the inner subtended by a more or less well defined protrusion; slightly pointed distally with flat rounded lips on either side. Perithecia 75–84 × 18–22 μ. Secondary appendage 75–90 × 9 μ. Total length to tip of primary appendage 75–100 × 9 μ, of perithecia 100–120 μ.

On the elytra of *Crypticus scriptipennis* Fairm., No. 3078, Kamerun, W. Africa.

This small species is distinguished by its undifferentiated primary appendage, which is a direct continuation of the receptacle as in *D. Thaxteri* Maire, as well as by the single secondary appendage which is similar to and about as large as the combined receptacle and primary appendage, though less tapering distally.

Dimeromyces Dhanyae nov. sp.

Male individual dull pale olivaceous brown throughout. Receptacle consisting of three very unequal and oblique cells, the basal extending upward anteriorly more than half its length: the appendage consisting of three cells, the two basal concolorous, the second minute, flat and horizontal, the terminal cell contrasting in form and color, reddish brown, twice as long as broad, slightly constricted below a terminal knob-like bluntly apiculate termination (cell?). Antheridium appressed, the venter short, erect, the neck large, irregularly bent in two directions, with a prominence on the inner side, distally pointed. Total length to tip of appendage, including foot, 65–70 μ. Terminal cell of appendage 11–13 × 5.7 μ. Antheridium about 44 × 13 μ.

Female individual compact, partly suffused with blackish brown. Receptacle consisting of usually four cells very obliquely superposed and somewhat irregular: the basal extending upward posteriorly more than half its length, bulging distally above an external blackish brown suffusion, the subbasal small and narrow, nearly vertical, bearing a two-celled, deeply suffused, short, broad appendage, the narrowed extremity of which is bent abruptly inward ending in a hyaline papilla: the fourth cell bearing a rather short curved hyaline cylindrical unicellular appendage; the primary appendage consisting of a subtriangular subtending basal cell and two free cells; the lower small, flattened, concolorous; the upper similar to that of the male, deep reddish brown and ending in a darker knob-like apiculate termination subtended by a dark transverse line (possibly a septum?). Perithecium erect, the stalk opaque except its hyaline base, rather abruptly and clearly distinguished from the ascigerous portion; which is strongly inflated, somewhat more convex on the inner side, opaque or deeply suffused below, transversely mottled above and translucent; the distal half subconical; the termination somewhat oblique, subtended by deeper suffusions, one of the lips on one side hyaline and slightly projecting. Perithecium, stalk-portion 48–58 × 20–22 μ: ascigerous portion 75 × 38–42 μ. Receptacle 48 × 38 μ. Terminal cell of primary appendage 15 × 7.5 μ: upper secondary appendage 38–50 × 6 μ: lower 28 × 19 μ. Total length to tip of perithecium 190 μ.

On the inferior tip of abdomen of *Dhanya* sp. Mindanao, P. I. No. 2389 (Webber Coll.)

I am indebted to Professor Oakes Ames for the host of this very peculiar and interesting form, which is so unlike any other thus far known that it cannot be mistaken. In the general form and coloration of its perithecium it bears a most remarkable resemblance to *Cucujomyces Diplocoeli* and *C. curtipes*. Both secondary appendages are anomalous, especially the lower which might be mistaken for a suffused compound antheridium. The knob-like termination of the primary appendage may prove in both sexes to be a minute cell.

Dimeromyces Amazonicus nov. sp.

Male individual hyaline, slender, erect, the receptacle consisting of three cells obliquely superposed; the basal longer; the width nearly uniform. Terminal appendage three-celled, the basal somewhat broader, the distal pointed, all of about equal length. Antheridium long and rather slender, erect, the stalk-cell relatively long and free, the neck nearly as long as the venter, diverging or slightly curved outward, its apex extending some distance above the tip of the appendage. Total length to tip of antheridium 42 × 8 μ. Antheridia 24 × 9 μ. Appendage 14 × 3 μ.

Female individual hyaline, erect, slightly curved: the receptacle consisting of four cells very obliquely superposed; the basal longer and extending upward almost as far as the two succeeding ones, the subbasal bearing the single secondary appendage, close to the base of which the third cuts off a small cell; the fourth less flattened, narrower and longer, bearing the single perithecium. Primary appendage consisting of two large cells of uniform width, the upper longer; the rest of the appendage abruptly narrower and faintly brownish at the base, slightly tapering and rather elongate. The secondary appendage elongate, consisting of a large nearly free basal cell, above which a faintly brownish narrower basal portion is distinguished composed of six to ten small flattish cells; the distal portion variably elongated and tapering, the walls of the cells tending to disorganize. Perithecium sessile, or very short-stalked, slightly curved, its inner margin rather strongly convex; the apex somewhat sulcate, distinguished by a rather abrupt constriction from the tip-region. Perithecia 60–75 × 14 μ. Longer appendages secondary, 114–130 × 4 μ (basal part) primary appendage 75–100 × 3.5 μ above basal cells, the latter 24 × 6 μ. Receptacle 28 × 15 μ. Total length to tip of perithecium 75–100 μ.

On the elytra of *Platydema* sp.? No. 2243, Independencia, Amazon, (Mann Coll.).

This species is clearly distinguished by its peculiar appendages and the abruptly differentiated apex of its perithecium, the large lip-cells in some individuals separate and irregularly divergent.

Dimeromyces Heterophylli nov. sp.

Male individual tinged with pale brownish yellow. Receptacle consisting of four cells, the basal much longer, the rest somewhat flattened and obliquely superposed. Appendage consisting of a large free basal cell prominent distally and externally; the rest of the appendage usually three-celled, abruptly narrower, its basal cell brownish, the whole converging more or less conspicuously inward from the oblique distal surface of the basal cell. The antheridia short and stout, almost sessile, with short necks strongly curved outward. Receptacle and foot 30–35 μ. Appendage 20 μ. Antheridia 18 × 7.5 μ.

Female individual tinged with pale brownish yellow. Receptacle consisting of usually six cells, somewhat flattened and oblique, except the longer irregularly triangular basal cell. Secondary appendage usually single and arising from the third or fourth cell, usually somewhat divergent. Primary appendage similar to that of the male, less conspicuously bent toward the perithecium, consisting of six or seven cells, the subbasal slightly brownish and abruptly distinguished from the much larger broader basal cell which bulges more or less conspicuously below it externally. Perithecium short-stalked, subsigmoid; its rather broad termination slightly bent upward, blunt, unmodified, sometimes slightly oblique. Perithecia 45–48 × 14–15 μ. Receptacle 38–44 × 12 μ. Secondary appendage, longest 65 μ: primary 28–35 μ. Total length to tip of perithecium 75–80 μ.

On the elytra of a bluish black immaculate species of *Heterophyllus*, Hayti, No. 2760 (Mann Coll).

This species is closely allied to *D. proximus*, but appears to differ in the characters of both sexes to which reference is made under that species.

Dimeromyces proximus nov. sp.

Male individual hyaline or faintly tinged with brownish yellow. Receptacle erect consisting of four cells; the basal narrower and somewhat longer; the rest subequal, slightly flattened and oblique. The appendage usually four celled, the basal much larger, the subbasal abruptly narrower and brownish, suberect and hardly reaching the tip of the upper antheridium. The latter relatively long and slender, one to three in number, the stalk-cell well defined, the venter slightly inflated, the neck slightly curved outward. Receptacle 28 μ. Antheridia 28 × 6 μ. Appendage 20–24 μ. Total length to tip of antheridia 58 μ.

Female individual faintly tinged with pale brownish yellow, erect, compact. Receptacle consisting of six or seven cells somewhat obliquely superposed and somewhat flattened; the basal subtriangular, short; the two above it, rarely a third, producing secondary appendages which are appressed, bending upward against the perithecium, the apex of which they seldom reach, rather stout, hardly tapering, six or seven celled; the primary appendage similar to that of the male, usually five celled; the subbasal abruptly narrower and slightly brownish; the distal as long as the rest of the appendage, or nearly so. Perithecium usually single, erect, straight or nearly so, stout, sessile, the tip tapering to the apex which is slightly bent sidewise toward a terminal pair of minute auricle-like projections. Perithecia 38–50 × 12–14 μ. Receptacle 30–38 μ. Primary appendage 28–38 μ; secondary 45–60 μ. Total length to tip of perithecium 75–100 μ.

On the elytra of a smaller species of *Heterophyllus* with an orange spot on the elytra. No. 2759 Hayti (Mann).

This species is so closely allied to the preceding that I have hesitated to separate them. Since both male and female show distinct differences, however, I have concluded to do so provisionally, at least. The perithecium of the present form not only bears the two terminal projections lacking in *D. Heterophylli*, but is differently shaped, erect and almost symmetrical; while in the latter the apex, which is turned slightly upward, has no such appendages, the inner margin of the perithecium being strongly convex, while the outer is straight or concave. In the male the primary appendage is not bent conspicuously inward, as in *Heterophylli*, which has shorter, relatively stouter, almost sessile antheridia, with shorter more strongly curved necks.

Dimeromyces Derispiae nov. sp.

Male individual pale yellowish. Receptacle consisting of from five to seven cells, the basal larger, the rest somewhat oblique and flattened, the distal less so; the primary appendage slightly bent inward, its basal cell somewhat larger, its distal half subcylindrical with a blunt termination. Antheridia from two to four superposed, arising from the four upper cells; slightly divergent, the stalk-cell minute, the neck well distinguished, short, stout, curved, slightly outward. Receptacle 55–59 × 12.5 μ. Appendage 34–42 μ. Antheridia 25–29 × 9–12.5 μ. Total length to tip of appendage including small pointed foot, 80–92 μ. Appendage 34–42.

Female individual yellow, becoming slightly tinged with brown, straight, erect. Receptacle with small pointed foot; consisting of about twelve to sixteen cells, short and much flattened above the basal one, not at all oblique, becoming gradually broader and somewhat rounded above, their posterior margins rather strongly convex; all but the lowest separating a small cell distally and externally. The subterminal cell, also rarely the second cell below it, giving rise to an outcurved secondary appendage, about seven to ten-celled, tapering more or less, slightly shorter than the similar, six to nine-celled, erect, tapering primary appendage. Perithecium erect, nearly symmetrical, subfusiform, sessile, relatively short; the tip relatively long, well distinguished, more convex anteriorly; the apex short, rounded. Perithecia 105–22 μ. Receptacle 125–170 × 25 μ. Primary appendage 90–125 × 12 μ; secondary 80–95 μ. Total length to tip of perithecium 210–275 μ.

On the left elytron of *Derispia moquerysi* Pic. No. 2568, Kamerun, West Africa.

In general habit this species recalls *D. nanomasculus*, although the perithecia and appendages are quite different and are produced from one side only. Rarely in the present species, a second perithecium and appendage may arise from the cells immediately below those normally present.

Dimeromyces luteus nov. sp.

Male individual pale yellowish, slender; the receptacle consisting of three obliquely superposed cells, the basal long and slender, the others subequal, one or both producing antheridia. Appendage of three or four cells, the basal larger rather clearly distinguished from the slightly brownish narrower cells above. Antheridia relatively long and slender, the stalk-cell well defined, curved inward, the upper one in contact more or less continuously with the appendage, extending nearly to its tip, the necks usually strongly curved inward. Receptacle 55–65 × 12.5 μ., Appendage 40–45 μ. Antheridia 46–50 × 9 μ. Total length 95–100 × 14 μ.

Female individual clear pale yellow, becoming somewhat suffused with brown, erect. Receptacle consisting of four cells; the basal somewhat elongate, the second and third oblique and somewhat flattened; producing normally one perithecium from the third and one secondary appendage from the second. Primary appendage shorter than the perithecium and erect beside it, or rarely slightly divergent; the basal cell much larger, long, and broader than the subbasal; which is clearly distinguished by its brown color. Secondary appendage seldom as long as the perithecium, and lying close against it; the basal cell longer, the following six or more squarish, the subdistal ones broader, the lowest tinged with brown. Perithecium becoming tinged with brown, straight, erect, rather stout, sessile, subsymmetrical; the tip- and apex-regions abruptly distinguished, tapering to a rather broad bluntly rounded termination. Perithecia 90–100 × 24–30 μ. Receptacle 60–65 × 20 μ. Longest appendages 75–100 μ. Total length to tip of perithecium 140–160 μ.

On the elytra of *Leiochrodes medianus* Westw. and *Leiochrodes* sp. No. 2950, 2951, 2996 and 2997, Auki, Solomon Is. (Mann Coll.).

In the group of forms occurring on species of *Leiochrodes* this species is easily distinguished by its very different color, sessile perithecium of a different form and lying between the two erect appendages which are usually in close contact with it and appear to hold it as in a socket. The male differs from that of the others in its long slender form, incurved necks and pale yellowish color. It occurs scattered over the elytra, sometimes in company with *D. rugosus*.

Dimeromyces rugosus nov. sp.

Male individual rather dark brown. Receptacle four-celled, the cells irregular, somewhat obliquely superposed, the basal longer. Appendage erect, three to four-celled, the basal much larger and broader. Antheridia one or rarely two, the stalk-cell well developed, often longer than broad, the venter rather stout, erect beside the appendage, the neck curved abruptly sidewise, or even recurved. Total length 75–92 × 17–21 μ. Antheridia 42–50 × 11–12.5 μ. Appendage 40 μ.

Female individual becoming rather dark brown, or partly opaque. Receptacle relatively small, four-celled, the cells obliquely superposed, the basal somewhat longer; the two others flattened and subequal; the subbasal producing the single secondary appendage; which is stout, five to nine-celled, distally usually somewhat broader, the third and fourth cells usually broader than long, the basal long and narrower: the third and sometimes the fourth cells producing perithecia; the primary appendage shorter than the secondary, which may reach to the middle of the perithecium, consisting of usually four cells, the second and third shorter and becoming darker brown. Perithecium relatively large, becoming nearly opaque above the hyaline base, long, straight, subsymmetrical, rather slender, becoming very dark and finely transversely rugose, the tip well distinguished, relatively short, tapering to the hyaline apex which usually appears as a shallow cup, or sometimes blunt and irregularly oblique. Perithecia 125–200 × 26–30 μ. Receptacle 42–64 × 35–45 μ. Primary appendage 60–80 μ, secondary 100–160 × 16 μ. Total length to tip of perithecia 125–200 μ.

On the elytra of *Leiochrodes medianus* Westw. Nos. 2998 and 2951, Auki, Solomon Islands (Mann Coll.). *Leiochrodes minutus* Pic, No. 1836, Sarawak, Borneo.

This species is well distinguished by its finely rugose perithecial wall, the granulosity forming transverse lines which are hardly visible as the mature perithecium becomes nearly opaque. The apex usually appears flat and hyaline with a slight elevation symmetrically placed on either side, giving the effect of a shallow cup. This form is not always apparent, however, probably owing to a slight turn of the perithecium. The brown suffusion of different individuals varies considerably and the receptacle may be either translucent or opaque.

Dimeromyces longicollis nov. sp.

Male individual more or less uniformly tinged with yellowish brown, erect. Receptacle three-celled, the cells somewhat obliquely superposed, the basal longer. Appendage erect, consisting of four cells, the three lower subequal, the subbasal becoming rich deep brown; the terminal one somewhat longer, slightly tapering. Antheridia one or two, relatively large and long, the stalk well developed, the slender long necks bent slightly inward, and reaching far above the tip of the appendage. Receptacle 50 × 17 μ. Appendage 50–60 × 7 μ. Antheridia 85 × 14 μ, the necks 42 × 3 μ. Total length about 140 × 18 μ.

Female individual tinged with yellowish brown, the perithecium becoming rich dark brown. Receptacle consisting of four cells; the two middle ones flattened and obliquely superposed. Primary appendage consisting of five or six cells, the basal slightly longer and broader, the subbasal small and distinguished by constrictions and dark septa. Secondary appendage consisting of six or more cells, somewhat longer, extending a short distance above the stalk-portion of the perithecium. Perithecium long and slender, erect, blackish brown; the stalk portion pale, and distinctly indicated; the distal portion curved inward; the short hyaline, slightly oblique apex turned outward. Perithecia 250 × 30 μ. Receptacle 64 × 22 μ. Primary appendage 64–84 × 12 μ; secondary 100–120 × 12 μ. Total length 275–310 μ.

On the elytra of *Leiochrodes medianus* Westw. Nos. 2951 and 2997, Auki, Solomon Islands. No. 2998 Fulakora, Solomons (Mann Collection).

This species is closely related to *D. decumbens*, but differs very clearly in the form of its perithecium and the length and character of its appendages, as well as in the great elongation of the neck of the antheridium in the male. One pair shows a maculation of the basal cell in both sexes, similar to that seen in *D. decumbens*, but there is no indication of it in the other individuals.

Dimeromyces decumbens nov. sp.

Male individual dull brown, straight, the receptacle consisting of three cells, the basal transversely rather coarsely punctate, the distal end free from maculation. Appendage erect, four celled, the two middle cells squarish and deeply suffused or opaque. Antheridium single, erect, in contact with the appendage, the stalk well developed, the neck clearly distinguished and bent sidewise distally. Receptacle 50–60 μ. Appendage 38 μ. Antheridia 80 × 18 μ. Total length 125 μ.

Female individual more or less uniform dull brown. Receptacle consisting of four cells obliquely superposed, the anterior face of the basal maculate as in the male. Primary appendage short, four-celled, the terminal cell hyaline, emerging from a cup-like base formed by the partial disorganization of the blackish wall below; the three other cells small, subequal; the subbasal nearly opaque, the others brown. Secondary appendage single, arising from the subbasal cell, very large, subsigmoid: consisting of six or seven cells, the basal long, stout, and curved; the rest successively shorter; the terminal one tapering to a blunt end. Perithecium darker dull brown, somewhat more so above the paler, rather short stalk-portion which curves upward at an angle, even at right angles, to the prostrate receptacle: the body of the perithecium elongate, slender, asymmetrical; the distal portion slightly bent outward, the tip-region distinguished by slight elevations; the coarse blunt apex bent slightly inward, snout-like. Perithecium 235–310 × 38–42 μ; the stalk-portion 60–80 μ. Receptacle 70 × 34 μ. Primary appendage 45–60 × 12–14 μ; secondary 250–290 × 20–22 μ. Total length to tip of perithecium 365–380 μ.

At the base of the posterior legs of *Leiochrodes* sp., Auki, Solomon Islands (Mann Coll.).

The receptacle and the male individual appear to lie flat in the host, the secondary appendage and the perithecium projecting upward at a variable angle, the axis of the primary appendage coinciding with that of the receptacle. A very large species clearly distinguished from its allies on hosts of the same genus.

Dimeromyces sulcatus nov. sp.

Male individual hyaline, erect; the receptacle consisting of three very obliquely superposed cells, the basal extending up beside the subbasal nearly to its upper margin. Appendage erect, subcylindrical, of five or six cells, considerably longer than the receptacle, relatively stout. Antheridia one or two, suberect, the stalk-cell slender and well developed, the body rather long and narrow, the neck clearly defined and bent abruptly at right angles backward or sidewise. Receptacle 25 × 8 μ, exclusive of the small pointed foot. Antheridia, to bend of neck, 34 × 5–6 μ. Appendage 38–45 × 4.5 μ.

Female individual almost perfectly hyaline, erect, the receptacle consisting of four cells, the protoplasmic region of the basal similar to that of the two cells above, the three lying very obliquely, and much flattened; the primary appendage subcylindrical, four- or five-celled, erect; the secondary appendage single, arising from the subbasal cell, seven or eight celled, erect and close beside the perithecium; the upper third of which is curved outward beyond its apex. Perithecium straight, except for the distal curvature which is subtended on the inner margin by a slight protuberance, long and slender, the tip slightly inflated and distinguished above and below by prominent constrictions; the apex deeply sulcate, almost as long as the tip, the margins convex. Perithecia 90–105 × 12–15 μ. Primary appendage 40–45 × 5 μ. Secondary 65–75 × 5 μ. Receptacle 30–34 × 12 μ. Total length to tip of perithecium 120–148 μ.

On the elytra of *Leiochrodes* sp. Los Baños, Luzon, P. I. No. 3043 (W. H. Weston).

This small and almost perfectly hyaline form is more nearly related to *D. luteus* and its allies which occur on similar hosts and is most easily distinguished by the peculiar conformation of the curved termination of its perithecium, and its sulcate apex.

Dimeromyces appendiculatus nov. sp.

Male individual faintly tinged with yellowish. Receptacle short and broad, foot large; basal cell flattened, extending upward on the anterior side above the base of the stalk-cell of the lower of the two antheridia; which arise from the distal and subbasal cells: that from the latter, with long stalk-cell, which bears it higher than the upper curved backward or sidewise, almost at right angles. Appendage long, slender and about five-celled above the large basal cell. Receptacle 19 × 20 μ. Antheridia, upper 35 μ, lower 45 μ. Appendage, longest examined, 45 μ. Total length to tip of lower antheridium including foot 60 μ.

Female individual pale yellowish. Receptacle as in the male, but four-celled, the lower cells becoming somewhat distorted and misplaced, the basal also forming a narrow margin which extends up to the end of the basal cell of the secondary appendage to which it is adherent. Both appendages similar, slender, tapering slightly. Perithecium relatively very large, sessile, larger at the base, tapering gradually, bent or curved inward or arcuate; the tip somewhat narrower and well distinguished; the apex subtended by a slight indentation bearing a short tooth-like projection on one side and a long slender usually upcurved distally clavate appendage on the other. Perithecia 115–150 × 14–17 μ, its appendage 40-50 × 7.5 μ at tip. Receptacle 15 × 22 μ (exclusive of large foot). Appendages longest, 95 μ. Total length to tip of perithecium 135-170 μ.

On the antennae of *Leiochrodes medianus* Westw. and *Leiochrodes* sp., Nos. 2996–98, Auki and Fulakora, Solomon Islands (Mann Coll.).

The very peculiar receptacle and perithecial appendage clearly distinguish this species. The cells of the former become so displaced by growth that it is usually almost impossible to determine their limits. The extension upward of the anterior edge of the basal cell to form a margin gives the receptacle a characteristic appearance.

Dimeromyces rigidus nov. sp.

Male individual yellowish, becoming dark brown. Receptacle consisting of from five to seven cells, those above the basal oblique and much flattened, especially below, both margins convex; the basal cell of the primary appendage rather abruptly narrower, and like the secondary distinguished from the rather short, slightly tapering two- to three-celled terminal portion by a thick black septum. Antheridia one to several, associated with stiff erect brown secondary appendages: long and slender, with well developed slender stalks, the neck relatively long, slender, straight, or somewhat curved. Receptacle, exclusive of large foot 30–40 × 10.5–14 μ. Primary appendage 30 μ, secondary, larger 50 × 5 μ, distinguished above the basal cell by a black septum. Antheridia 50 × 6.5 μ.

Female individual becoming more or less suffused with dark brown. Receptacle straight, or but slightly curved, consisting of usually six cells, the basal paler, short and subtriangular; the rest more or less compressed and oblique, the subterminal normally giving rise to the usually single perithecium; the rest, except the subbasal, producing secondary appendages; the broadened basal cells of which bear from two to usually four erect, stout, rigid deep reddish brown branches distinguished by broad black constricted basal septa: the basal cell of the shorter and more slender primary appendage pale and slightly inflated, often turned inward, very abruptly distinguished from the rest of the appendage. Perithecium relatively large, bent abruptly upward at its insertion, the stalk-portion long and stout, straight or distally somewhat curved, pale, rather clearly distinguished from the darker purplish brown ascigerous portion; the tip tapering, pale, abruptly distinguished by subtending elevations on both sides, the inner margin more convex; the apex abruptly distinguished, distally compressed; its abruptly broader base forming a conspicuous prominence on the inner margin, and a lesser one somewhat higher on the outer; the termination much compressed; the lips small, the inner bulging and slightly prominent; but when turned at right angles the apex appears spathulate or ligulate, flat and distally broadly rounded. Perithecia 275–355 × 25–30 μ. Receptacle, without foot, 65–75 × 25 μ. Appendages 150 × 8 μ or longer. Total length to tip of perithecium 390–425 μ.

On the inferior abdomen of *Aulacophora* sp. No. 2948, Auki, Solomon Is. (Mann Coll.)

This large species is well distinguished by its dark color, the form of its perithecium, its brown rigid appendages arising in transversely arranged groups and also by the characters of the male. The appendages are for the most part broken distally and may be longer than indicated, but do not appear considerably to exceed one third the length of the perithecium. It is most nearly related to some of the variations of *D. Aulacophorae* of which it may prove to be a variety.

Dimeromyces nigricaulis nov. sp.

Male individual subhyaline; the receptacle consisting of two or three cells; erect, of almost uniform diameter, the septa hardly oblique, the subbasal bearing the antheridium, a second sometimes arising from the basal cell of the appendage; which is not distinguished from the receptacle, and is separated from the short one or two celled abruptly narrower distal portion by a thick black constricted septum. Antheridium erect, curved outward distally, the stalk-cell well developed and conspicuously blackened distally; the outcurved neck relatively rather stout and short. Receptacle including basal cell of appendage and exclusive of foot 25 × 4.5 μ. Antheridia 25 × 7.5 μ. Free appendage above black septum 12 × 4 μ.

Female individual uniform dirty pale yellowish brown, the receptacle somewhat darker. Receptacle consisting of usually six subequal cells, the four middle ones somewhat flattened and oblique, the subterminal one producing the usually single perithecium; the rest, except the basal, giving rise to appendages the basal cells of which each produce two branches transversely but somewhat irregularly paired, rather short, stout, and divergent from the base of the perithecium, distinguished by thick black constricted septa: the basal cell of the primary appendage hardly longer than broad, the free appendage short, few- or even one-celled, a secondary basal cell normally separated obliquely on the inner side and bearing a usually four-celled free appendage separated by a similar black septum. Perithecia as a rule slightly and evenly arcuate throughout, abruptly bent outward from the insertion, stout, very slightly broader distally, the stalk-portion not distinguished, the tip clearly differentiated by slight subtending elevations, associated with a slightly deeper suffusion of this region, the inner margin more convex, somewhat abruptly narrower and slightly tapering to the short apex, which is distinguished only by the prominently rounded inner lip-cell. Perithecia 200–235 × 20–22 μ. Receptacle 70–85 × 22 μ. Appendages 40–60 × 8–8.5 μ. Total length to tip of perithecium 300 μ, more or less.

On the elytra of *Diacantha flavescens* Ws. No. 3069, Kamerun, W. Africa.

Apart from other characters which might be variable, this species seems distinguished by the blackened termination of the stalk-cell of the antheridium, which is present and characteristic in the three individuals examined. A dozen females are available which show no essential variations from the type described; but more abundant material would no doubt show deviations in form or size. It is undoubtedly most nearly related to *D. Aulacophorae*, from which the female differs in the form of its perithecium and its uniform short transversely paired branches.

DIMEROMYCES AULACOPHORAE Thaxter.

The original material of this species growing on *Aulacophora postica* from the Straits Settlements, which, though sufficiently abundant, was not in very good condition, has been supplemented by specimens from the Solomon and Fiji Islands, Madagascar and Africa on species of *Aulacophora*, *Platyxantha*, *Monolepta* and *Hyperacantha*. The material is abundant and in the best condition, and illustrates the very considerable range of variation in this type. The male is almost always, though not invariably, distinguished by the presence of secondary appendages: more often one, subtending the antheridium, if it is single; or, if there are several, two or three such appendages may be associated with them in various positions. The variations of the female are associated more particularly with the secondary appendages, their number, position and association, and in the development of the basal cell of the primary appendage which is usually relatively large and often much elongated, and may even be curved or recurved. In a majority of individuals, the secondary appendages are shorter and stouter than in the type, and tend to diverge or recurve from the base of the perithecium. They may arise from only one, or from three of the cells below the perithecium, and may be quite simple or less frequently, as in the type, be multiplied by proliferation from the left side of the basal cell, which may thus appear to bear two or three appendages, each distinguished by a basal blackened septum. The appendage above the perithecium may be similarly modified, and even the basal cell of the primary appendage may occasionally produce a subterminal adventitious branch. The apex of the perithecium, when viewed sidewise, is asymmetrical, and somewhat compressed; but, if viewed at right angles to this position, appears symmetrical, more or less trilobed, the middle lobe more prominent and much broader. The perithecium as a whole is variably developed, straight or sometimes falcate, the distal portion characteristically tinged with reddish brown.

Dimeryomyces geminandrus nov. sp.

Male individual hyaline to yellowish, sometimes with a slight brown tinge. Receptacle slender, erect; consisting of from two to six cells: the basal long, the rest somewhat flattened and obliquely superposed: basal cell of primary appendage long, symmetrical, of nearly uniform diameter, separated from the slender usually three-celled free appendage by a broad black constricted septum. Antheridia two to a dozen or more, often adventitious, even from the basal cell of the primary appendage which is usually enclosed by the venters of two paired antheridia which arise symmetrically and subterminally from the terminal cell of the receptacle, the necks curved outward in opposite directions: the rest formed in an often irregular series from the cells below, on the anterior side, sometimes associated with a secondary appendage from the basal cell of which one or more adventitious antheridia may also develop. Antheridia long and slender, the long stalk-cell abruptly erect or but slightly divergent, the neck curved outward. Receptacle, exclusive of foot and basal cell of appendage, 40–80 × 8–12 μ. Free primary appendage 40 × 4 μ, the basal cell 18–20 × 6 μ. Total length to tips of highest antheridia 80–150 μ.

Female individual very variable in size, usually distinctly yellow or with variable brown suffusions. Receptacle rather short, consisting of from six to eight somewhat flattened obliquely superposed cells, the upper becoming tinged with brown: the subterminal giving rise to the usually single perithecium, below which two to four cells, as well as the terminal one, give rise to secondary appendages; the latter single and simple, or their basal cells producing two to four branches distinguished by very thick black constricted septa, hyaline, slender, flexuous, often very greatly elongated. Basal cell of the primary appendage usually producing one, sometimes more, adventitious branches similar to the other secondary appendages, on the inner side. Perithecium normally single, rarely two or three, very variable, often of great size, sometimes sessile with a broad base, more often with the stalk-portion much elongated, pale yellowish, distally undistinguished from the purplish brown ascigerous region; the tip usually well distinguished by slight elevations and tapering symmetrically to the clearly defined apex, which, in face view, is more or less broadly spreading, symmetrically rounded with submarginal indentations; the side view asymmetrical, compressed, blunt, usually with a slight subterminal enlargement. Perithecia 150–925 × 20–34 μ. Receptacle 70–125 × 20–40 μ. Longest appendage seen, 715 × 5 μ.

On the elytra and inferior abdomen, the larger at the tips of the elytra, of *Hyperacantha Kolbei* Ws. No. 3072; and *Diacantha Deusseni*, Kamerun, West Africa.

Although this common species varies enormously in size its growth being apparently influenced by its position on the host, it does not vary essentially in fundamental characters. It is distinguished from the nearly allied *D. Aulacophorae* by the absence of any peculiar modification of the basal cell of the primary appendage, the extraordinary development of its perithecia, its fundamentally yellow color, and very slender and elongate appendages. The male is almost always readily distinguished by the presence of symmetrically paired antheridia arising from the distal cell on either side of the basal cell of the appendage which they enclose in a characteristic fashion.

Dimeromyces auriculatus nov. sp.

Male individual almost hyaline. Receptacle erect, consisting of three cells, the two upper bearing superposed antheridia; the basal cell of the appendage slightly longer than broad, separated from the more slender free appendage by a black constricted septum. Antheridia slender with long stalk-cells, hyaline, the efferent region purplish; the neck hyaline, rather stout; the whole slightly curved outward throughout. Receptacle, exclusive of foot and base of appendage, 20 × 8 μ. Basal cell of appendage 8.5 × 4.5 μ. Antheridia 38 × 6 μ.

Female individual rather short and stout. Receptacle erect, subtriangular, consisting of five cells; the three middle ones similar, flattened, oblique and somewhat curved; the subterminal bearing the single perithecium; the second, third and fifth, secondary appendages; the latter rarely double, separated by a large black constricted septum, stout, hardly reaching above the middle of the perithecium from the base of which they may curve outward, especially those from the terminal cell: basal cell of the primary appendage, relatively small, bell-shaped, producing a single terminal free appendage separated by a black septum and constriction. Perithecia relatively short and stout, yellow and abruptly narrower above the insertion, slightly asymmetrical, somewhat straighter on the inner side, subfusiform, brownish yellow; the stalk- and tip-regions not abruptly distinguished, the former tapering rapidly to the abruptly distinguished hyaline apex, which is usually symmetrical, a distal rounded protruding termination subtended on either side by similar ear-like appendages. Perithecia 135–170 × 29–34 μ. Receptacle 42–55 × 22–25 μ. Longer appendages 85 × 8 μ. Total length to tip of perithecium 190–230 μ.

On the mid distal surface of the right elytron of *Diacantha Deussenii* Karsch, No. 3073; Kamerun, W. Africa.

This species is represented by only seven males and an equal number of females; but although it is related rather closely to the other species on this type of host, it seems well distinguished by the form of the perithecia, and the prominent and characteristic auricles which characterize the apex. Its comparatively short and stout appendages recall those of *D. Aulacophorae*, but are quite unlike those of *D. geminandrus* which occurs on the same host.

Dimeromyces helicoideus nov. sp.

Male individual erect, straight, or slightly curved. Receptacle hyaline, becoming slightly yellowish, its two cells of uniform diameter, the basal cell of the appendage narrower distally, separated from the free two- to three-celled appendage by a black constricted septum. Antheridia normally solitary, pale purplish brown, except the hyaline outcurved termination of the neck; the stalk well developed; erect, appressed or but slightly divergent. Receptacle, exclusive of foot, 34 × 8 μ. Appendage, above black septum, 25 × 5 μ. Antheridia 38 × 6 μ.

Female individual. Receptacle hyaline becoming suffused with brown except at the base; consisting of normally five cells; the three middle ones somewhat flattened and oblique, the fourth producing the normally single perithecium, the second, third and fifth normally single secondary appendages, the lowest rarely double. Primary appendage shorter but similar to the secondary which are long, slender and tapering, distinguished above the basal cell by a black septum. Perithecium strongly curved or recurved so that the tip may touch the foot; the stalk-portion well developed but usually not distinguished, pale yellowish, merging into the rich dark purplish red-brown of the ascigerous portion: tip usually not distinguished; the apex subhyaline, short, blunt, unmodified. Perithecia 85–100 × 18 μ. Receptacle 42 × 12.5 μ. Appendages 50–170 μ.

Near the margin of the right elytron of *Crepidodera* sp. Nos. 3061, 3057 and 3074. Metet. Kamerun, W. Africa.

The strongly curved or helicoid habit of this species is very characteristic. It is much smaller than a majority of the forms on related hosts, and has only been seen in the position indicated; except that in one instance a few smaller more nearly straight individuals were found on the posterior right leg.

Dimeromyces Hyperacanthae nov. sp.

Male individual subhyaline to faintly brownish, relatively small, the foot large. Receptacle consisting of three cells, one or both of the distal bearing antheridia; which are slender, erect, somewhat flaccid, the neck slightly curved, relatively long; the appendage erect, distinguished from its basal cell by a broad black constricted septum. Receptacle 30 × 12 μ, exclusive of foot which is 16–18 μ. Appendage with basal cell 42 μ. Antheridia 42 × 5.5 μ.

Female individual uniform rather dark translucent reddish brown; erect. Receptacle consisting of normally five cells, the basal larger, the rest slightly flattened and oblique, a single perithecium usually arising from the fourth; the rest, except the basal, producing appendages the basal cells of which give rise to one or often two free branches distinguished by black constricted basal septa, suberect, rigid, appressed or hardly divergent, the slightly tapering extremities becoming disorganized and seldom reaching to the tip of the perithecium. Basal cell of primary appendage not distinguished from those of the receptacle, its outer margin enlarged and bulging so as to throw the insertion of the free appendage inward; the latter short, distinguished by a black septum. Perithecium quite sessile, short and stout, rarely longer and more slender, the outer margin nearly straight, or less convex than the inner, the tip undifferentiated, or sometimes indicated by a slight depression; the apex subhyaline, its distal margin straight and oblique, viewed sidewise, owing to a finger-like or tooth-like protrusion upward and inward from its inner angle. Perithecia 100–135 × 30–36 μ. Receptacle, excluding foot and basal cell of primary appendage, 50-85 × 25 μ. Appendages 100 × 6 μ, longest 150 μ. Total length to tip of perithecium 160–210 μ; the more slender form up to 335 μ, through the elongation of the perithecium.

Forming a dense group at the tip of the free upper surface of the abdomen of *Hyperacantha robusta* Ws. and *Hyperacantha* Sp., Nos. 3070–71, Kamerun, W. Africa.

The more slender and elongate form of this species, above referred to, was found protruding from just beneath the tips of the elytra. The material is somewhat scanty and it is barely possible that it may prove specifically distinct. The projection, or appendage, from the tip of its perithecium is much longer and more slender, and the perithecium itself is long and slender. The difference is probably due, however, to its position of growth.

Dimeromyces Necrotalis nov. sp.

Male individual (single specimen) pale yellowish. Receptacle consisting of four obliquely superposed cells, the three upper producing antheridia; the basal short, its pointed distal anterior angle protruding below the base of the lowest antheridium. The appendage straight, tapering, distally attenuated; the basal cell large and distally symmetrically rounded. Antheridia with a well defined short stout stalk, the efferent tube region and neck both clearly defined, about as long as the stalk and venter; the neck straight, rather stout.

Female individual (single specimen) pale yellowish throughout; consisting of ten cells, the basal narrow and extending horizontally to, or slightly beyond, the base of the first secondary appendage; the rest of the receptacle forming a horizontal series, all the cells except the eighth and ninth, wedge shaped, the points downward; the terminal cell larger, subtriangular, externally abruptly prominent below the base of the primary appendage; which is similar but slightly stouter and somewhat shorter than the secondary appendages, which tend to alternate with perithecia, diverging slightly from one another, straight, tapering throughout, distally usually attenuated, about seven or eight celled; forming, with the perithecia and receptacle, a fan-like structure. Perithecia straight or the stalk curved; the latter well developed, nearly uniform, about as long as the ascigerous portion, which is abruptly distinguished, symmetrically fusiform; the distal half with two or three slight marginal depressions and elevations, the upper distinguishing the tip rather clearly; the apex well defined, about as long as broad, distally rounded. Perithecia 106 × (stalk) 8 × (ascigerous portion) 15μ. Longer appendages 75 × 7.5 μ at base. Horizontal cells of receptacle 35 × 10 μ.

On antennae of *Necrota Africana* Gr., Kamerun, West Africa. No. 2577.

This species is represented by two mature and perfect individuals and an additional female, in which the perithecia are still young. Its chief interest lies in the fact that it illustrates very clearly the ease with which the *Dimeromyces*-type may run into the *Dimorphomyces*-

type: since a very slight variation in the method by which the intercalary cells of the horizontal axis are cut off from the basal cell, would make it conform entirely to the characters of the last mentioned genus. In the present instance, however, the subbasal cell which bears the primary appendage becomes displaced so as to end the horizontal series on one side, the extension of the basal cell keeping pace only with the last two or three cells separated from it at the other extremity of the series, below which it forms a short margin.

Dimeromyces unguipes nov. sp.

Male individual rather evenly tinged with pale brown: slender, erect, the foot long, slender and claw-like. Receptacle consisting of three cells; the basal much longer, and but slightly broader distally; the subbasal narrow and oblique, prominent below the base of the erect antheridium; which arises from the distal cell and reaches above the apex of the 3-celled somewhat tapering erect terminal appendage. Antheridium with a well developed free stalk somewhat shorter than the parts above: the venter stout and rather short, more prominent distally and externally, the antheridial cell tube region and the efferent neck short, of about equal length, clearly distinguished and bent slightly inward. Antheridium 38 × 9 μ. Receptacle, without foot, 34 × 8 μ. Appendage 27 μ. Total length to tip of antheridium 80–90 μ.

Female individual rather slender, erect; the foot long and claw-like; the receptacle consisting of five obliquely superposed cells; the three mid-cells much flattened and obliquely superposed, the basal much longer, the third producing a secondary appendage, the fourth a perithecium. Primary appendage of three successively smaller cells, the distal darker; secondary appendage nearly twice as long, the basal cell longer than the three terminal ones combined, the two mid cells darker brown. Perithecium straight, erect, slender; its greatest width just below the tip, whence it tapers gradually and evenly to the narrow insertion; the stalk well developed, but not distinguished clearly from the body, which is more deeply tinged with brown for some distance below the tip, the lower half of this darker area marked by fine transverse lines which correspond to slight ridges which extend nearly to the insertion of the stalk: the tip and apex blunt conical, subsymmetrical. Perithecium 135 × 16 μ. Secondary appendage 66 × 8.5 μ; primary 34 × 6 μ. Receptacle without foot, 42 × 16 μ. Total length to tip of perithecium 180 μ.

On the elytra of *Eustilbus apicalis* Shp., Agua Caliente, Guatemala. (Kellerman).

Among the few specimens of this species which are available only two males and a single female are uninjured, so that more copious material will probably necessitate some modification of the above description. The species is so well defined, however, that I have felt it safe to include it in the present enumeration. Its claw-like foot, transversely ridged perithecium, the form of which is otherwise characteristic, the characters of the secondary appendage with its greatly elongated basal cell, as well as the structure of the slender male with its relatively large, distally and externally hunched antheridium serve clearly to separate it from other known forms.

Dimeromyces eximius nov. sp.

Male individual relatively large. Receptacle nearly opaque, except the basal and the lower portion of the subbasal cell; consisting of about five superposed cells, the lower two much flattened, terminated by an undifferentiated appendage consisting of two cells; the lower much larger, the upper shorter and narrower, slightly translucent, especially its flattish distal margin. The successive cells of the receptacle above the basal, producing an anterior series of unicellular branches, or protrusions, from each of which may arise one or more antheridia or appendages or both combined; the appendages stiff, simple, faintly brownish, rather closely septate, the septa darker and slightly constricted, slightly divergent; the antheridia sessile, rather short and stout, the three regions very clearly distinguished; the neck short, stout, distally rounded and slightly broader, curved slightly inward; the venter large, faintly brownish; the middle efferent region contrasting red-brown: the insertion blackened. Receptacle, including large foot, 58 × 15 μ. Primary appendage 22 × 12 μ; secondary 130 μ or more, × 8 μ. Antheridia 38–44 × 13 μ. Total length to tip of appendage 82 μ.

Female individual similar in structure to the male, the secondary appendages more numerous and elongate, curved upward and inward, distinctly brownish, the septa darker, with a slight constriction. Primary appendage indistinguishable, the axis quite opaque above the basal cell. Perithecia one or two, lying at the left, like the antheridia, while the secondary appendages, except two or three of the lowest, lie at the right: the stalk very short, nearly hyaline, abruptly bent;

the rest deeply suffused with reddish brown, large, broader distally, almost opaque; the tip-region translucent, defined above and below by a clear fine dark line; the apex larger and longer than the tip-region, asymmetrical, tapering rather abruptly, the inner two thirds much darker, especially the prominent overarching upper lips; the outer third concolorous with the tip-region. Perithecia 135–160 × 32–38 μ. Secondary appendages, longest, 275 × 9 μ. Axis, including foot and primary appendage, 100–110 × 23–25 μ.

On the anterior legs of *Pachyteles* sp., Verdant Vale, Arima, Trinidad, No. 2821.

The characters of this striking species are very aberrant owing to the presence of short unicellular branches, or protrusions, from which, in the male, both appendages and antheridia, and in the female appendages and perithecia, may be formed. In general appearance it recalls to a certain extent certain individuals of *D. pinnatus*, although the structure differs fundamentally, since in this form appendages arise on opposite sides of the receptacle. The form and coloration of the antheridia is very striking and the antheridial cells are unusually numerous, at least six being present.

Dimeromyces Caribbaeus nov. sp.

Male individual hyaline, erect, slender. Receptacle consisting of from four to seven cells, straight or slightly curved backward. Appendage four- or five-celled, the rounded subbasal cell distinguished by a black septum. Antheridia two to six, hyaline throughout, rather slender, the stalk relatively long, the neck well distinguished, rather stout slightly bent. Antheridia 28 × 5.5 μ. Receptacle 38–45 × 8 μ. Appendage 26–32 μ.

Female individual almost exactly similar to that of *D. Petchi*, the perithecium is however less strongly bent to one side, less deeply colored, the whole noticeably larger than the receptacle, the tip-portion slightly but characteristically inflated, distinguished by a distinct constriction and contrasting black ring at its base. The basal cell of the primary appendage broadly oval, not flattened and contracted distally, as in *D. Petchi*. Perithecium 65–76 × 20–22 μ. Receptacle 48 × 18 μ. Primary appendage 25–30 μ, secondary up to 55 μ. Total length 100–120 μ.

On the legs and inferior anterior thorax of *Perigona* sp., No. 2920, Grand Etang, Grenada, B. W. I.

Were it not for the striking differences between the males of this species and of *D. Petchi* it would be difficult to separate them satisfactorily by means of the female. They seem, however, to differ rather constantly in the points above referred to, and individuals of the two may be sorted out in almost every case by the relative size of the perithecium and receptacle. Were it not for the striking differences seen in the males I should not, however, have hesitated to unite them under one name. This difference rests in part on the general form of the receptacle, but especially on the form and coloration of the antheridia; which, in the present species, are quite hyaline, relatively long and slender, with well developed necks straight or but slightly curved. In *D. Petchi*, on the other hand, the antheridia are short and stout, the region occupied by the individual discharge tubes quite black, ending in a hood-like conformation from which the hyaline neck, stout and hardly longer than broad, projects almost at right angles. In all the considerable material examined from Ceylon, Java, Borneo and the Philippines there are no individuals which show a transitional tendency.

Dimeromyces Gonocnemalis nov. sp.

Male individual erect, slender. Receptacle consisting of from two to five cells, subequal above, the basal cell much elongated, the rest somewhat oblique, slightly flattened. Appendage consisting of four to five successively smaller cells, distally more or less conspicuously enlarged and tinged with dark brown. Antheridia borne on a long slender hyaline stalk, slightly divergent, straight, the neck and venter more or less deeply tinged with purplish brown, the neck well distinguished, subulate. Receptacle 34–50 × 9 μ. Appendage 64 × 8 μ at base. Antheridia 50 × 9 μ, the stalk 20–22 μ: the antheridia and appendage usually diverging slightly from one another.

Female individual very variable in size. Receptacle usually curved strongly backward, hyaline to pale yellowish, the upper half or more becoming rather deeply tinged with smoky brown: consisting of eight to sixteen cells, more or less, greatly flattened below, oblique only through the general curvature. Primary appendage not abruptly distinguished from the receptacle, soon broken, consisting of not more than six or seven cells, the two or three lower becoming brownish, deeper, and contrasting distally: secondary appendages longer, tending to alternate with perithecia, erect and rather rigid or somewhat sinuous, often elongate, two or three to ten, two to four or more of the terminal cells often rather abruptly somewhat smaller, and forming a more or less characteristic termination; the cells eighteen, more or less, in number, many of them abruptly somewhat broader distally than the base of the cell next in order. Perithecia one or two to eight, more or less; the stalk-portion well developed, elongate, but not usually clearly distinguished from the ascigerous portion, which becomes rather deep red-brown; the stalk yellowish, or tinged with pale brown; the whole straight and rigid, or slightly sigmoid, tapering distally to the undifferentiated tip, and rather bluntly pointed apex. Perithecia, largest 300 × 25 μ, the ascigerous portion about 140 μ, but varying to much smaller dimensions. Receptacle about 60–80 × 15–20 μ. Primary appendage 125–170 × 12 μ: longer secondary appendages 300–425 μ, more or less.

On the elytra of *Gonocnemis* sp. Nos. 2356 and 2633, Kamerun, W. Africa.

This species is most nearly allied to *D. maculatus* from which it differs in its immaculate appendages, short perithecia and quite different receptacle. The male is also at once distinguished by its comparatively short appendage and long-stalked antheridia.

Dimeromyces maculatus nov. sp.

Male individual pale yellowish, becoming tinged with brownish. Receptacle rather stout, clavate or brush-shaped, consisting of from five to ten cells greatly flattened and very oblique; the basal larger and usually abruptly bent. Primary appendage erect, rather elongate, tapering; the basal cell larger, the subbasal dark red-brown and distinguished by dark septa and constrictions; the third somewhat smaller, often dark and slightly inflated; the rest of the appendage pale, tapering, its lower cells sometimes darker and slightly inflated. Antheridia one or two to seven, sometimes associated with one, or rarely two, secondary appendages similar to the primary, but longer and stouter, with longer basal cells. Antheridia nearly sessile, the regions clearly distinguished; nearly straight, the neck long and slender, its base and tip hyaline, the rest brown, as is the clearly distinguished efferent region below it. Receptacle 50–65 × 20–25 μ. Appendages, primary, 115–150 μ; secondary, longer, 190 μ. Antheridia slightly divergent, average, 48 × 9 μ.

Female individual showing extreme variations in size and development of appendages. Receptacle faintly yellowish, becoming tinged with brown, the posterior margin narrowly edged with deep blackbrown, tending to become short and stout with the upper part of the anterior margin horizontal, or nearly so; consisting of eight to thirteen cells, very greatly flattened, very obliquely superposed, or the transverse axis of the upper almost vertical. Primary appendage similar to that of the male: secondary appendages two or three to six or seven, stouter; a basal portion consisting of from one to rarely six stouter, paler cells marked by scattered brown spots, and a more slender, darker terminal portion one or more of the lower cells of which are abruptly darker, the basal sometimes almost opaque, with straight or slightly concave margins; those just above it often distinguished by dark septa and slight constrictions, the whole terminal portion resembling the primary appendage in a general way. Perithecia variably elongated, one to several, the stalk-portion sometimes much longer than the ascigerous part and rather clearly distinguished from it, but not always: at first pale, except the contrasting clear reddish brown tip which subtends the short, rather flat hyaline apex. Perithecia 100-360 × 20-30 μ. Receptacle 80-110 × 25-45 μ. Primary appendage 125-200 μ, secondary 150-550 μ.

On the elytra and inferior thorax of *Sphaerostylus Wylici* Murr. No. 3089, Kamerun, W. Africa.

This striking species is most nearly related to *D. Africanus* and *D. Gonocnemalis,* from which it is most easily distinguished by the form of its receptacle and the peculiar maculation of the basal cells of its secondary appendages. In the individuals of a group growing at the base of the mid-legs the perithecia and appendages are very greatly developed, the maculate basal cells especially, being larger and more numerous. The individuals occurring elsewhere are almost uniformly much smaller, with rarely more than one maculate basal cell, and the differentiation between the basal and terminal portions usually far more pronounced.

Dimeromyces decipiens nov. sp.

Male individual (seen edgewise) rather slender; receptacle of two cells, deeply tinged with reddish brown, except the anterior margin and the base of the much larger basal cell. Appendage three-celled, the basal evenly brown, contrasting abruptly with the other two which are hyaline, taper slightly to the blunt tip, and are separated by a slightly oblique thin blackish septum. Antheridium somewhat longer than the receptacle and appendage combined, the neck very long and slender, slightly curved. Antheridium 57 × 7.6 μ. Appendage 19 × 5.5 μ at base. Receptacle without foot, 28 μ. Total length to tip of antheridium 84 μ.

Female individual irregularly furcate in habit. Axis consisting of twenty cells, more or less, obliquely superposed, the basal much longer and hyaline, or translucent below its distal end, and the cells above it deep red-brown, more or less opaque, becoming dirty yellowish translucent brown distally; the series tapering slightly above the middle and ending in a blunt cell without any recognizable primary appendage; the cells bearing rather short, stiff, two- to three-celled secondary appendages, the lower more or less deeply tinged with reddish brown, those above subhyaline; the lowest heterogeneous, deeply suffused, blunt, curved, nearly uniform, of about six hardly distinguishable cells; subtended by the single perithecium which arises from the opaque region of the subbasal cell, its short stalk-portion abruptly curved turning the perithecium sidewise and away from the curvature of the axis: the stalk-region not distinguished, the ascigerous region strongly convex distally and externally, the inner margin straight, or slightly concave; the tip abruptly distinguished by indentations, more abrupt on the inner side, its margins slightly convex; the apex oblique distally and externally rounded and prominent. Perithecia 118–140 × 22–27 μ. Basal (heterogenous) appendage 50–66 × 5–5.7 μ; the rest about 35 × 6 μ. Total length to tip of axis 140–190 μ.

On the inferior surface of the tip of the abdomen of *Eleusis* (*Isomalus*) sp. No. 2359, Kamerun, West Africa.

Owing to the fact that the axis of the female in this peculiar species is so turned that it is viewed for the most part edgewise, it is very difficult to obtain a clear idea of its exact structure. It is perhaps more nearly allied to *D. Thaxteri*, but in the adults, only two of which have been examined, there is no indication of the presence of a primary appendage and the heterogeneous appendage that is present just above the insertion of the perithecium, seems to have no counterpart in other species. It is possible that the axis is the result of a secondary proliferation, but there are no young individuals from which this might be determined. In general habit it is not unlike a somewhat anomalous form of *Rhachomyces*. The male has only been seen viewed edgewise, and owing to the partial opacity of the receptacle the characters of the latter cannot be made out with certainty. The very long slender neck of the single antheridium and the rather peculiar appendage are its chief peculiarities.

Dimeromyces ametrothecalis nov. sp.

Male individual very pale yellowish, almost hyaline. Receptacle three-celled; including the foot, roughly an inequilateral triangle, bearing a two-celled subulate short appendage from its upper angle, and normally two antheridia from its upper side arising from the nearly equal subbasal and terminal, smaller, triangular cells. Antheridia nearly straight and symmetrical, the upper inclined against the appendage, the lower often slightly divergent; the stalk-cell short and well distinguished; the venter stout, nearly symmetrical; the neck rather long, straight and well differentiated, extending far above the tip of the appendage. Receptacle without foot 21–22 × 9–10 μ. Appendage 13–17 μ. Antheridia 30–38 × 10 μ. Total length to tips of necks 63–66 μ.

Female individual pale yellowish, becoming tinged with brown. Receptacle relatively minute, consisting of four cells, the basal much larger, the rest successively smaller, somewhat flattened and oblique. Primary appendage short, three-celled, subulate; the secondary single, arising from the subbasal cell, nine or ten-celled, tapering, lying obliquely against the base of the perithecium, the cells mostly broader than long, separated by somewhat oblique septa, and decreasing slightly in size from below upward. Perithecia normally single, arising from the third cell, monstrously developed, becoming more or less deeply tinged with reddish brown; the stalk well developed, curved, but not clearly distinguished from the ascigerous portion, which becomes gradually broader; the tip rather clearly defined, more convex on one side, paler; the apex short, slightly bent, rounded, clearly distinguished. Perithecia 170–400 × 25–27 μ. Receptacle and foot 38–42 × 12–15 μ. Primary appendage 16–20 μ; secondary 40–60 μ. Total length to tip of perithecium 200–430 μ.

On the upper surface of the abdomen of a flat Cucujid, belonging to an undescribed genus, No. 2339, Kamerun, West Africa.

This form does not appear to be nearly related to any other and is very clearly distinguished by its monstrous perithecia, small subulate primary, and straight closely and obliquely septate secondary appendages.

Dimeromyces Copropori nov. sp.

Male individual becoming tinged with brown, short and stout. Receptacle consisting of three cells; the distal more deeply tinged with brown as is the relatively large basal cell of the appendage, which bears a small button-like terminal cell. Antheridia usually two, faintly brownish, short and stout, with short hardly distinguished necks, straight, the stalk short but distinct. Receptacle, without foot, 20 × 12 μ. Appendage 13 μ. Antheridia 30 × 9 μ. Total length to tips of antheridia 58–62 μ.

Female individual rather uniformly tinged with pale dirty yellowish brown. Receptacle erect, distally somewhat curved backward, consisting of about sixteen to twenty cells, those above the basal flattened, horizontal, except for the distal curvature; those in the mid-region broader, producing either perithecia, or short two-celled abortive appendages which may be mere unicellular protuberances and alternate more or less with the perithecia: the primary appendage two-celled, similar to the secondary, and forming a blunt termination to the receptacle from which it is hardly differentiated. Perithecia paler below and at the apex, nearly sessile, subcylindrical, the tip rather clearly distinguished, darker, tapering to the slightly asymmetrical, broad, blunt termination. Perithecia 100–125 × 20–25 μ. Receptacle, including foot and primary appendage 125–180 × 25–28 μ. Secondary appendages 18 × 10 μ. Total length to tips of perithecia 225–260 μ.

On legs of *Coproporus* sp. No. 2620, Kamerun, West Africa.

This very distinct species recalls to some extent the habit of *D. pinnatus* although the organs arise from one side only and the form of the perithecia and appendages is quite different. The host is said by Mr. Arrow to be an undescribed species.

Dimeromyces Roreri nov. sp.

Male individual hyaline, very faintly tinged with yellowish brown. Receptacle consisting of two cells, or rarely three; erect, the appendage three-celled, the middle cell roundish and distinguished by slight constrictions and dark septa. Antheridia one or two, slightly asymmetrical, the stalk free and well developed, the neck moderate and well distinguished. Receptacle, without foot, 12–13 × 8 μ, the foot small. Appendage 12 μ. Antheridia 38 × 8.5 μ. Total length 48–54 μ.

Female individual nearly hyaline with faint yellowish brown suffusions. Receptacle consisting of six or seven flattened, somewhat oblique and rather irregular cells; the fourth, usually producing the single secondary appendage; the fifth, and sometimes the third, a perithecium. Primary appendage similar to that of the male; secondary appendage of five or six cells, the second and third squarish and distinguished by dark septa. Perithecium asymmetrical, the stalk not distinguished and bent somewhat sidewise; subclavate below the tip, which is clearly distinguished and subtended by a conspicuous transverse brown suffusion associated with a slight constriction; above which the tip is slightly inflated and bent outward below the hyaline slightly inflated apex; which is clearly distinguished by a slight constriction, bent slightly inward, somewhat longer than broad, bluntly rounded. Perithecium 80–100 × 17–19 μ. Receptacle 20–25 × 12–13 μ. Primary appendage 16–20 μ; secondary 42 × 5 μ. Total length to tip of perithecium 105–125 μ.

On the inferior extremity of the abdomen of a species of *Gyrophaena* received from Mr. J. B. Rorer from Port of Spain, Trinidad, No. 2306.

Owing to the basal curvature of the perithecium individuals of this species lie as a rule with the receptacle shown edgewise, so that it is difficult to represent the normal sequence of the cells. The form, and subterminal suffusion of the perithecium, which bears a certain resemblance to that of *D. Carriboeus*, clearly distinguishes it from other species.

Dimeromyces gracilis nov. sp.

Male individual very minute, the receptacle consisting apparently of but two cells, terminating in a two-celled appendage, the basal cell of which is small and nearly round, the distal cylindrical and several times as long. The single minute antheridium arising from the subbasal cell, apparently sessile, with a short neck. Total length to tip of appendage 22 μ, the foot 12 μ.

Female individual almost hyaline. The receptacle consisting of four cells and terminated by a two-celled appendage similar to that of the male and quite undifferentiated. The third cell of the receptacle producing the single perithecium, which is long and slender, the stalk-portion not distinguished, of nearly uniform diameter to the relatively long tapering tip-region, which is not well distinguished, the apex small and rounded. Perithecia 150–190 × 20 μ. Receptacle, including foot, 48–56 × 12 μ. Appendage 20 × 18 μ.

On a myrmecophilus aleocharid, No. 2612, Kamerun, West Africa.

Two mature females of this curious and very simple form have been examined, neither of which, though well developed, bear a secondary appendage although a small cell is separated from the subbasal cell below the insertion of the perithecium. The single male examined is still adherent to the foot of the female, with the antheridium so turned that its structure can only be surmised. The host was taken by Mr. Schwab among a number of Staphilinidae following a procession of soldier ants.

Dimeromyces Gyrophaeniae nov. sp.

Male individual hyaline, faintly tinged with yellowish. Receptacle consisting of four or five somewhat oblique cells; the basal nearly as long as the rest combined. Appendage short, stout, of two subequal cells; the distal thick-walled, slightly broader and rounded. Antheridia three or four, diverging at about 45°, the stalk slightly shorter than the venter and neck; the latter short, straight, conical, hardly distinguished. Receptacle 34 × 12–13 μ. Appendage 17 × 8 μ. Antheridia 30 × 8.5 μ. Total length to tips of antheridia 64 μ.

Female individual pale yellowish. Receptacle turned sometimes at right angles to the axis of its basal cell: consisting of usually eighteen to twenty cells, those above the basal very greatly flattened, especially the lower ones: giving rise alternately to perithecia and appendages, which diverge left and right, respectively, projecting upward from the nearly horizontal receptacle. Primary appendage like that of the male; secondary usually four-celled, the three lower subequal, hardly longer than broad, the distal one broader, rounded and thick-walled. Perithecia subclavate, curved, tapering to the slightly bent short tip, and rather small asymmetrical apex. Perithecia 75–80 × 21 μ. Receptacle, above basal cell, 55–70 × 20–25 μ. Primary appendage 17–19 μ; secondary 46 μ.

On the elytra of a species of *Gyrophaena*: No. 2610, Kamerun, West Africa.

The material of this peculiar species is somewhat scanty, and the perithecia are so irregularly bent that it is difficult to determine their exact outline. The species is allied to *D. Roreri* and *D. Coprophori*, but its differences from either are so great as to render a comparison superfluous.

Dimeromyces Platycilibis nov. sp.

Male individual pale yellowish tinged with brown. Axis of the receptacle straight, erect, five- or six-celled; the basal longer subtriangular; the rest somewhat flattened, the middle ones broader: terminal appendage simple, three-celled, the middle cell darker, subspherical, distinguished above and below by blackish septa. Antheridia one to three, strongly divergent, sometimes almost at right angles; stout, with four or more antheridial cells; the necks short, well distinguished, slightly curved outward, stout. Length to tip of appendage 38–48 μ. Antheridia 15–18 × 7.5 μ.

Female individual yellowish, rather strongly tinged with brownish at maturity. Axis of the receptacle nearly straight and erect, five-to eight-celled; the basal subtriangular, much larger; the rest flattened, the upper somewhat oblique: the terminal primary appendage relatively very small, similar to that of the male; the secondary appendages usually single and subterminal, a second sometimes arising below the usually single perithecium, stout, simple; a basal portion of about six to nine cells slightly longer than broad distinguished from a terminal portion by a constriction associated with a deeper brown suffusion; the terminal part unicellular, or several celled; variably elongate, the walls variably thickened and disorganized, tapering somewhat from a broader base. Perithecia somewhat more distinctly brown, rather short and stout, broader below the tip, thence tapering somewhat asymmetrically to the broad blunt termination, the stalk short or obsolete. Spores about 20–24 × 2.5 μ. Perithecia 58–68 × 20 μ. Receptacle, less foot, 25–28 μ. Primary-appendage 15–28 μ. Secondary appendages 45–85 × 10–12 μ, terminal part 45–130 μ.

On the upper surface of *Platycilibe* sp., No. 2986, M. C. Z., Wainoni Bay, Solomon Islands (Mann Coll.).

This species is abundantly distinguished by its minute primary appendage, and the differentiation of its secondary appendages, the terminal portion of which may vary considerably in length, a few short stout forms occurring.

Dimeromyces Aphanocephali nov. sp.

Male individual colorless, erect, the receptacle three-celled, the cells subequal, the upper long-oval, tilted and externally prominent. Antheridium erect, the stalk-cell free only externally, the short neck hardly divergent and not exceeding the tip of the appendage: total length 35 μ. Antheridium 20 μ. Appendage 15 × 3.5 μ, free, of two subequal cells.

Female individual colorless. The receptacle and appendages sprawling on the substratum: the former four-celled, the cells flattened and somewhat oblique, the subbasal producing a secondary appendage, which usually tapers and consists of seven or eight cells, arising from the anterior side; while a second buffer-appendage of a similar character grows from the same cell on the anterior side: the terminal primary appendage similar, somewhat larger: all the appendages straight or variously curved, the primary and secondary usually directed away from the buffer-appendage: the whole lying flat on the host, the basal cell much compressed. Perithecium with an elongate stalk arising perpendicularly from the substratum, erect or strongly curved, the ascigerous portion moderately well distinguished, not quite symmetrical, slightly inflated, rather long and slender, terminating in a short, finger-like, erect projection: the lumen of the lower part of the stalk becoming more or less obliterated. Ascigerous part of perithecium 60–75 × 15–18 μ: the stalk × 10–11.5 μ: the total length 150–240 μ. Appendages 50–70 μ.

At the tips of the elytra of *Aphanocephalus pubescens* Grouv. No. 1834, Sarawak, Borneo.

In its sprawling habit and the form of its perithecium this species closely resembles *D. appressus*. The characters of the male are, however, quite different and the appendages of the female lack the dark septa and rigid habit seen in this species, while the form of the receptacle is also dissimilar.

Dimeromyces aberrans nov. sp.

Male individual consisting of a small basal and subbasal cell, the latter bearing distally a bluntly pointed terminal cell and subterminally a short stalked antheridium, the neck very short and blunt or nearly obsolete, the antheridial cells apparently two. The subbasal cell sometimes producing two or more branches consisting of a basal cell bearing distally a bluntly pointed terminal cell and subterminally a short stalked antheridium. Total length about 30 μ. Antheridium and stalk 11 × 3.5 μ.

Female individual pale yellowish, the broad short basal cell for the most part blackened and indistinguishable from the foot, somewhat overlapping the small subbasal cell which appears to produce a branch to the right and left; the two similar, rather closely septate, often broader distally, curved upward on either side of the one to several perithecia; one, the secondary appendage, often slightly longer and stouter, than the other, which represents the primary appendage, and is subtended by the first perithecium; a second usually developing from the basal cell of the secondary appendage, accessory perithecia not infrequently arising from cells adjacent to those which produce the first, seldom more than four maturing. Perithecium asymmetrical, irregularly fusiform, the stalk-portion relatively short and tapering to its narrow insertion, not distinguished from the gradually much broader ascigerous portion, which is slightly bent inward distally, strongly convex externally and straight or concave on the inner side below the tip; the apex often bent rather abruptly upward, broad, flat or blunt. Perithecia 55–75 × 13 μ. Appendages 30–48 × 6–7 μ. Total length to tip of perithecium 70–92 μ.

On the elytra of *Tomarus atomarius* Sharp. No. 1527, Columbus, Ohio; No. 1609, Los Amates, Guatemala; No. 1599, El Rancho, Guatemala.

The hosts bearing this parasite were collected for me by the late Professor Kellerman to whose kindness I am indebted for numerous other new and interesting forms. The species is nearly allied to *D. Tomari*, from which it differs especially in the form of its perithecia and appendages. Although in specimens which bear two or more perithecia, the general development appears to be more or less bilaterally symmetrical, the structure does not seem to be fundamentally different from that of other species of the genus. The presence of adventitious perithecia might be compared to the much more striking phenomenon seen in *D. adventitiosus*, while the development of short branches bearing antheridia finds an occasional parallel in *D. Thaxteri*. The males adhere so closely to the base of the female that they can only be separated with the greatest difficulty, and but few have been clearly seen. The antheridium is not typical, the neck being hardly developed at all, and the antheridial cells but two in number, as far as it has been possible to determine. The antherozoids hardly seem to enter a common tube, but rather to make their exit directly from the neck of the antheridial cell. They are so small, however, and the outlines are so vague, that a definite determination of these points has been impossible.

Dimeromyces Tomari nov. sp.

Male individual nearly colorless: the basal cell about as broad as long; the subbasal much smaller, squarish, bearing terminally a relatively stout three to four celled appendage-like termination, and laterally a single rather stout and short antheridium, its stalk-cell clearly developed, the antheridial cells apparently two, the termination short stout and blunt, the efferent neck hardly if at all developed. Foot somewhat larger than the basal and subbasal cells combined, which measure about 8.4 × 4.2 μ. The appendage-portion 20 × 4 μ. The antheridium, including stalk, about 12 × 5 μ.

Female individual faintly tinged with yellowish, becoming faintly brownish with age. Receptacle consisting of a large basal and smaller subbasal cell, the former tinged with brown below and overlapping the latter distally; the subbasal cell giving rise to a simple somewhat irregular secondary appendage more closely septate below, and on the opposite side to the third cell of the receptacle from which the normally single perithecium arises: the rest of the axis not distinguished from the primary appendage by which it is continued. Perithecium relatively large, the stalk well developed, often broader than the ascigerous portion from which it is not otherwise distinguished, the tip slightly asymmetrical, tapering to the rather broad blunt apex; the ascigerous portion nearly straight on the inner and more convex on the outer margin. Perithecia 80–100 × 12 μ; largest 140 μ. Appendages 50–90 μ. Basal and subbasal cells 10.5 × 12.5 μ. Total length to tip of perithecia 90–150 μ.

On the elytra of *Tomarus bellus* Gronv. No. 2776, Grand Etang, Grenada, B. W. I.

Except for the presence of a secondary appendage, and the different form of its perithecium, this species closely resembles that of *Eudimeromyces Chiliotis*, and it is possible that the difference in the male on which the latter genus was based may be less fundamental than was at first supposed. In the present species the male has been clearly seen in several instances to possess a compound antheridium borne on a short stalk-cell from the subbasal cell. A reëxamination of the material of *Eudimeromyces*, however, seems clearly to show that no lateral compound antheridium is produced by the male; but the terminal cell of the four-celled individual seems to become an antheridium as in *Dioicomyces* and *Amorphomyces*. Examination with the highest powers available indicate, however, that the basal part of this flask shaped terminal cell is longitudinally divided, and that the structure may after all prove to be a compound antheridium. It should be borne in mind, however, that these appearances may be misleading and the terminal cell referred to may be neither a simple nor a compound antheridium; and that, if abundant material were examined, compound antheridia might be developed laterally in some instances. In the dozen individuals examined, no indications have been seen of such an organ. A second type, on *Chiliotis*, *Eudimeromyces Andicolus*, has apparently been described by Spegazzini under *Corethromyces;* but he apparently overlooked the male, and was misled as to the generic relationships of his form. His figure clearly shows the coalescence of the stalk- and basal-cells of the perithecium which is known only in a few unisexual genera.

Dimeromyces Tomoderi nov. sp.

Male individual, hyaline or becoming faintly tinged with brownish yellow. Receptacle two-celled, the basal flattened, oblique, somewhat bulging and blackened anteriorly; the subbasal hardly larger, its sides converging to the base of the free appendage; which consists of two subequal cells, about as broad as long, the extremity symmetrically rounded. Antheridium single, relatively large and elongate; the free stalk-cell almost twice as long as broad; the antheridial cells two or three, large; the neck about as long as the venter, rather stout, straight and not abruptly distinguished. Receptacle 8 × 7 μ. Foot 8–10 μ. Appendage 8–9 × 3.5 μ. Antheridium 28 × 5 μ.

Female individual pale brownish yellow, the perithecium darker brown. Receptacle consisting of four somewhat obliquely superposed cells, somewhat flattened, the distal bearing the simple three- to four-celled primary appendage subterminally on the right side; the subbasal cell bearing the single secondary appendage; which is simple, four- to five-celled, nearly uniform, blunt, similar to the primary one: the subterminal cell bearing the sessile perithecium, which is short and stout, asymmetrical, straight or slightly concave posteriorly, strongly convex anteriorly, tapering from above the middle to the snout-like termination which forms a more or less prominent anterior projection and is more or less oblique inward. Perithecia 50–55 × 20–22 μ. Appendages 18–25 × 4 μ, the primary shortest. Receptacle 20 × 14 μ, exclusive of foot. Total length to tip of perithecium 80–90 μ.

On the femur of the anterior leg of *Tomoderus* sp. No. 2590, Kamerun, West Africa.

This small and compact form is not to be confused with any of the described species, and is characterized by its simple short undifferentiated appendages, and peculiarly shaped stout sessile perithecium. The male is especially characteristic from the relatively great size of its erect antheridium, which seems usually to possess but two antheridial cells.

DIMEROMYCES ANISOLABIS Thaxter.

This species has been found, since its original publication, on *Euborellia Janeirensis* Dohrn, Nos. 2257 and 2263, from Independencia and Ceara Mirim, Amazon, in the Mann collection, M. C. Z., the only evident difference from the type appearing in the more prominently geniculate habit of the basal cell of the lower secondary appendage. Specimens have also been examined on *Euborellia* from the Solomon and Fiji Islands, Nos. 3015, 3017 and 3018 also collected by Mr. Mann. This last form is smaller than the type, the basal cell of the primary appendage always producing two branches, distinguished by dark septa, radially placed. The lower secondary appendage also produces two branches from its basal cell in about half the individuals.

Dimeromyces Prorei nov. sp.

Male individual slender, hyaline; the basal much longer than the obliquely separated subbasal cell; the basal cell of the appendage somewhat smaller, separated from the erect three-celled free portion by a dark septum. Antheridium usually single, somewhat divergent, the venter rather well distinguished from the neck and tube, which is curved outward. Total length to tip of appendage, including foot 50 μ. Antheridium 20 × 5 μ.

Female individual rather short and stout. Basal cell somewhat triangular; subbasal much smaller, flattened, triangular; bearing anteriorly the short externally rather prominent basal cell of the secondary appendage; which is separated from the free portion by a constriction and dark septum, above which the first three cells are usually somewhat differentiated, tinged with smoky brown, the lowest especially, strongly convex below on the inner side, so that the appendage diverges; the distal portion more slender, hyaline and tapering.

Basal cell of the primary appendage free, diverging from the distal end of the small short fourth cell of the receptacle; which is slightly bent outward distally, and bears from one to three radially disposed branchlets distinguished by dark septa; the branchlets hyaline, simple, slightly tapering, the two appendages often diverging almost symmetrically on either side of the perithecium which rises between them from the third cell of the receptacle; and is short stalked, or nearly sessile, straight, somewhat asymmetrical, very slightly inflated; the apex bent inward and distinguished, especially on the inner side, by a variably well marked indentation; the inner lip slightly more prominent. Spores about 30 × 4 μ. Perithecium 65–80 × 8–9 μ. Receptacle 30–34 × 18 μ. Secondary appendage 80–125 μ. Total length to tip of perithecium 85–105 μ.

On the inferior abdomen and forceps of *Proreus simulans* Stal. from cane sheaths. No. 3134, Los Baños, Laguna, P. I., W. H. Weston.

This species is most nearly related to *D. Forficulae* from which it is most readily distinguished by the nearly horizontal septum which separates the basal and subbasal cell, the latter being subtriangular with its distal angle nearly median, while the receptacle of *D. Forficulae* consists of four successively narrower cells, their septa very oblique and nearly parallel. There is but one secondary appendage, the lower, developed in the present species, the character of which as well as of the perithecium is somewhat different.

Dimeromyces Australasiae nov. sp.

Male individual colorless, straight, erect, rather slender; the axis consisting of three cells; the basal much longer, terminated by the basal cell of the appendage, which may bear one or two short, two-celled branches distally, each separated by a dark basal septum. Stalk-cell of the antheridium small, the regions rather clearly distinguished by indentations on the inner margin: neck subgeniculate bent or curved more or less strongly outward. Axis, including basal cell of appendage, 38–44 × 9 μ. Antheridia 24–28 × 7 μ. Total length to tips of appendages, including foot, 60–75 μ.

Female individual very faintly tinged with purplish brown. Receptacle consisting of four obliquely superposed cells; the basal as long as the rest combined, more or less strongly curved above the foot, and bulging on the posterior side: the subbasal cell bearing the first secondary appendage, the basal cell of which is rounded, distally narrowed, and bears a single peculiar divergent two-celled branch, the basal cell of which is strongly convex on the inner side and separated by a usually conspicuous constriction from the somewhat longer tapering stiff distal cell: the third cell of the receptacle bearing the single perithecium, its stalk hyaline, sometimes longer than the body, mostly straight, the body nearly symmetrically subfusiform; a slight but variable indentation of the outline below the tip on the inner margin, and a rather abrupt indentation below the apex externally; the latter oblique, the outer lip very prominent, extending up over the inner, distally rounded. The terminal cell giving rise to a second secondary appendage on the inner side, the large basal cell of which produces an oblique series of three to five simple three-septate, nearly cylindrical, hyaline branches distinguished by closely set dark basal septa; the whole being entirely similar to the basal cell of the terminal primary appendage, which it overlaps. Spores about 30 × 3.5 μ. Perithecia, body 50–75 × 20 μ, stalk-portion 30–100 × 9–12 μ. First secondary appendage, including basal cell, 38–42 μ. Branches of primary appendage, 22–38 × 3.5 μ.

On the inferior surface of the abdomen and forceps of *Chelisoches morio* (Fabr.) Nos. 2949 and 3012, Auki, No. 3026, Florida; Solomon Islands. Nos. 3014, Nausori, No. 3013, Viti Leon. No. 3027, Nadarivata, Fiji Is., Mann Coll., M. C. Z.

This species is well distinguished by the basal cell of the primary and upper secondary appendage, which resemble the basal cells of the branches of the appendages in forms like *Laboulbenia pusilla*. The tip of the perithecium, and the lower secondary appendage, are also quite distinctive.

DIMEROMYCES APPRESSUS Thaxter.

This species was first obtained from east Indian species of *Labia*, and has since been collected in Trinidad, B. W. I., on a similar host. The West Indian material does not differ essentially from the type.

Dimeromyces Chaetospaniae nov. sp.

Male individual similar to that of *D. appressus*. Somewhat larger. The basal cell almost wholly blackened, extending beyond the stalk-cell of the antheridium, and separating a small distal cell which forms a subtending rounded prominence: the subbasal cell large, obliquely separated. The basal cell of the appendage giving rise to a terminal and sometimes also to one or two subterminal external branchlets, their basal cells distinguished by dark septa, their subbasal (terminal) cells hardly exceeding the tip of the stout antheridium, which is abruptly curved outward. Total length to tip of antheridium, including foot, 35-38 μ. Antheridium 15–17 × 7 μ.

Female individual somewhat blackened above the foot, the suffusion involving part of cell one and three. At first somewhat spathulate, becoming prostrate; compact, the subbasal cell small and misplaced by the third cell, which bears the single perithecium and appears to be subbasal in position: somewhat obliquely separated from the fourth cell which is distally abruptly prominent below the basal cell of the primary appendage; which, in addition to the terminal branch, may bear two or even three accessory branchlets externally, three-celled, the two lower distinguished by dark septa; the fourth cell also producing distally and inwardly a second secondary appendage, the basal cell of which may bear from one to three branchlets, similar to those of the primary. The first secondary appendage arising from an undifferentiated basal cell, externally prominent, like the subbasal cell of the receptacle which lies below it; bearing a single three-celled termination, the two lower cells distinguished by dark septa. The receptacle, and the appendages with their branches, lying at maturity in the same plane and seen edgewise; the perithecium usually relatively stout, the neck about as long as the body and often not very clearly distinguished from it; the tip slightly and abruptly bent inward, snout-like, inconspicuously sulcate, broad; the whole projecting upward from the prostrate receptacle at an angle of usually 45°. Spores about 20 × 3 μ. Perithecium including stalk, 75–130 × 13-15 μ. Appendages, including basal cell, 20-30 μ. Receptacle, seen flatwise, including foot, 28 × 20 μ.

On the inferior surface of the abdomen and forceps of *Chaetospania thoracica* Dohrn., No. 2126, Sarawak, Borneo.

Although closely allied to *D. appressus* and having much the same habit of growth, this species is clearly distinguished by its appendage, as well as by other points. The stalk of the perithecium, in some instances, may be more slender and clearly distinguished from the shorter body, as in *D. appressus*, though never to such a degree. The actual structure of the receptacle and the origin of perithecium and appendages can only be made out in young individuals.

Dimeromyces moniliformis nov. sp.

Male individual. Axis prostrate, consisting of three cells, continuous, with the short straight four-celled terminal appendage, the subbasal cell of which is slightly inflated and distinguished below by a slight constriction and black septum. Axis of the antheridium parallel to that of the receptacle, its stalk-cell externally convex, bulging strongly and extending down so as to overlap the distal end of the basal cell; its venter rather stout, slightly inflated, the neck rather short and stout, curved outward distally. Antheridium and stalk 30 × 6 μ. Appendage and receptacle, including foot, 50 μ.

Female individual nearly hyaline, with faint tinges of purplish brown. Receptacle prostrate, the axis consisting of four subtriangular cells, the second and third wedge-shaped. The primary appendage similar to that of the male, four-celled, the subbasal cell more prominently distinguished by a constriction and black septum; its upper septum also slightly blackened. Subbasal cell of the receptacle bearing the single secondary appendage, its basal cell overlapping that of the receptacle, strongly prominent and rounded externally; its distal end abruptly narrower, and turned upward to join the nearly hyaline subbasal, which is distinguished from it by a blackened oblique septum: the three succeeding cells slightly suffused, rounded, with strong intervening constrictions, and very broad black septa, the two lower distally oblique; the fourth somewhat larger, wholly and more deeply suffused with purplish brown, the blackened septa narrower; the fifth still larger; the rest of the appendage two- to three-celled, without prominent constrictions, distally tapering slightly to a blunt termination; the whole curved outward and diverging more or less from the perithecium. Perithecium slightly curved inward, normally single, and arising from the third cell of the receptacle; the stalk hyaline, not well distinguished, about as long as the main body which is pale brownish, modified by four successive nearly equidistant and symmetrical enlargements, the upper, which involves the base of the tip, less prominent. Spores about 30 × 4 μ. Perithecia, including stalk, 125–145 × 12–20 μ. Primary appendage 25–34 μ: secondary 100–110 × 8 μ, at base, × 12 μ distally. Total length to tip of primary appendage 58–66 μ, including foot.

On forceps of *Labia mucronata* Stal., No. 2139A, Sarawak, Borneo.

This species is very clearly distinguished by the moniliform base of its secondary appendage and the successive enlargements of the perithecium. The latter is solitary in all the specimens examined, but the fourth cell of the receptacle separates a small cell which may perhaps sometimes develop a second perithecium or secondary appendage.

Dimeromyces annulatus nov. sp.

Male individual very faintly tinged with brownish, the receptacle of two cells, the subbasal subtriangular, bearing the undifferentiated basal cell of the simple appendage distally, and laterally and obliquely the solitary antheridium; which is rather stout, erect, the stalk-cell oblique, the venter somewhat inflated, the neck rather short and stout, distally strongly curved outward. The appendage erect, simple, three-celled, the subbasal cell slightly narrower, squarish, distinguished by dark septa. Length to tip of appendage 28 × 7.5 μ. The antheridium 17 × 5.5 μ.

Female individual pale brownish, the receptacle spathulate or subtriangular, consisting of four superposed cells, which become more or less displaced by a prominent bulging of the third cell on the posterior side, and by the enlargement of the basal cells of the three appendages. Primary appendage four-celled, the subbasal short, somewhat narrower, and distinguished by dark septa; the three terminal cells of the receptacle flattened and broad; the third bearing the perithecium and protruding posteriorly; the second producing the first lateral appendage; the third the primary and a second lateral appendage, both the latter distinguished below by the presence of broad short cells separated by dark annular septa, of which there may be five to six in the second and somewhat fewer in the first; the termination of the appendages hyaline, slightly inflated, their total length not half that of the perithecium, from the base of which they may diverge slightly. Perithecium very slightly curved, somewhat asymmetrical, its base broad without any differentiated stalk-region, tapering slightly, distally, to a blunt slightly asymmetrical termination. Spores about 32 × 2.5 μ. Perithecia, larger 56 × 10 μ. Primary appendage 20 × 5 μ; larger secondary 30 × 3.5 μ. Receptacle 13–15 × 13 μ.

On the inferior abdomen of *Chaetospania paederina* Gerst. No. 2205, Kamerun, W. Africa.

This minute species is most nearly related to *D. appressus* from which it is readily separated by its annular appendages and sessile perithecium. The compact and relatively small receptacle appears quite differently from different points of view, and through secondary displacement and the presence of a dorsal outgrowth of its third cell, the original cell-arrangement is often difficult to recognize.

Dimeromyces lobatus nov. sp.

Male individual consising of three cells; the basal abruptly more than twice as broad distally, forming a shelf on which the broad flat subbasal cell rests; the third cell narrower: appendage erect, terminal, its basal cell undifferentiated, slightly narrower distally, separated from the subbasal by a somewhat constricted deep reddish cup-like septum, the distal portion two-celled, tapering to a blunt point. Antheridium arising from the third cell of the receptacle, sometimes also from the second, parallel to and in close contact with the appendage, the neck rather long and stout, slightly curved outward, subtended by a slight prominence above the somewhat inflated but narrow venter. Total length, including foot (18 μ), 65 μ. Basal cell × 8 μ below and × 14–16 μ distally.

Female individual. Receptacle four-celled, otherwise similar to that of the male, the basal cell of the primary terminal appendage undifferentiated, the subbasal distinguished by a more conspicuous constriction and dark septum. Secondary appendages arising from the second and fourth cells, the latter tapering, subhyaline, distinguished above its somewhat rounded basal cell by a dark septum; the former large, externally and distally tinged with deep blackish brown; its basal cell somewhat rounded, separated from the subbasal by a constriction and broad black septum: straight, slightly divergent, five-celled; the terminal cell producing three or four short, stout, recurved, deep brown branchlets or lobes. Perithecium arising from the third cell, the stalk well developed (the ascigerous region immature). Primary appendage 25–50 μ; lower secondary, 40–50 μ. Receptacle, including foot, 40–50 μ.

On forceps of *Echinosoma Congolense* Bor. No. 3104, Kamerun.

Owing to the fact that the perithecia are immature, I have described this species with reluctance. Its peculiarities are such, however, that it is not likely to be confounded with any other, the lobed branching of the distal cell of its lower secondary appendage being unique. The males are mature and peculiar. About twenty specimens have been examined.

Dimeromyces 1-flagellatus nov. sp.

Male individual colorless, the axis straight, three-celled, the basal cell very large, of nearly uniform diameter; the subbasal usually smaller than the basal cell of the appendage, which is obliquely separated from it: both cells more often bearing slender, somewhat divergent antheridia, slightly curved outward throughout; appendage evanescent, slender, two-celled. Axis (three cells) 25–30 × 5.5. μ. Antheridia 25 × 4 μ. Basal cell 20 × 5.5 μ.

Female individual erect, straight nearly hyaline. Receptacle consisting of five-cells, the basal very large, sometimes longer than the rest combined, the second or third cell bearing the single sessile perithecium; the two obliquely separated, subequal, more flattened than the two above; all separated by slight marginal indentations; the subbasal often cutting off a small cell distally and anteriorly; the only appendage primary, terminal, slightly divergent, somewhat moniliform and reddish brown below; its basal cell large, usually slightly concave externally, the subbasal very small almost square, more deeply suffused, the two or three cells above larger, more rounded, somewhat less suffused; the three to five distal cells pale, the terminal one usually much longer. Stalk of the perithecium usually wholly undistinguished, straight, an external elevation often submedian, thence tapering asymmetrically to the blunt termination, the outer lip-cell large and somewhat spoon-shaped, lying over the inner. Spores (female) 28 × 2.5 μ. Perithecia 50–60 × 13 μ. Appendage 75–90 μ. Total length to tip of perithecium 95–105 μ. Basal cell, largest 42 × 10 μ.

On the head and prothorax of *Spongorostox alter* Burr. Porto Vehlo, Amazon, Mann Coll., No. 2242, M. C. Z.

This peculiar species differs in the absence of any secondary appendages, in its elongate basal cell and usually quite sessile perithecium. The appendage of the male has only been seen in a shriveled condition, and is quickly destroyed. The antheridia may be solitary or more often two superposed, and are slender, with long attenuated necks.

DIMEROMYCES THAXTERI Maire.

This very peculiar and somewhat anomalous species which was first observed on *Gryllus* from Java, and inadvertently described by me under the preoccupied name *D. falcatus*, already used by Paoli for a species on mites, has been again found on a species of *Gryllus* from Wainoni Bay, Solomon Islands, and also on a small cricket of an apparently different genus from the Zamboanga District, Mindanao, P. I.

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SOME GEOMETRIC INVESTIGATIONS ON THE GENERAL PROBLEM OF DYNAMICS.

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§1. **Physical Interpertations.** 1) *Kinetics.* Consider the motion of a system in a conservative field of force from one position P_0 to another P_1, with the sum of its kinetic and potential energies equal to a given constant; the system will move naturally or unguided along the path for which the *action* is least. This is known as Jacobi's *principle of least action*. Action is defined as twice the time-integral of the kinetic energy T, i. e.

$$Action = \int_{(P_0)}^{(P_1)} 2\ T\ dt.$$

If U is the potential energy, W is the work function (negative potential), h is the given constant of energy, v is the velocity of the system, then, for a unit mass, we have the energy equation

$$T + U = h, \qquad \text{or} \qquad \tfrac{1}{2}\, v^2 - W = h, \qquad \text{or} \qquad v^2 = 2(W + h),$$

and

$$\int 2T\, dt = \int v^2\, dt = \int v\cdot v\, dt = \int v\, ds = \int \sqrt{2\,(W + h)}\, ds.$$

The totality of trajectories for all initial conditions are thus determined according to the principle of least action by

$$Action = \int_{(P_0)}^{(P_1)} v\, ds = \int_{(P_0)}^{(P_1)} \sqrt{2\,(W + h)}\, ds = minimum.$$

2) *Brachistochrones.* If the conservative system described above moves not according to the principle of least action but so that the time of motion from P_0 to P_1 is least, the path is called a *brachistochrone*, and the totality of trajectories for all initial conditions are thus determined by

$$Time = \int_{(P_0)}^{(P_1)} dt = \int_{(P_0)}^{(P_1)} \frac{ds}{v} = \int_{(P_0)}^{(P_1)} \frac{ds}{\sqrt{2\,(W + h)}} = minimum.$$

3) *Optics.* The paths of light in an isotropic medium in which the index of refraction ν varies from point to point, are determined according to Fermat's principle by

$$\int_{(P_0)}^{(P_1)} \nu \, ds = minimum.$$

Here the integral is proportional to the time.

4) *Catenaries.* When a homogeneous, flexible, inextensible string is acted on by conservative forces, the forms of equilibrium (general catenaries) are determined by

$$\int_{(P_0)}^{(P_1)} 2\,(W + h)\, ds = minimum.$$

5) *General case; natural families.* We note that each one of the above problems leads to a set of curves determined by an expression of the form

$$\int_{(P_0)}^{(P_1)} F\, ds = minimum.$$

where F is a function of the coördinates of position only, and ds is the element of length in the space under consideration. The curves defined by such an expression may be called, following Painlevé, a natural family. For the dynamical trajectories, brachistochrones, and catenaries, there are ∞^1 such families corresponding to all possible values of the constant h; but in the optical case there is only one such family.

§2. **The general problem of dynamics.** Consider a system with n degrees of freedom, i. e. the position of the system at any instant is determined by n independent parameters or coördinates; thus

$$x = f_1(x_1, x_2, \ldots, x_n), \quad y = f_2(x_1, x_2, \ldots, x_n), \quad z = f_3(x_1, x_2, \ldots, x_n).$$

Then

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_1}{\partial x_n}\frac{dx_n}{dt}, \quad \frac{dy}{dt} = \frac{\partial f_2}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_2}{\partial x_n}\frac{dx_n}{dt}, \quad \text{etc.,}$$

and twice the kinetic energy is

$$2T = v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \sum_{ik} a_{ik}\frac{dx_i}{dt}\frac{dx_k}{dt} \quad (i, k = 1, 2, \ldots, n),$$

where the a's are functions of the n parameters $x_1, x_2, \ldots, x_n$. If the acting forces are conservative, there will exist a work function $W(x_1, x_2, \ldots, x_n)$ independent of the time, and a constant of energy h so that $T - W = h$. The trajectories are then determined by

$$\int_{(P_0)}^{(P_1)} \sqrt{2(W+h)} \quad \sqrt{\sum_k a_{ik}\, dx_i\, dx_k} = \textit{minimum}.$$

Employing geometric language, we may regard $x_1, x_2, \ldots, x_n$ as the coördinates of a point in a space of n dimensions, and P_0 and P_1 as two points in such a space. As the element of arc length in such a space is given by

$$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,$$

we are led to the problem of minimizing $\int F\, ds$, where F is a function of the coördinates $x_1, x_2, \ldots x_n$ and ds is the element of arc in a space of n dimensions. The space may be euclidean (of zero curvature), or non-euclidean, of constant or variable curvature. The natural family consists of $\infty^{2(n-1)}$ curves, one through each point in each direction.

If the work function W is zero, then F is a constant, and our problem reduces to finding the curves for which

$$\int_{(P_0)}^{(P_1)} ds = \textit{minimum};$$

this evidently defines the *geodesics* in our space.

§3. **Geometric Properties of a dynamical system.** In 1869, Beltrami [1] proved a remarkable theorem concerning the geodesics in a space of n dimensions, V_n. This theorem may be stated as follows:

The ∞^{n-1} geodesics which pass out normally from points of any hypersurface, V_{n-1}, form a normal hypercongruence, i. e. admits ∞^1 normal hypersurfaces, which are the loci of equal arc lengths measured from any one of the system of hypersurfaces.

The theorem is a generalization of Gauss's theorem for the geodesics on an ordinary surface, V_2.

In 1871, Lipschitz [2] announced the corresponding theorem for the general problem of dynamics; using the geometric language of natural families, this theorem may be stated as follows:

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[1] E. Beltrami, *Sulla teorica generale dei parametri differenziali.* Memorie dell' Academia delle Scienze dell' Instituto di Bologna, series 2a, vol. 8, p. 549.

[2] R. Lipschitz, *Untersuchung eines Problems der Variationsrechnung in welchem das Problem der Mechanik enthalten ist.* Crelle's Journal, vol. 74.

The ∞^{n-1} *curves of a natural family which pass out normally to any hypersurface,* V_{n-1}, *form a normal hypercongruence, i. e. admit of* ∞^1 *normal hypersurfaces, which are the loci of equal values of* $\int F\,ds$ *measured from any one of the system of hypersurfaces.*

This theorem is true for space of any dimensionality or any curvature.[3]

Thomson and Tait[4] had given this theorem for dynamical trajectories in a conservative field of force in a euclidean space of three dimensions. We quote: "If from all points on an arbitrary surface, particles not mutually influencing one another be projected with the proper velocities [so as to make the sum of the kinetic and potential energies have a given value] in the direction of the normals; points which they reach with equal actions lie on a surface cutting the paths at right angles." The ∞^1 orthogonal surfaces appear as surfaces of equal action. A similar theorem may be stated using the language of brachistochrones, or of optical light paths; in these cases the ∞^1 orthogonal surfaces appear as surfaces of equal time.

The question arises whether the orthogonal properties stated above are *characteristic* of the trajectories under the principle of least action, or the brachistochrones, or the optical light waves, or general catenaries, or, in short, of *any* natural family? In other words, does the orthogonal property belong exclusively to natural families of curves? This question may be answered in the affirmative. Using geometric language we may state the theorem:

If a system of $\infty^{2(n-1)}$ *curves in space of n dimensions (euclidean or curved) is such that* ∞^{n-1} *curves of the system meet an arbitrary hypersurface (space of* $n-1$ *dimensions) orthogonally, and always form a normal hypercongruence, i. e. admit of* ∞^1 *orthogonal hypersurfaces, the system is of the natural type.*

Thus the system may be considered as any one of the types, dynamical or optical, discussed above. This *converse* of the Lipschitz theorem or of a generalized Thomson and Tait theorem was first proved for three dimensional spaces of constant curvature by Kasner.[5] In the following sections the author proves this theorem for a euclidean space of four dimensions (§9), then for a euclidean space of n dimensions, (§10), and finally for a general curved space of n dimensions, (§11). §5 contains a derivation of the differential equations of the natural family of curves by the methods of the calculus of variation. In §6 the author gives a very simple and direct proof of the Lipschitz theorem. It will be noted that although the proof of the direct theorem is very brief, the proof of the converse theorem is quite long.

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[3] For a detailed discussion of these theorems and of the general problem of dynamics, see G. Darboux, Leçons sur la théorie générale des surfaces, vol. 2, chapt. 8. See also, P. Appell, Traité de mécanique rationelle, vol. 2, chapt. 25.

[4] Thomson and Tait, *Treatise on Natural Philosophy*, vol. 1, part 1, p. 353, 1879.

The methods adopted for proving the converse theorems are similar to those used by Kasner for the three dimensional case, except that the use of the arc length as the parameter along a trajectory introduces a symmetry without which the rather long and complex algebraic expressions could hardly be handled.[6]

§4. **Distance and angle.** The totality of points determined by assigning all possible values to n variables or coördinates $x_1, x_2, \ldots, x_n$ are said to constitute a space of n dimensions, V_n. If the x's are expressed as functions of m variables, $u_1, u_2, \ldots, u_m$,

$$x_i = f_i(u_1, u_2, \ldots\ldots, u_m), \quad (i = 1, 2, \ldots, n)$$

the totality of points thus determined are said to form a space of m dimensions, V_m, contained in V_n. In particular, if $m = n - 1$, we have a hypersurface of $n - 1$ dimensions; if $m = 1$, we have a curve.

The element of arc length or distance between two points x_i and $x_i + dx_i$ in V_n is defined by a quadratic differential form

$$(1) \qquad ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,^{7}$$

where the a's are functions of the x's. The parameter curve x_i is such that along it only the coördinate x_i varies, the other coördinates remaining constant; along such a curve the element of arc length evidently takes the form

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[5] E. Kasner, *The theorem of Thomson and Tait and natural families of trajectories.* Trans. Am. Math. Soc., vol. 11 (1910), pp. 121–140.

[6] Natural families of curves have been characterized geometrically in other ways, for euclidean spaces by E. Kasner: *Natural families of trajectories: conservative fields of force:* Trans. Am. Math. Soc., vol. 10 (1909), pp. 201–219; and for general curved spaces by the present writer: *Natural families of curves in a general curved space of n-dimensions;* Trans. Am. Math. Soc., vol. 13 (1912), pp. 77–95.

[7] We write only $\sum_{ik}$ and understand that the summation is to be carried out from 1 to n for each of the indicated subscripts, unless otherwise specified.

$$(2) \qquad ds_i = \sqrt{a_{ii}}\, dx_i.$$

A direction passing out from a point x_i is determined by the n direction cosines

$$(3) \qquad \xi_i = \frac{dx_i}{ds}, \quad (i = 1, 2, \ldots, n)$$

which are related by the identity

$$(4) \qquad \sum_{ik} a_{ik}\, \xi_i\, \xi_k = 1.$$

If $\xi_i^{(1)}$ and $\xi_i^{(2)}$ determine two directions passing out from a point x_i, the angle between them is given by

$$(5) \qquad \cos \omega = \sum_{ik} a_{ik}\, \xi_i^{(1)}\, \xi_k^{(2)},$$

and hence the angle between two parameter curves x_i and x_k is

$$(6) \qquad \cos \omega_{ik} = \frac{a_{ik}}{\sqrt{a_{ii}\, a_{kk}}}.$$

If the space is euclidean, the parameter lines at any point may be chosen as n mutually orthogonal straight lines, hence, from (6) and (2),

$$(7) \qquad a_{ik} = 0 \quad \text{if} \quad i \neq k, \quad \text{and} \quad a_{ii} = 1 \quad (i, k = 1, 2, \ldots, n)$$

and the element of arc length takes the form

$$(8) \qquad ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2.$$

§5. **Differential equations of a natural family of curves.** Our problem is to find the differential equations of the curves for which

$$(9) \qquad t = \int_{(P_0)}^{(P_1)} F\, ds = \int_{t_0}^{t_1} F \sqrt{\sum_{ik} a_{ik}\, dx_i\, dx_k} = minimum.$$

The variation of this integral, keeping the end points P_0 and P_1 fixed must then be equal to zero. Using the usual symbol δ as the symbol of variation, we have

$$dt^2 = F^2 ds^2 = \sum_{ik} (F^2 a_{ik})\, dx_i\, dx_k,$$

$$2\,dt\,\delta(dt) = \sum_{ikl} \frac{\partial (F^2 a_{ik})}{\partial x_l}\, dx_i\, dx_k\, \delta x_l + 2 \sum_{il} (F^2 a_{il})\, dx_i\, \delta(dx_l)$$

$$\therefore \delta(dt) = \tfrac{1}{2} \sum_{ikl} \frac{\partial (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt}\, dt\, \delta x_l + \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, \delta(dx_l).$$

Integrating both sides, and noting that, by partial integration,

$$\int_{t_0}^{t_1} \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, d(\delta x_l) = \left| \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, \delta x_l \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, dt\, \delta x_l$$

and that the first term of the right member vanishes since δx_l is zero at the fixed end points P_0 and P_1, we finally get

$$\delta \int_{t_0}^{t_1} dt = \int_{t_0}^{t_1} \sum_{l} \left[\sum_{ik} \tfrac{1}{2} \frac{\partial (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt} - \frac{d}{dt} \sum_{i} (F^2 a_{il}) \frac{dx_i}{dt} \right] dt\, \delta x_l.$$

Since δx_l is arbitrary, this expression will vanish if

$$\frac{d}{dt} \sum_{i} F^2 a_{il} \frac{dx_i}{dt} = \tfrac{1}{2} \sum_{ik} \frac{\partial (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt}, \qquad (l = 1, 2, \ldots, n).$$

Replacing dt by $F\, ds$, this becomes

$$(10) \qquad \frac{d}{ds} \sum_{i} F a_{il} \frac{dx_i}{ds} = \frac{1}{2F} \sum_{ik} \frac{\partial (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{ds} \frac{dx_k}{ds}, \qquad (l = 1, 2, \ldots, n).$$

For a *euclidean* space, we may use the relations (7), and (10) becomes

$$(11) \qquad \frac{d}{ds}\left(F \frac{dx_l}{ds} \right) = \frac{1}{2F} \sum_{i} \frac{\partial F^2}{\partial x_l} \left(\frac{dx_i}{ds} \right)^2, \qquad (l = 1, 2, \ldots, n)$$

or, expanding and solving for $\frac{d^2 x_l}{ds^2}$,

$$\frac{d^2 x_l}{ds^2} = \frac{1}{F} \left[\frac{\partial F}{\partial x_l} \sum_{i} \left(\frac{dx_i}{ds} \right)^2 - \frac{dx_l}{ds} \sum_{k} \frac{\partial F}{\partial x_k} \frac{dx_k}{ds} \right], \qquad (l = 1, 2, \ldots, n).$$

If we let $L = log\, F$, write[8] $x_l' = \frac{dx_l}{ds}$, $x_l'' = \frac{d^2 x_l}{ds^2}$, and choose the arc length s as the parameter along the curves, so that

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[8] Throughout this paper, primes refer to total derivatives with respect to s.

$$(12) \qquad \sum_i (x'_i)^2 = 1,$$

we get

$$(13) \qquad x''_l = \frac{\partial L}{\partial x_l} - x'_l \sum_k \frac{\partial L}{\partial x_k} x'_k, \quad (l = 1, 2, \ldots, n),$$

as the differential equations of the natural family in euclidean space.

For *any* space, expanding (10), we get

$$\sum_i a_{il} x''_i + \sum_{ik} \left(\frac{\partial a_{il}}{\partial x_k} - \tfrac{1}{2} \frac{\partial a_{ik}}{\partial x_l} \right) x'_i x'_k = \frac{1}{F} \left[\frac{\partial F}{\partial x_l} \sum_{ik} a_{ik} x'_i x'_k - \sum_{ik} \frac{\partial F}{\partial x_k} a_{il} x'_i x'_k \right].$$

Choosing the arc length as parameter along the curves, so that

$$(4) \qquad \sum_{ik} a_{ik} x'_i x'_k = 1,$$

writing $L = \log F$, and introducing the Christoffel three-indices symbol of the first kind[9]

$$\begin{bmatrix} ik \\ l \end{bmatrix} = \tfrac{1}{2} \left(\frac{\partial a_{il}}{\partial x_k} + \frac{\partial a_{kl}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_l} \right),$$

this becomes

$$\sum_i a_{il} x''_i + \sum_{ik} \begin{bmatrix} ik \\ l \end{bmatrix} x'_i x'_k = \frac{\partial L}{\partial x_l} - \sum_{ik} \frac{\partial L}{\partial x_k} a_{il} x'_i x'_k \qquad (l = 1, 2, \ldots, n).$$

To solve these equations for x'', we multiply by A_{ml}, the minor of a_{ml} in the determinant $a = | a_{\lambda\mu} |$ divided by a itself, and sum with respect to l. From the properties of the determinant $| a_{\lambda\mu} |$ we have

$$(14) \qquad \sum_l a_{il} A_{ml} = 0, \quad \text{if } i \neq m; \qquad \sum_l a_{il} A_{il} = 1.$$

Our equations then become

$$(15) \qquad x''_m + \sum_{ik} \begin{Bmatrix} ik \\ m \end{Bmatrix} x'_i x'_k = \sum_l \frac{\partial L}{\partial x_l} (A_{ml} - x'_m x'^{\,l}), \quad (m = 1, 2, \ldots, n),$$

where we have introduced the Christoffel symbol of the second kind

$$\begin{Bmatrix} ik \\ m \end{Bmatrix} = \sum_l A_{ml} \begin{bmatrix} ik \\ l \end{bmatrix}.$$

Equations (15) reduce immediately to equations (13) if we place

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[9] Bianchi, *Geometria differenziale*, vol. 1, chapt. II.

$$(16)\qquad A_{ml} = 0, \text{ if } l \neq m; \qquad A_{mm} = 1, \qquad (l, m = 1, 2, \ldots, n).$$

Thus *any natural family of curves is represented by differential equations of the form* (13) *for euclidean space and of the form* (15) *for any space. Here L is an arbitrary point function.* From these equations it is evident that, given a point x_i of V_n and a direction ξ_i through this point, one and only one curve of a given natural family of curves is determined. There are thus ∞^{n-1} curves passing through a point and a totality of $\infty^{2(n-1)}$ curves composing the family.

If the function F is constant, our problem reduces to finding the curves for which

$$(17)\qquad \int ds = \mathit{minimum},$$

and equations (15) become

$$(18)\qquad x_m'' + \sum_{ik} \left\{ {ik \atop m} \right\} x_i' x_k' = 0, \quad (m = 1, 2, \ldots, n)$$

the well-known equations of the *geodesics* in any space.[10]

§6. **Proof of the direct or Lipschitz theorem.** Darboux[11] proves this by noting that the determination of the *geodesics* in any space V_n, where

$$ds^2 = \sum_{ik} F^2 a_{ik}\, dx_i\, dx_k,$$

leads to a minimizing of

$$\int \sqrt{\sum_{ik} F^2 a_{ik}\, dx_i\, dx_k} = \int F \sqrt{\sum_{ik} a_{ik}\, dx_i\, dx_k}.$$

If, now, the *geodesics* in the space V_n are *conformally* represented in a space V'_n in which

$$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,$$

the representing curves will evidently form a *natural* family in V'_n. Since conformal representation leaves angles unchanged, the orthogonal property expressed by Beltrami's theorem for the geodesics will be immediately carried over into the orthogonal property expressed by Lipschitz's theorem for a natural family. Further, since ds is transformed into $F\,ds$, the hypersurfaces which are the loci of equal arc lengths are transformed into the hypersurfaces which are the loci of equal actions.

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[10] Bianchi, ibid., vol. 1, p. 334.
[11] Darboux, ibid., vol. 2, p. 510.

We shall now give a more direct proof of the Lipschitz theorem without using the Beltrami Theorem as a basis.

Consider the equations of the natural family given by

$$(10)\qquad \frac{d}{ds}\sum_i F a_{il}\frac{dx_i}{ds} = \frac{1}{2F}\sum_{ik}\frac{\partial(F^2 a_{ik})}{\partial x_l}\frac{dx_i}{ds}\frac{dx_k}{ds}, \qquad (l = 1, 2, \ldots, n).$$

Let the parameter curves x_1 be curves of the natural type, and let the parameter x_1 represent the action along these curves measured from the corresponding intersection with a definite orthogonal parameter hypersurface of the system $x_1 =$ constant, e. g. $x_1 = 0$. Thus

$$x_1 = \int F_1\, ds_1 = \int F_1 \sqrt{a_{11}}\, dx_1,$$

since, along the x_1 parameter curves, we have by (2)

$$ds_1 = \sqrt{a_{11}}\, dx_1,$$

so that

$$F_1\sqrt{a_{11}} = 1.$$

Applying equations (10) to our x_1 parameter curves (along which $x_2, x_3, \ldots, x_n$ are all constants), we have

$$(19)\qquad \frac{\partial}{\partial x_1}\left(\frac{a_{1l}}{a_{11}}\right) = 0, \quad (l = 2, 3, \ldots, n).$$

Since the parameter curves x_1 are normal to the hypersurface $x_1 = 0$, we have from (6) for the angle between the parameter curve x_1 and any other parameter curve x_l,

$$\cos\omega_{1l} = \frac{a_{1l}}{\sqrt{a_{11}}\sqrt{a_{ll}}} = 0, \text{ for } x_1 = 0$$

or

$$\frac{a_{1l}}{a_{11}} = 0, \quad (l = 2, 3, \ldots, n), \text{ for } x_1 = 0.$$

But as shown by (19), a_{1l}/a_{11} is independent of x_1, so that

$$\frac{a_{1l}}{a_{11}} = 0, \quad (l = 2, 3, \ldots, n), \text{ for all values of } x_1$$

and hence the curves x_1 are normal to all hypersurfaces $x_1 =$ constant, and they thus admit of ∞^1 normal hypersurfaces, i. e. they form a normal hypercongruence; the hypersurfaces appear as surfaces of equal action.

§7. **Conditions for a normal hypercongruence.** Consider first a *euclidean space of four dimensions*, S_4. We wish to derive the conditions that a system of ∞^3 curves in S_4 are such that there exists a system of ∞^1 hypersurfaces (V_3) which are orthogonal to the curves. We write the equations of the ∞^3 curves in the form

$$(20)\quad X = X(u, v, w, t), \quad Y = Y(u, v, w, t), \quad Z = Z(u, v, w, t), \quad R = R(u, v, w, t)$$

where u, v, w are three arbitrary parameters whose particular values u_0, v_0, w_0 determine uniquely a curve C_0, while the variable t determines the points on this curve.

We shall now assume that there is a hypersurface H which is orthogonal to the curves; let this be determined by placing

$$(21)\qquad t = t(u, v, w)$$

in equations (20). Through a point (X, Y, Z, R) of such a hypersurface, determined by the values $u = u_0, v = v_0, w = w_0$, there passes the curve

$$(22)\quad X = X(u_0, v_0, w_0, t), \quad Y = Y(u_0\, v_0, w_0, t), \quad Z = Z(u_0, v_0, w_0, t), \quad R = R(u_0, v_0, w_0, t)$$

and the direction-cosines of its tangent are proportional to

$$\frac{\partial X}{dt}, \quad \frac{\partial Y}{\partial t}, \quad \frac{\partial Z}{\partial t}, \quad \frac{\partial R}{\partial t}.$$

Since this direction is perpendicular to our surface, we must have

$$(23)\qquad \frac{\partial X}{\partial t}\,dX + \frac{\partial Y}{\partial t}\,dY + \frac{\partial Z}{\partial t}\,dZ + \frac{\partial R}{\partial t}\,dR = 0,$$

where dX, dY, dZ, dR are computed from (20) and t is replaced by its value from (21). Now

$$dX = \frac{\partial X}{\partial u}\,du + \frac{\partial X}{\partial v}\,dv + \frac{\partial X}{\partial w}\,dw + \frac{\partial X}{\partial t}\,dt,$$

and similarly for dY, dZ, and dR. Introducing these values in (23), we get

$$T\,dt + U\,du + V\,dv + W\,dw = 0 \tag{24}$$

where

$$T = \Sigma\left(\frac{\partial X}{\partial t}\right)^2,\quad U = \Sigma\,\frac{\partial X}{\partial t}\,\frac{\partial X}{\partial u},\quad V = \Sigma\,\frac{\partial X}{\partial t}\,\frac{\partial X}{\partial v},\quad W = \Sigma\,\frac{\partial X}{\partial t}\,\frac{\partial X}{\partial w}, \tag{25}$$

the summation extending over X, Y, Z, and R. The value of t must therefore be a function of u, v, w which satisfies equation (24). In order that we may have a system of hypersurfaces which are orthogonal to these curves, it is necessary and sufficient that the conditions for integrability of equation (24) be satisfied for all values of t, u, v, w. These conditions are

$$\begin{aligned}
&T\left(\frac{\partial U}{\partial v} - \frac{\partial V}{\partial u}\right) + U\left(\frac{\partial V}{\partial t} - \frac{\partial T}{\partial v}\right) + V\left(\frac{\partial T}{\partial u} - \frac{\partial U}{\partial t}\right) = 0\\
&T\left(\frac{\partial V}{\partial w} - \frac{\partial W}{\partial v}\right) + V\left(\frac{\partial W}{\partial t} - \frac{\partial T}{\partial w}\right) + W\left(\frac{\partial T}{\partial v} - \frac{\partial V}{\partial t}\right) = 0\\
&T\left(\frac{\partial W}{\partial u} - \frac{\partial U}{\partial w}\right) + W\left(\frac{\partial U}{\partial t} - \frac{\partial T}{\partial u}\right) + U\left(\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t}\right) = 0,
\end{aligned} \tag{26}$$

where the second and third of these conditions may be derived from the first by a cyclical interchange of U, V, W and of u, v, w.

For a *euclidean space of n dimensions*, S_n, the equations of the ∞^{n-1} curves may be written

$$X_i = X_i\,(u_1, u_2, \ldots, u_{n-1}, t), \quad (i = 1, 2, \ldots, n). \tag{20'}$$

The orthogonal hypersurface H is determined by placing

$$t = t\,(u_1, u_2, \ldots, u_{n-1}) \tag{21'}$$

in (20′), and through a point $X_i^{(0)}$ of such a hypersurface, there passes the curve

$$X_i = X_i\,(u_1^{(0)}, u_2^{(0)}, \ldots, u_{n-1}^{(0)}, t), \quad (i = 1, 2, \ldots n), \tag{22'}$$

and the direction-cosines of its tangent are proportional to

$$\frac{\partial X_i}{\partial t}, (i = 1, 2, \ldots, n).$$

Since this direction is perpendicular to our surface, we must have

$$\sum_i \frac{\partial X_i}{\partial t} dX_i = 0. \tag{23'}$$

where dX_i is computed from (20′) and t is replaced by its value from (21′). Now

$$dX_i = \sum_{r=1}^{n-1} \frac{\partial X_i}{\partial u_r} du_r + \frac{\partial X_i}{\partial t} dt, \quad (i = 1, 2, \ldots, n).$$

Introducing these values in (23′), we get

$$T\,dt + U_1\,du_1 + U_2\,du_2 + \ldots + U_{n-1}\,du_{n-1} = 0, \tag{24'}$$

where

$$T = \sum_i \left(\frac{\partial X_i}{\partial t}\right)^2, \qquad U_r = \sum_i \frac{\partial X_i}{\partial t}\frac{\partial X_i}{\partial u_r}, \quad (r = 1, 2, \ldots, n-1). \tag{25'}$$

The conditions of integrability of equation (24′) are

$$T\left(\frac{\partial U_k}{\partial u_r} - \frac{\partial U_r}{\partial u_k}\right) + U_k\left(\frac{\partial U_r}{\partial t} - \frac{\partial T}{\partial u_r}\right) + U_r\left(\frac{\partial T}{\partial u_k} - \frac{\partial U_k}{\partial t}\right) = 0, \tag{26'}$$

$$(r, k = 1, 2, \ldots, n-1).$$

Equations (26′) *give* $\frac{(n-1)(n-2)}{2}$ *independent conditions; these must be satisfied if our system of curves is to form a normal hypercongruence.*

Equations (26′) are also the conditions for a normal hypercongruence in *any space of n dimensions*, V_n, if we note that the condition (23′) for perpendicularity is replaced by

$$\sum_{ik} a_{ik} \frac{\partial X_i}{\partial t} dX_k = 0, \tag{23''}$$

so that equations (25′) are replaced by

$$T = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t}, \quad U_r = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial u_r}, \quad (r=1,2,\ldots,n-1). \tag{25''}$$

§8. **Analytic statement of the converse theorem.** Consider an arbitrary system of $\infty^{2(n-1)}$ curves in a euclidean space of n dimensions, assuming that one curve passes through each point in each direction. Such a system may be defined by a set of n differential equations of the second order—

$$(27)\quad x_i'' = F_i(x_1, x_2, \ldots, x_n;\quad x_1', x_2', \ldots x_n'),\quad (i = 1, 2, \ldots, n),$$

where $x_i' = \dfrac{dx_i}{ds}, x_i'' = \dfrac{d^2x_i}{ds^2}$, and the arc length s is chosen as parameter along the curves, so that

$$(28)\qquad \sum_i (x_i')^2 = 1.$$

Here the F_i are uniform functions, analytic in the $2n$ arguments. *Our problem is to find the form of the function F_i so that the ∞^{n-1} curves of the system which pass out orthogonally to any hypersurface in the space, should form a normal hypercongruence.*

Denoting the initial values of x_i, x_i', which may be taken at random, subject to (28), by x_i, p_i respectively, and employing $X_1, X_2, \ldots, X_n$ as current coördinates, we may write the solution of (27) in the form

$$(29)\quad X_i = x_i + p_i(S - s) + \tfrac{1}{2} F_i (S - s)^2 + \tfrac{1}{6} M_i (S - s)^3 + \ldots,\quad (i = 1, 2, \ldots, n),$$

$$(30)\qquad \sum_i p_i^2 = 1.$$

Here the F_i are expressed as functions of x_i, p_i, and the M_i found by differentiating (27) are given by

$$(31)\qquad M_i = \sum_l \left(\frac{\partial F_i}{\partial x_l} p_l + \frac{\partial F_i}{\partial p_l} F_l\right),\quad (i = 1, 2, \ldots, n).$$

Consider, now, an arbitrary hypersurface

$$(32)\qquad x_i = f_i(u_1, u_2, \ldots, u_{n-1}),\quad (i = 1, 2, \ldots, n).$$

At each point of the surface and normal to it a *definite* curve of the given system (27) may be constructed. A certain hypercongruence will thus be determined. We wish to express the conditions that this shall be of the normal type. If the direction of the normal at any point of the surface is given by $P_i(u_1, u_2, \ldots, u_{n-1})$, $(i = 1, 2, \ldots, n)$, then these P_i are determined by the equations

$$(33)\qquad \sum_i P_i^2 = 1;\quad \sum_i P_i \frac{\partial f_i}{\partial u_r} = 0,\quad (r = 1, 2, \ldots, n - 1).$$

The second of these conditions merely expresses the fact that the normal is perpendicular to the parameter curves of the surface (32). By differentiating (33) we have the further relations

$$(34)\qquad \sum_i P_i \frac{\partial P_i}{\partial u_r} = 0,\ (r = 1, 2, \ldots, n-1);\ \sum_i \frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r} = \sum_i \frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k},$$
$$(r, k = 1, 2, \ldots, n-1).$$

We may now identify the normal at any point of the surface with the tangent line to the curve, so that

$$(35)\qquad p_i = P_i\,(u_1, u_2, \ldots, u_{n-1}),\quad (i = 1, 2, \ldots, n).$$

The equations of the ∞^{n-1} curves corresponding to the given initial conditions may now be written

$$(36)\qquad X_i = f_i + P_i t + \tfrac{1}{2}\overline{F}_i t^2 + \tfrac{1}{6}\overline{M}_i t^3 + \ldots,\quad (i = 1, 2, \ldots, n),$$

where t replaces $S - s$, and where, e. g.,

$$(37)\qquad \overline{F}_i\,(u_1, u_2, \ldots, u_{n-1}) = F_i\,(f_i, f_2, \ldots, f_n;\quad P_1,\quad P_2, \ldots, P_n).$$

The coefficients of the powers of t in (36) are thus functions of $(n-1)$ parameters $u_1, u_2, \ldots, u_{n-1}$, so that the equations of the ∞^{n-1} curves are expressed in the form (20′). In order that the system (36) form a normal hypercongruence, equations (26′) must be satisfied.

Applying (25′) and (26′) to (36), and using the conditions (33) and (34), we get

$$\frac{\partial X_i}{\partial t} = P_i + \overline{F}_i t + \ldots;\quad \frac{\partial X_i}{\partial u_r} = \frac{\partial f_i}{\partial u_r} + t\frac{\partial P_i}{\partial u_r} + \ldots;\quad \frac{\partial X_i}{\partial u_k} = \frac{\partial f_i}{\partial u_k} + t\frac{\partial P_i}{\partial u_k} + \ldots$$

$$T = \sum_i P_i^2 + 2t\sum_i P_i F_i + \ldots = 1 + 2t\sum_i P_i F_i + \ldots,$$

$$U_r = \sum_i P_i\frac{\partial f_i}{\partial u_r} + t\left(\sum_i P_i\frac{\partial P_i}{\partial u_r} + \sum_i \overline{F}_i\frac{\partial f_i}{\partial u_r}\right) + \ldots = t\sum_i \overline{F}_i\frac{\partial f_i}{\partial u_r} + \ldots,$$

$$U_k = \sum_i P_i\frac{\partial f_i}{\partial u_k} + t\left(\sum_i P_i\frac{\partial P_i}{\partial u_k} + \sum_i F_i\frac{\partial f_i}{\partial u_k}\right) + \ldots = t\sum_i F_i\frac{\partial f_i}{\partial u_k} + \ldots$$

$$\frac{\partial T}{\partial u_r} = 2t\left(\sum_i P_i\frac{\partial f_i}{\partial u_r} + \sum_i \overline{F}_i\frac{\partial P_i}{\partial u_r}\right) + \ldots;\ \frac{\partial T}{\partial u_k} = 2t\left(\sum_i P_i\frac{\partial f_i}{\partial u_k} + \sum_i \overline{F}_i\frac{\partial P_i}{\partial u_k}\right) + \ldots$$

$$\frac{\partial U_r}{\partial t} = \sum_i F_i \frac{\partial f_i}{\partial u_r} + \ldots; \quad \frac{\partial U_r}{\partial u_k} = t\left(\sum_i \overline{F}_i \frac{\partial^2 f_i}{\partial u_r \partial u_k} + \sum_i \frac{\partial \overline{F}_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) + \ldots.$$

$$\frac{\partial U_k}{dt} = \sum_i \overline{F}_i \frac{\partial f_i}{\partial u_k} + \ldots; \quad \frac{\partial U_k}{\partial u_r} = t\left(\sum_i \overline{F}_i \frac{\partial^2 f_i}{\partial u_k \partial u_r} + \sum_i \frac{\partial \overline{F}_i}{\partial u_r}\frac{\partial f_i}{\partial u_k}\right) + \ldots.$$

Substituting in (26′) we shall get our condition in the form

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \ldots\ldots\ldots = 0. \tag{38}$$

Since this must vanish independent of t, every coefficient, α, must vanish. We find that α_0 vanishes identically, which is as it should be since the hypercongruence is normal to the initial surface $t = 0$ by hypothesis. Let us first discuss the condition that α_1 vanish. This gives

$$\sum_i \frac{\partial \overline{F}_i}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \sum_i \frac{\partial \overline{F}_i}{\partial u_k}\frac{\partial f_i}{\partial u_r} = 0, \quad (r, k = 1, 2, \ldots, n-1). \tag{39}$$

Now by differentiating (37), we get

$$\frac{\partial \overline{F}_i}{\partial u_r} = \sum_l \frac{\partial F_i}{\partial x_l}\frac{\partial f_l}{\partial u_r} + \sum_l \frac{\partial F_i}{\partial p_l}\frac{\partial P_l}{\partial u_r}$$

$$\frac{\partial \overline{F}_i}{\partial u_k} = \sum_l \frac{\partial F_i}{\partial x_l}\frac{\partial f_l}{\partial u_k} + \sum_l \frac{\partial F_i}{\partial p_l}\frac{\partial P_l}{\partial u_k} \tag{40}$$

and (39) becomes

$$\sum_{il} \frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_i}{\partial u_r} - \frac{\partial f_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) + \sum_{il} \frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) = 0, \tag{41}$$
$$(r, k = 1, 2, \ldots, n-1).$$

These $\frac{(n-1)(n-2)}{2}$ *conditions are therefore necessary conditions in order that the* ∞^{n-1} *curves belonging to the system* (27) *and orthogonal to the hypersurface* (32) *shall form a normal hypercongruence.*

Our problem is to find the forms of the functions F_i in order that this condition of orthogonality should hold for *every* hypersurface (32) taken as a base. Equations (41) must then necessarily hold for n arbitrary functions f_i ($i = 1, 2, \ldots, n$). These functions may be so chosen that for any assigned values of $u_1, u_2, \ldots, u_{n-1}$, the quantities

f_i and all partial derivatives of all orders of f_i with respect to the u's, shall take on *arbitrary* numerical values, or so that the quantities f_i, P_i, $\frac{\partial f_i}{\partial u_r}$, $\frac{\partial P_i}{\partial u_r}$, shall take on arbitrary numerical values, subject only to the relations (33) and (34). Before applying these conditions in a euclidean space of n dimensions, it will be of advantage to apply them to a euclidean space of four dimensions.

§9. **Proof of the converse theorem in a euclidean space of 4 dimensions.** For a space of four dimensions we may write the equations of the base hypersurface (32) in the form

$$(32') \qquad x_i = f_i\,(u, v, w), \quad (i = 1, 2, 3, 4),$$

and the six conditions (33) and (34) as

$$(33') \qquad P_1\frac{\partial f_1}{\partial u} + P_2\frac{\partial f_2}{\partial u} + P_3\frac{\partial f_3}{\partial u} + P_4\frac{\partial f_4}{\partial u} = 0, \quad (u \to v \to w)^{12}$$

$$(34') \quad \left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right) + \left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right)$$
$$+ \left(\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}\right) = 0, \quad (u \to v \to w)^{12}$$

The three necessary conditions (41) become

$$(41') \quad \sum_{il}^{1\ldots 4}\frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial v}\frac{\partial f_i}{\partial u} - \frac{\partial f_l}{\partial u}\frac{\partial f_i}{\partial v}\right) + \sum_{il}^{1\ldots 4}\frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial v}\frac{\partial f_i}{\partial u} - \frac{\partial P_l}{\partial u}\frac{\partial f_i}{\partial v}\right) = 0,$$
$$(u \to v \to w).$$

We may evidently place each of the two summations in the left member of this equation equal to zero.

Expanding the first summation in (41′) we may evidently group the terms into the following eighteen:

$$(42) \quad \left(\frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2}\right)\left(\frac{\partial f_3}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial f_3}{\partial u}\frac{\partial f_2}{\partial v}\right) + \left(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\right)\left(\frac{\partial f_1}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial f_1}{\partial u}\frac{\partial f_3}{\partial v}\right)$$
$$+ \left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial f_2}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}\right)\left(\frac{\partial f_1}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial f_1}{\partial u}\frac{\partial f_4}{\partial v}\right)$$

---

[12] We shall use the symbol $(u \to v \to w)$ to mean that similar relations may be written by a cyclical interchange of u, v, w.

$$+\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)\left(\frac{\partial f_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial f_2}{\partial u}\frac{\partial f_4}{\partial v}\right)+\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)\left(\frac{\partial f_3}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial f_3}{\partial u}\frac{\partial f_4}{\partial v}\right)$$
$$=0,\ (u\rightarrow v\rightarrow w)$$

To eliminate the f's we proceed as follows. Solve (33′) for the ratios of the p's, thus

(43) $p_1 : p_2 : p_3 : p_4 =$

$$-\begin{vmatrix}\frac{\partial f_4}{\partial u}&\frac{\partial f_2}{\partial u}&\frac{\partial f_3}{\partial u}\\ \frac{\partial f_4}{\partial v}&\frac{\partial f_2}{\partial v}&\frac{\partial f_3}{\partial v}\\ \frac{\partial f_4}{\partial w}&\frac{\partial f_2}{\partial w}&\frac{\partial f_3}{\partial w}\end{vmatrix} : -\begin{vmatrix}\frac{\partial f_1}{\partial u}&\frac{\partial f_4}{\partial u}&\frac{\partial f_3}{\partial u}\\ \frac{\partial f_1}{\partial v}&\frac{\partial f_4}{\partial v}&\frac{\partial f_3}{\partial v}\\ \frac{\partial f_1}{\partial w}&\frac{\partial f_4}{\partial w}&\frac{\partial f_3}{\partial w}\end{vmatrix} : -\begin{vmatrix}\frac{\partial f_1}{\partial u}&\frac{\partial f_2}{\partial u}&\frac{\partial f_4}{\partial u}\\ \frac{\partial f_1}{\partial v}&\frac{\partial f_2}{\partial v}&\frac{\partial f_4}{\partial v}\\ \frac{\partial f_1}{\partial w}&\frac{\partial f_2}{\partial w}&\frac{\partial f_4}{\partial w}\end{vmatrix} : \begin{vmatrix}\frac{\partial f_1}{\partial u}&\frac{\partial f_2}{\partial u}&\frac{\partial f_3}{\partial u}\\ \frac{\partial f_1}{\partial v}&\frac{\partial f_2}{\partial v}&\frac{\partial f_3}{\partial v}\\ \frac{\partial f_1}{\partial w}&\frac{\partial f_2}{\partial w}&\frac{\partial f_3}{\partial w}\end{vmatrix}$$

Multiply the three equations of (42) by $\frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial u}, \frac{\partial f_1}{\partial v}$, respectively, and add, remembering the expansions of the determinants (43) for the p's in terms of minors of the second order. We get

$$p_4\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_3\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)-p_2\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0.$$

Similarly, multiply the three equations (42) by $\frac{\partial f_2}{\partial w}, \frac{\partial f_2}{\partial u}, \frac{\partial f_2}{\partial v}$, respectively, by $\frac{\partial f_3}{\partial w}, \frac{\partial f_3}{\partial u}, \frac{\partial f_3}{\partial v}$, respectively, and by $\frac{\partial f_4}{\partial w}, \frac{\partial f_4}{\partial u}, \frac{\partial f_4}{\partial v}$, respectively, and add. We thus get the four conditions

$$(44)\quad\left\{\begin{aligned}
&p_4\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_3\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)-p_2\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0\\
&p_4\left(\frac{\partial F_3}{\partial x_1}-\frac{\partial F_1}{\partial x_3}\right)-p_3\left(\frac{\partial F_4}{\partial x_1}-\frac{\partial F_1}{\partial x_4}\right)+p_1\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0\\
&p_4\left(\frac{\partial F_1}{\partial x_2}-\frac{\partial F_2}{\partial x_1}\right)+p_2\left(\frac{\partial F_4}{\partial x_1}-\frac{\partial F_1}{\partial x_4}\right)-p_1\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)=0\\
&p_1\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_2\left(\frac{\partial F_3}{\partial x_1}-\frac{\partial F_1}{\partial x_3}\right)+p_3\left(\frac{\partial F_1}{\partial x_2}-\frac{\partial F_2}{\partial x_1}\right)=0
\end{aligned}\right.$$

Now, expanding the second summation in (41′), we get the following 48 terms:

$$\begin{aligned}
&\frac{\partial F_1}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right)+\frac{\partial F_1}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_1}{\partial v}\right)\\
&+\frac{\partial F_1}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_1}{\partial v}\right)+\frac{\partial F_1}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_1}{\partial v}\right)\\
&+\frac{\partial F_2}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right)+\frac{\partial F_2}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_2}{\partial v}\right)\\
&+\frac{\partial F_2}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_2}{\partial v}\right)+\frac{\partial F_2}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_2}{\partial v}\right)\\
&+\frac{\partial F_3}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right)+\frac{\partial F_3}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
&+\frac{\partial F_3}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_3}{\partial v}\right)+\frac{\partial F_3}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
&+\frac{\partial F_4}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}\right)+\frac{\partial F_4}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_4}{\partial v}\right)\\
&+\frac{\partial F_4}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right)+\frac{\partial F_4}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_4}{\partial v}\right),\\
&\qquad\qquad\qquad\qquad\qquad\qquad(u \rightarrow v \rightarrow w).
\end{aligned}$$

We may reduce the number of terms by eliminating the multipliers of $\frac{\partial F_1}{\partial p_2}, \frac{\partial F_2}{\partial p_3}, \frac{\partial F_3}{\partial p_4}, \frac{\partial F_4}{\partial p_1}$. Thus to eliminate the coefficient of $\frac{\partial F_1}{\partial p_2}$, we solve (33′) for

$$\begin{aligned}
\frac{\partial f_1}{\partial u}&=-\frac{p_2}{p_1}\frac{\partial f_2}{\partial u}-\frac{p_3}{p_1}\frac{\partial f_3}{\partial u}-\frac{p_4}{p_1}\frac{\partial f_4}{\partial u},\\
\frac{\partial f_1}{\partial v}&=-\frac{p_2}{p_1}\frac{\partial f_2}{\partial v}-\frac{p_3}{p_1}\frac{\partial f_3}{\partial v}-\frac{p_4}{p_1}\frac{\partial f_4}{\partial v},
\end{aligned}$$

and thus get

$$\begin{aligned}
\frac{\partial P_2}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_1}{\partial v}=-\frac{p_2}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right)-\frac{p_3}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
-\frac{p_4}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right).
\end{aligned}$$

If we further eliminate

$$\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}$$

by means of (34′), the above 48 terms reduce to 33, viz.:

$$
\begin{aligned}
(45)\quad & \left[\frac{\partial F_1}{\partial p_3} - \frac{p_1}{p_2}\frac{\partial F_2}{\partial p_3}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left[\frac{\partial F_2}{\partial p_4} - \frac{p_2}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_4}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial p_1} - \frac{p_3}{p_4}\frac{\partial F_4}{\partial p_1}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_3}{\partial v}\right) + \left[\frac{\partial F_4}{\partial p_2} - \frac{p_4}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right) \\
& + \left[\frac{\partial F_1}{\partial p_4} - \frac{p_1}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_4}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left[\frac{\partial F_2}{\partial p_1} - \frac{p_2}{p_4}\frac{\partial F_4}{\partial p_1}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial p_2} - \frac{p_3}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right) + \left[\frac{\partial F_4}{\partial p_3} - \frac{p_4}{p_2}\frac{\partial F_2}{\partial p_3}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_4}{\partial v}\right) \\
& + \left[\frac{\partial F_1}{\partial p_1} - \frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right) \\
& + \left[\frac{\partial F_2}{\partial p_2} - \frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial f_3} - \frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right) = 0, \\
& \hfill (u \rightarrow v \rightarrow w).
\end{aligned}
$$

The coefficients in parentheses in (45) may now be taken as independent quantities, and for (45) to vanish identically, we must have the vanishing of all the expressions in brackets. Hence, the first 8 terms give

$$
\begin{aligned}
& p_2\frac{\partial F_1}{\partial p_3} - p_1\frac{\partial F_2}{\partial p_3} = 0; \quad p_3\frac{\partial F_2}{\partial p_4} - p_2\frac{\partial F_3}{\partial p_4} = 0; \quad p_4\frac{\partial F_3}{\partial p_1} - p_3\frac{\partial F_4}{\partial p_1} = 0; \\
& p_1\frac{\partial F_4}{\partial p_2} - p_4\frac{\partial F_1}{\partial p_2} = 0; \quad p_3\frac{\partial F_1}{\partial p_4} - p_1\frac{\partial F_3}{\partial p_4} = 0; \quad p_4\frac{\partial F_2}{\partial p_1} - p_2\frac{\partial F_4}{\partial p_1} = 0; \\
& p_1\frac{\partial F_3}{\partial p_2} - p_3\frac{\partial F_1}{\partial p_2} = 0; \quad p_2\frac{\partial F_4}{\partial p_3} - p_4\frac{\partial F_2}{\partial p_3} = 0.
\end{aligned}
$$

By combinations of these we get four additional relations

$$p_1\frac{\partial F_4}{\partial p_3}-p_4\frac{\partial F_1}{\partial p_3}=0;\quad p_2\frac{\partial F_1}{\partial p_4}-p_1\frac{\partial F_2}{\partial p_4}=0;\quad p_3\frac{\partial F_2}{\partial p_1}-p_2\frac{\partial F_3}{\partial p_1}=0;$$
$$p_4\frac{\partial F_3}{\partial p_2}-p_3\frac{\partial F_4}{\partial p_2}=0.$$

These 12 relations may be written in the equivalent forms

$$\frac{\partial}{\partial p_3}\left(p_2F_1-p_1F_2\right)=0;\quad \frac{\partial}{\partial p_4}\left(p_2F_1-p_1F_2\right)=0;\quad \frac{\partial}{\partial p_1}\left(p_3F_2-p_2F_3\right)=0;$$
$$\frac{\partial}{\partial p_4}\left(p_3F_2-p_2F_3\right)=0;\quad \frac{\partial}{\partial p_1}\left(p_4F_3-p_3F_4\right)=0;\quad \frac{\partial}{\partial p_2}\left(p_4F_3-p_3F_4\right)=0;$$
$$\frac{\partial}{\partial p_2}\left(p_1F_4-p_4F_1\right)=0;\quad \frac{\partial}{\partial p_3}\left(p_1F_4-p_4F_1\right)=0;\quad \frac{\partial}{\partial p_2}\left(p_3F_1-p_1F_3\right)=0;$$
$$\frac{\partial}{\partial p_4}\left(p_3F_1-p_1F_3\right)=0;\quad \frac{\partial}{\partial p_1}\left(p_4F_2-p_2F_4\right)=0;\quad \frac{\partial}{\partial p_3}\left(p_4F_2-p_2F_4\right)=0.$$

Hence

$$(46)\qquad \begin{cases} p_2F_1-p_1F_2=\alpha_{12};\ p_3F_2-p_2F_3=\alpha_{23};\ p_4F_3-p_3F_4=\alpha_{34};\\ p_1F_4-p_4F_1=\alpha_{41};\ p_3F_1-p_1F_3=\alpha_{13};\ p_4F_2-p_2F_4=\alpha_{24};\end{cases}$$

where, e. g.,

$$(47)\qquad \alpha_{12}=\alpha_{12}(p_1,p_2,x_1,x_2,x_3,x_4);\ \alpha_{23}=\alpha_{23}(p_2,p_3,x_1,x_2,x_3,x_4);\ \text{etc.}$$

The vanishing of the last three brackets in (45) gives

$$(48)\qquad \frac{\partial F_1}{\partial p_1}-\frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1}=\frac{\partial F_2}{\partial p_2}-\frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2}=\frac{\partial F_3}{\partial p_3}-\frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3}=\frac{\partial F_4}{\partial p_4}-\frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}.$$

These are conditions on the forms of the α's appearing in (46). To find these conditions we may proceed as follows: in

$$\frac{\partial F_1}{\partial p_1}-\frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1}=\frac{\partial F_3}{\partial p_3}-\frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3}$$

we may replace $\frac{\partial F_4}{\partial p_1}$ by $\frac{p_4}{p_2}\frac{\partial F_2}{\partial p_1}$, and get

$$p_2\frac{\partial F_1}{\partial p_1}-p_1\frac{\partial F_2}{\partial p_1}=p_2\frac{\partial F_3}{\partial p_3}-p_3\frac{\partial F_2}{\partial p_3}$$

Adding F_2 to both members of this equation, we evidently have

$$\frac{\partial}{dp_1}(p_2F_1 - p_1F_2) = \frac{\partial}{\partial p_3}(p_2F_3 - p_3F_2)$$

or

$$\frac{\partial \alpha_{12}}{\partial p_1} + \frac{\partial \alpha_{23}}{\partial p_3} = 0.$$

By similar procedure we finally get the conditions

$$(49)\quad \left\{\begin{array}{lll} \dfrac{\partial \alpha_{12}}{\partial p_1} + \dfrac{\partial \alpha_{23}}{\partial p_3} = 0; & \dfrac{\partial \alpha_{23}}{\partial p_2} + \dfrac{\partial \alpha_{34}}{\partial p_4} = 0; & \dfrac{\partial \alpha_{34}}{\partial p_3} + \dfrac{\partial \alpha_{41}}{\partial p_1} = 0; \\ \dfrac{\partial \alpha_{41}}{\partial p_4} + \dfrac{\partial \alpha_{12}}{\partial p_2} = 0; & \dfrac{\partial \alpha_{34}}{\partial p_4} + \dfrac{\partial \alpha_{13}}{\partial p_1} = 0; & \dfrac{\partial \alpha_{13}}{\partial p_3} - \dfrac{\partial \alpha_{12}}{\partial p_2} = 0; \\ \dfrac{\partial \alpha_{41}}{\partial p_1} + \dfrac{\partial \alpha_{24}}{\partial p_2} = 0; & \dfrac{\partial \alpha_{24}}{\partial p_4} - \dfrac{\partial \alpha_{23}}{\partial p_3} = 0. & \end{array}\right.$$

Hence the α's are given by

$$(50)\quad \left\{\begin{array}{l} \alpha_{12} = p_2\phi_1 - p_1\phi_2;\ \alpha_{23} = p_3\phi_2 - p_2\phi_3;\ \alpha_{34} = p_4\phi_3 - p_3\phi_4; \\ \alpha_{41} = p_1\phi_4 - p_4\phi_1;\ \alpha_{13} = p_3\phi_1 - p_1\phi_3;\ \alpha_{24} = p_4\phi_2 - p_2\phi_4; \end{array}\right.$$

where the ϕ's are arbitrary functions of the coördinates x_1, x_2, x_3, x_4 only. Combining (50) and (46), we must have

$$(51)\qquad \frac{F_1 - \phi_1}{p_1} = \frac{F_2 - \phi_2}{p_2} = \frac{F_3 - \phi_3}{p_3} = \frac{F_4 - \phi_4}{p_4}$$

Letting K stand for these equal ratios, (51) may be written

$$(52)\qquad F_i = \phi_i + p_iK, \quad (i = 1, 2, 3, 4).$$

Multiplying by p_i and summing with respect to i, we get

$$\sum_i p_iF_i = \sum_i p_i\phi_i + K\sum_i p_i^2,$$

and remembering that

$$\sum_i (x_i')^2 = 1, \quad \sum_i x_i' x_i'' = 0$$

or

$$\sum_i p_i^2 = 1, \quad \sum_i p_iF_i = 0,$$

we have

$$K = -\sum_i p_i \phi_i,$$

and (52) becomes

$$(53) \qquad F_i = \phi_i - p_i \sum_k p_k \phi_k, \quad (i = 1, 2, 3, 4).$$

We still have to satisfy equations (44). Substituting the values of the F's as given by (52) into (44), these reduce to

$$(54) \quad \begin{cases} p_4\left(\dfrac{\partial\phi_2}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_2}\right) + p_3\left(\dfrac{\partial\phi_4}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_4}\right) - p_2\left(\dfrac{\partial\phi_4}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_4}\right) = 0 \\ p_4\left(\dfrac{\partial\phi_3}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_3}\right) - p_3\left(\dfrac{\partial\phi_4}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_4}\right) + p_1\left(\dfrac{\partial\phi_4}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_4}\right) = 0 \\ p_4\left(\dfrac{\partial\phi_1}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_1}\right) + p_2\left(\dfrac{\partial\phi_4}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_4}\right) - p_1\left(\dfrac{\partial\phi_4}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_4}\right) = 0 \\ p_1\left(\dfrac{\partial\phi_2}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_2}\right) + p_2\left(\dfrac{\partial\phi_3}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_3}\right) + p_3\left(\dfrac{\partial\phi_1}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_1}\right) = 0 \end{cases}$$

Since these equations are to hold for arbitrary values of the p's, (the only condition being $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$), and since the expressions in parentheses are independent of the p's, we must have

$$(55) \quad \begin{cases} \dfrac{\partial\phi_1}{\partial x_2} = \dfrac{\partial\phi_2}{\partial x_1}; \quad \dfrac{\partial\phi_2}{\partial x_3} = \dfrac{\partial\phi_3}{\partial x_2}; \quad \dfrac{\partial\phi_3}{\partial x_4} = \dfrac{\partial\phi_4}{\partial x_3}; \\ \dfrac{\partial\phi_4}{\partial x_1} = \dfrac{\partial\phi_1}{\partial x_4}; \quad \dfrac{\partial\phi_1}{\partial x_3} = \dfrac{\partial\phi_3}{\partial x_1}; \quad \dfrac{\partial\phi_2}{\partial x_4} = \dfrac{\partial\phi_4}{\partial x_2}. \end{cases}$$

Hence, the ϕ's are partial derivatives of a single function $L(x_1, x_2, x_3, x_4)$, i. e.

$$(56) \qquad \phi_1 = \frac{\partial L}{\partial x_1}; \quad \phi_2 = \frac{\partial L}{\partial x_2}; \quad \phi_3 = \frac{\partial L}{\partial x_3}; \quad \phi_4 = \frac{\partial L}{\partial x_4}.$$

Introducing these values into (53), we finally get

$$(57) \qquad x_i'' = \frac{\partial L}{\partial x_i} - x_i' \sum_k \frac{\partial L}{\partial x_k} x_k', \quad (i = 1, 2, 3, 4),$$

and on comparison with (13), we note that these are the differential equations of our natural family in a euclidean space of four dimensions. We may therefore state the converse

THEOREM. *If a system of ∞^6 curves (one passing through each point in each direction) in a euclidean space of four dimensions is such that those ∞^3 curves of the system which meet an arbitrary hypersurface (space of three dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. In our discussion we have only considered the vanishing of α_1 in the condition (38) for a normal hypercongruence. We may therefore state a much stronger converse.

THEOREM. *If a system of ∞^6 curves in a euclidean space of four dimensions is such that those ∞^3 curves of the system which meet an arbitrary hypersurface orthogonally, always meet some infinitesimally neighboring hypersurface orthogonally, the system is of the natural type.*

As stated in the last theorem the weaker requirement of orthogonality must hold for *all* hypercongruences of the system, for a hypercongruence may meet two neighboring surfaces orthogonally (i. e. be approximately normal) without meeting ∞^1 hypersurfaces orthogonally.

§10. **Proof of the converse theorem in a euclidean space of n dimensions.** We return to our $(n-1)(n-2)/2$ conditions

$$(41)\quad \sum_{il} \frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial f_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) + \sum_{il} \frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) = 0,$$
$$(r, k = 1, 2, \ldots, n-1)$$

for a normal hypercongruence. We may evidently place each of the two summations in the left member of this equation equal to zero.

By grouping the terms in the first summation, this may be written

$$(58)\quad \sum_{il}\left(\frac{\partial F_i}{\partial x_l} - \frac{\partial F_l}{\partial x_i}\right)\left(\frac{\partial f_i}{\partial u_k}\frac{\partial f_l}{\partial u_r} - \frac{\partial f_i}{\partial u_r}\frac{\partial f_l}{\partial u_k}\right) = 0, (r, k = 1, 2, \ldots, n-1).$$

To eliminate the f's we proceed as follows. Solve the $(n-1)$ equations

$$(33)\qquad \sum_i P_i \frac{\partial f_i}{\partial u_r} = 0, \quad (r = 1, 2, \ldots, n-1)$$

for the ratios of the P's. If we use the notation

$$(59) \qquad c_{\lambda\mu} = \frac{\partial f_\lambda}{\partial u_\mu}, \qquad (c_{\lambda\mu} \pm c_{\mu\lambda})$$

equations (33) and (58) may be written

$$(60) \quad c_{1r}p_1 + c_{2r}p_2 + c_{3r}p_3 + \ldots\ldots + c_{nr}p_n = 0, (r = 1, 2, \ldots, n - 1);$$

$$(61) \qquad \sum_{il} \left(\frac{\partial F_i}{\partial x_l} - \frac{\partial F_l}{\partial x_i}\right) \begin{vmatrix} c_{ik} & c_{lk} \\ c_{ir} & c_{lr} \end{vmatrix} = 0, (r, k = 1, 2, \ldots\ldots, n - 1).$$

From the $(n - 1)$ homogeneous equations (60) we see that the p's are proportional to the determinants of the matrix

$$(62) \qquad \left\| \begin{array}{lllll} c_{11} & \ldots c_{i1} & \ldots c_{l1} & \ldots c_{m1} & \ldots c_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1k} & \ldots c_{ik} & \ldots c_{lk} & \ldots c_{mk} & \ldots c_{nk} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1r} & \ldots c_{ir} & \ldots c_{lr} & \ldots c_{mr} & \ldots c_{nr} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{1,\,n-1} & \ldots c_{i,\,n-1} & \ldots c_{l,\,n-1} & \ldots c_{m,\,n-1} & \ldots c_{n,\,n-1} \end{array} \right\|$$

so that, e. g., if we omit the column headed by c_{m1} we get

$$(63) \quad p_m : (-1)^{m+1} \begin{vmatrix} c_{11} & \ldots c_{m-1,\,2} & c_{m+1,\,1} & \ldots c_{n1} \\ c_{12} & \ldots c_{m-1,\,2} & c_{m+1,\,2} & \ldots c_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ c_{1,\,n-1} & \ldots c_{m-1,\,n-1} & c_{m+1,\,n-1} & \ldots c_{n,\,n-1} \end{vmatrix}$$

Consider two arrays, the minors of the 2*d* order and the corresponding co-minors of the $(n - 3)$ *d* order in q_m, where q_m is the determinant occurring in the above expression for p_m, so that

$$(64) \qquad p_m = (-1)^{m+1}\, q_m.$$

If we designate by $c_{ik,\,lr}$ the minor of 2*d* order,

$$(65) \qquad c_{ik,\,lr} = \begin{vmatrix} c_{ik} & c_{lk} \\ c_{ir} & c_{lr} \end{vmatrix}$$

and by $C_{ik,\,lr}$ its corresponding co-minor of the $(n-3)\,d$ order, and expand by Laplace's theorem[13] in terms of the $(n-1)\,(n-2)/2$ minors of the $2d$ order formed from the columns headed c_{i1} and c_{l1}, we get

$$(66)\quad q_m = (-1)^v\,[c_{i1,\,l2}\,C_{i1,\,l2} + c_{i1,\,l3}\,C_{i1,\,l3} + \ldots. + c_{i\,n-2,\,l\,n-1}\,C_{i\,n-2,\,l\,n-1}],$$

where ν is the sum of the columns in p_m headed by c_{i1} and c_{l1}.

From the matrix (62), we see that the determinants q_m, q_l, and q_i are formed by omitting the columns headed c_{m1}, c_{l1}, and c_{i1}, respectively, and that, therefore, the minors of the $2d$ order

$$c_{lk,\,mr} = \begin{vmatrix} c_{lk} & c_{mk} \\ c_{lr} & c_{mr} \end{vmatrix} \text{ and } c_{ik,\,mr} = \begin{vmatrix} c_{ik} & c_{mk} \\ c_{ir} & c_{mr} \end{vmatrix}$$

in the expansions for q_i and q_l, respectively, have the same co-minors of the $(n-3)\,d$ order that the minors of $2d$ order, $c_{ik,\,lr}$, have in the expansion for q_m. Designating these common co-minors by C_{12}, $C_{13}, \ldots., C_{n-2,\,n-1}$, we may now write the expansions

$$(67)\quad \begin{cases} q_m = (-1)^{i+l} & [c_{i1,\,l2}\,C_{12} + c_{i1,\,l3}\,C_{13} + \ldots. + c_{i\,n-2,\,l\,n-1}\,C_{n-2,\,n-1}] \\ q_l = (-1)^{i+(m-1)} & [c_{il,\,m2}\,C_{12} + c_{il,\,m3}\,C_{13} + \ldots. + c_{i\,n-2,\,m\,n-1}\,C_{n-2,\,n-1}] \\ q_i = (-1)^{(l-1)+(m-1)} & [c_{l1,\,m2}\,C_{12} + c_{l1,\,m3}\,C_{13} + \ldots. + c_{l\,n-2,\,m\,n-1}\,C_{n-2,\,n-1}] \end{cases}$$

On the other hand, if the minors of the $2d$ order formed from two columns which are not headed by two of the three terms c_{m1}, c_{l1}, c_{i1}, are multiplied by the above co-minors of the $(n-3)d$ order, C_{12}, C_{13}, $\ldots.$, $C_{n-2,\,n-1}$, and added, we shall have the expansion of a determinant with two columns alike, and hence the sum will be equal to zero.

Let us apply this discussion to the equations (61), involving the minors of the $2d$ order. If we expand these equations, and multiply the $(n-1)\,(n-2)/2$ equations by the above $(n-1)\,(n-2)/2$ co-minors of the $(n-3)\,d$ order, $C_{12}, C_{13}, \ldots., C_{n-2,\,n-1}$, in succession, and then add, we shall evidently get *three* terms only — all the sums will vanish except those giving q_m, q_l, and q_i. Equations (61) thus assume the form

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[13] See R. F. Scott, A treatise on the theory of determinants, 1880, chapt. III.

$$(68)\quad \left(\frac{\partial F_i}{\partial x_l}-\frac{\partial F_l}{\partial x_i}\right)(-1)^{i+l}q_m+\left(\frac{\partial F_i}{\partial x_m}-\frac{\partial F_m}{\partial x_i}\right)(-1)^{i+m-1}q_l+\left(\frac{\partial F_l}{\partial x_m}-\frac{\partial F_m}{\partial x_l}\right)(-1)^{l+m-2}q_i=0$$

Substituting

$$q_m=(-1)^{m+1}p_m;\quad q_l=(-1)^{l+1}p_l;\quad q_i=(-1)^{i+1}p_i,$$

we get, finally,

$$(69)\quad \left(\frac{\partial F_i}{\partial x_l}-\frac{\partial F_l}{\partial x_i}\right)p_m-\left(\frac{\partial F_i}{\partial x_m}-\frac{\partial F_m}{\partial x_i}\right)p_l+\left(\frac{\partial F_l}{\partial x_m}-\frac{\partial F_m}{\partial x_l}\right)p_i=0$$

where i, l, m may take all values from 1 to n.

Let us now consider the vanishing of the $2d$ summation in (41), viz.,

$$(70)\quad \sum_{il}\frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)=0,\ (r,k=1,2,\ldots,n-1)$$

We must introduce here the identities

$$(33)\qquad \sum_i P_i\frac{\partial f_i}{\partial u_r}=0$$

$$(34)\qquad \sum_i\left(\frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}-\frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k}\right)=0$$

Expanding (70) we get

$$(71)\quad \sum_i\frac{\partial F_i}{\partial p_1}\left(\frac{\partial P_1}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_1}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\sum_i\frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_r}-\frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\ldots$$
$$+\sum\frac{\partial F_i}{\partial P_m}\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\ldots=0,\ (r,k=1,2,\ldots,n-1),$$

and we may reduce the number of terms by eliminating the coefficients of $\frac{\partial F_n}{\partial p_1},\frac{\partial F_1}{\partial p_2},\frac{\partial F_2}{\partial p_3},\ldots,\frac{\partial F_{m-1}}{\partial p_m},\ldots.,$ from the sums in (71). Thus to eliminate

$$\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_1}{\partial u_k}-\frac{\partial P_2}{\partial u_k}\frac{\partial f_1}{\partial u_r}\right),$$

the coefficient of $\frac{\partial F_1}{\partial p_2}$ in the second sum, we have from (33)

$$\frac{\partial f_1}{\partial u_k} = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial f_i}{\partial u_k}; \quad \frac{\partial f_1}{\partial u_r} = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial f_i}{\partial u_r};$$

hence,

$$\frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_1}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_1}{\partial u_r}\right) = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right),$$

and

$$\sum_i \frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) = \sum_{i=2}^{n}\left[\frac{\partial F_i}{\partial p_2} - \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)$$
$$= \left[\frac{\partial F_2}{\partial p_2} - \frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_2}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_2}{\partial u_r}\right) + \sum_{i=3}^{n}\left[\frac{\partial F_i}{\partial p_2} - \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)$$

Similarly, we may eliminate the coefficient of $\frac{\partial F_{m-1}}{\partial p_m}$ in the mth sum in (71), and thus find

$$(72)\quad \sum_i \frac{\partial F_i}{\partial p_m}\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) = \left[\frac{\partial F_m}{\partial p_m} - \frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}\right) + \sum_{i=1\ldots m-1}^{m+1\ldots n}\left[\frac{\partial F_i}{\partial p_m} - \frac{p_i}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right).$$

We may consider this as a typical term, letting m assume all values from 1 to n cyclically, or $m = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n$. Substituting in (71) and noting that the coefficients

$$\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}$$

are still related by the identity (34), whereby

$$\frac{\partial P_n}{\partial u_r}\frac{\partial f_n}{\partial u_k} - \frac{\partial P_n}{\partial u_k}\frac{\partial f_n}{\partial u_r} = -\sum_{i=1}^{n-1}\left(\frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)$$

equations (71) may finally be written

$$(72)\quad \sum_{m=1}^{n-1}\left[\left(\frac{\partial F_m}{\partial p_m}-\frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right)-\left(\frac{\partial F_n}{\partial p_n}-\frac{p_n}{p_{n-1}}\frac{\partial F_{n-1}}{\partial p_n}\right)\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k}-\frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}\right)+\sum_{m=1}^{n}\sum_{i=1\ldots m-1}^{m+1\ldots n}\left[\frac{\partial F_i}{\partial p_m}-\frac{p_i}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)=0.$$

In (72) all the coefficients in parentheses may now be taken as independent quantities, so that for (72) to vanish identically, we must have the vanishing of all the expressions in brackets. Hence

$$p_{m-1}\frac{\partial F_i}{\partial p_m}-p_i\frac{\partial F_{m-1}}{\partial p_m}=0,\quad \begin{pmatrix} i=1,\ldots,m-1,\ m+1\ldots n\\ m=1\rightarrow 2\rightarrow\ldots\rightarrow n\end{pmatrix}.$$

By successive elimination of $\frac{\partial F_{m-1}}{\partial p_m}$, we may write these

$$p_k\frac{\partial F_i}{\partial p_m}-p_i\frac{\partial F_k}{\partial p_m}=0,\quad \begin{pmatrix} i,k=1,\ldots,m-1,\ m+1\ldots n\\ m=1,2,\ldots,n\end{pmatrix}$$

or

$$\frac{\partial}{\partial p_m}(p_k F_i-p_i F_k)=0,\quad \begin{pmatrix} i,k=1,\ldots,m-1,\ m+1\ldots n\\ m=1,2,\ldots,n\end{pmatrix}$$

so that $(p_k F_i - p_i F_k)$ can only contain p_i and p_k, and hence,

$$(73)\quad p_k F_i - p_i F_k = \alpha_{ik}(p_i, p_k, x_1, x_2, \ldots, x_n),\quad \begin{matrix}(i,k=1,2,\ldots,n)\\ (i\neq k)\end{matrix}$$

where

$$(74)\qquad \alpha_{ik}=-\alpha_{ki}.$$

From (72) we further have

$$\frac{\partial F_n}{\partial p_n}-\frac{p_n}{p_{n-1}}\frac{\partial F_{n-1}}{\partial p_n}=\frac{\partial F_m}{\partial p_m}-\frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m},\quad (m=1\rightarrow 2\rightarrow\ldots\rightarrow n).$$

These are conditions on the forms of the α's in (73). By successive subtractions, these may be written

$$(75)\qquad \frac{\partial F_i}{\partial p_1} - \frac{p_i}{p_{i-1}}\frac{\partial F_{i-1}}{\partial p_i} = \frac{\partial F_k}{\partial p_k} - \frac{p_k}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_k} \quad (i,\, k = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n).$$

If $k \pm i + 1$, we may replace $\dfrac{\partial F_{i-1}}{\partial p_i}$ by $\dfrac{p_{i-1}}{p_{k-1}}\dfrac{\partial F_{k-1}}{\partial p_i}$, and (75) becomes

$$\frac{\partial F_i}{\partial p_i} - \frac{p_i}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_i} = \frac{\partial F_k}{\partial p_k} - \frac{p_k}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_k}$$

and by subtraction of F_{k-1} from both members,

$$\frac{\partial}{\partial p_i}(p_{k-1} F_i - p_i F_{k-1}) = \frac{\partial}{\partial p_k}(p_{k-1} F_k - p_k F_{k-1}),$$

or

$$(76)\qquad \frac{\partial \alpha_{i,\,k-1}}{\partial p_i} = \frac{\partial \alpha_{k,\,k-1}}{\partial p_k}; \quad (k \pm i + 1).$$

If $k = i + 1$, we may replace $\dfrac{\partial F_{k-1}}{\partial p_k}$ by $\dfrac{p_{k-1}}{p_{i-1}}\dfrac{\partial F_{i-1}}{\partial p_k}$, and subtracting F_{i-1} from both members, (75) becomes

$$(77)\qquad \frac{\partial \alpha_{i,\,i-1}}{\partial p_i} = \frac{\partial \alpha_{k,\,i-1}}{\partial p_k}, \quad (k = i + 1).$$

From (76) and (77) and with the help of (74), we may now write the series of conditions on the α's:

$$(78)\qquad \begin{aligned} &\frac{\partial \alpha_{12}}{\partial p_1} + \frac{\partial \alpha_{23}}{\partial p_3} = 0; \quad \frac{\partial \alpha_{23}}{\partial p_2} + \frac{\partial \alpha_{34}}{\partial p_4} = 0; \quad \ldots\ldots \\ &\frac{\partial \alpha_{i+1,\,i}}{\partial p_{i+1}} + \frac{\partial \alpha_{ik}}{\partial p_k} = 0; \quad \frac{\partial \alpha_{ik}}{\partial p_i} + \frac{\partial \alpha_{k,\,k+1}}{\partial p_{k+1}} = 0; \quad \ldots\ldots \end{aligned}$$

Hence the α's are given by

$$(79)\qquad \alpha_{ik} = p_k \phi_i - p_i \phi_k, \quad (i, k = 1, 2, \ldots, n)$$

where the ϕ's are arbitrary functions of the coördinates $x_1, x_2, \ldots, x_n$ only.

Combining (79) and (73), we must have

$$p_k F_i - p_i F_k = p_k \phi_i - p_i \phi_k \qquad (r, k = 1, 2, \ldots, n)$$

or

$$(80) \qquad \frac{F_1 - \phi_1}{p_1} = \frac{F_2 - \phi_2}{p_2} = \ldots\ldots = \frac{F_i - \phi_i}{p_i} = \ldots\ldots = \frac{F_k - \phi_k}{p_k} = \ldots\ldots = \frac{F_n - \phi_n}{p_n}.$$

Letting K stand for these equal ratios, (80) may be written

$$(81) \qquad F_i = \phi_i + p_i K, \qquad (i = 1, 2, \ldots, n).$$

Multiplying by p_i and summing with respect to i, we get

$$\sum_i p_i F_i = \sum_i p_i \phi_i + K \sum_i p_i^2,$$

and remembering that

$$\sum_i p_i^2 = \sum_i (x_i')^2 = 1; \quad \sum_i p_i F_i = \sum_i x_i' x_i'' = 0,$$

we have

$$K = -\sum_i p_i \phi_i,$$

and (81) becomes

$$(82) \qquad F_i = \phi_i - p_i \sum_k p_k \phi_k \qquad (i = 1, 2, \ldots, n).$$

Now, we must still satisfy equations (69), and this will introduce conditions on the forms of the ϕ's. Substituting the values of the F's as given by (81) into (69), these reduce to

$$(83) \qquad \left(\frac{\partial \phi_i}{\partial x_l} - \frac{\partial \phi_l}{\partial x_i}\right) p_m - \left(\frac{\partial \phi_i}{\partial x_m} - \frac{\partial \phi_m}{\partial x_i}\right) p_l + \left(\frac{\partial \phi_l}{\partial x_m} - \frac{\partial \phi_m}{\partial x_l}\right) p_i = 0,$$
$$(i, l, m = 1, 2, \ldots, n).$$

Since these equations are to hold for abitrary values of the p's (the only condition being $p_1^2 + p_2^2 + \ldots + p_n^2 = 1$), and since the expressions in parentheses are independent of the p's, we must have

$$(84) \qquad \frac{\partial \phi_i}{\partial x_l} = \frac{\partial \phi_l}{\partial x_i}, \qquad (i, l = 1, 2, \ldots, n)$$

Hence, the ϕ's are partial derivatives of a single function $L(x_1, x_2, \ldots, x_n)$, or

$$(85) \qquad \phi_i = \frac{\partial L}{\partial x_i}, \quad (i = 1, 2, \ldots, n).$$

Introducing these values into (82), we finally get

$$(86) \qquad x_i'' = \frac{\partial L}{\partial x_i} - x_i' \sum_k \frac{\partial L}{\partial x_k} x_k' \qquad (i = 1, 2, \ldots, n),$$

and on comparison with (13), we note that these are the differential equations of our natural family in a euclidean space of n dimensions. We may therefore state the converse

THEOREM. *If a system of* $\infty^{2(n-1)}$ *curves (one passing through each point in each direction) in a euclidean space of n dimensions is such that those* ∞^{n-1} *curves of the system which meet an arbitrary hypersurface (space of* $n - 1$ *dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. Since we have only considered the vanishing of α_1 in the condition (38), we may here, analogous to the case of four dimensions, state a much stronger converse theorem.

§11. **Proof of the converse theorem in any space.** In §7 we derived the necessary analytic conditions

$$(26') \quad T\left(\frac{\partial U_k}{\partial u_r} - \frac{\partial U_r}{\partial u_k}\right) + U_k\left(\frac{\partial U_r}{\partial t} - \frac{\partial T}{\partial u_r}\right) + U_r\left(\frac{\partial T}{\partial u_k} - \frac{\partial U_k}{\partial t}\right) = 0,$$
$$(r, k = 1, 2, \ldots, n - 1)$$

where

$$(25'') \qquad T = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t}; \quad U_r = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t} = 0,$$
$$(r = 1, 2, \ldots, n - 1),$$

for a hypercongruence of curves

$$(20') \qquad X_i = X_i(u_1, u_2, \ldots, u_{n-1}, t), \qquad (i = 1, 2, \ldots, n)$$

to be of the normal type. We may proceed exactly as in §8 and apply these conditions to a family of ∞^{n-1} curves picked out from the system of $\infty^{2(n-1)}$ curves defined by

$$(27)\qquad x_i'' = F_i\,(x_1, x_2, \ldots, x_n;\; x_1', x_2', \ldots, x_n'), \quad (i = 1, 2, \ldots, n),$$

and passing out orthogonally to an arbitrary hypersurface

$$(32)\qquad x_i = f_i\,(u_1, u_2, \ldots, u_{n-1}), \quad (i = 1, 2, \ldots, n).$$

This hypercongruence may be written

$$(36)\qquad X_i = f_i + P_i\,t + \tfrac{1}{2}\,\overline{F}_i\,t^2 + \tfrac{1}{6}\,\overline{M}_i\,t^3 + \ldots, \qquad (i = 1, 2, \ldots, n),$$

where

$$(37)\qquad \overline{F}_i\,(u_1, u_2, \ldots, u_{n-1}) = F_i\,(f_1, f_2, \ldots, f_n;\; P_1, P_2 \ldots, P_n)$$

and the P's are connected by identities similar to (33) and (34) but of a more general form, viz.,

$$(33')\qquad \sum_{\lambda\mu} a_{\lambda\mu}\,P_\lambda\,P_\mu = 1;\quad \sum_{\lambda\mu} a_{\lambda\mu}\,P_\lambda\,\frac{\partial f_\mu}{\partial u_r} = 0, \; (r = 1, 2, \ldots, n-1);$$

$$(34')\qquad \sum_{\lambda\mu} P_\mu\left(2\,a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_r} + P_\lambda\frac{\partial a_{\lambda\mu}}{\partial u_r}\right) = 0, \quad (r = 1, 2, \ldots n-1);$$

$$\sum_{\lambda\mu}\frac{\partial f_\mu}{\partial u_\mu}\left(a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_k} + P_\lambda\frac{\partial a_{\lambda\mu}}{\partial u_k}\right) = \sum_{\lambda\mu}\frac{\partial f_\mu}{\partial u_k}\left(a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_r} + P_\lambda\,\frac{\partial a_{\lambda\mu}}{\partial u_r}\right),$$
$$(r, k = 1, 2, \ldots, n-1).$$

Before applying the conditions of orthogonality to the curves (36) we shall introduce considerable simplification in the work by introducing a set of new quantities, Q_i, defined by the linear transformation

$$(87)\qquad Q_\mu = \sum_\lambda a_{\lambda\mu}\,P_\lambda, \quad \text{and hence, } P_\mu = \sum_\lambda A_{\lambda\mu}\,Q_\lambda$$

where $A_{\lambda\mu}$ is the minor of $a_{\lambda\mu}$ in the determinant $a = |a_{\lambda\mu}|$ divided by a itself, so that

$$(88)\qquad \sum_\mu a_{i\mu}\,A_{\lambda\mu} = 0, \quad \text{if } i \neq \lambda\,; \quad \sum_\mu a_{\lambda\mu}\,A_{\lambda\mu} = 1.$$

Now, the identities (33′) and (34′) take the much simpler form

$$(89)\qquad \sum_\mu Q_\mu\,P_\mu = 1;\quad \sum_\mu Q_\mu\,\frac{\partial f_\mu}{\partial u_r} = 0, \quad (r = 1, 2, \ldots, n-1);$$

$$(90)\qquad \sum_\mu \left(Q_\mu \frac{\partial P_\mu}{\partial u_r} + P_\mu \frac{\partial Q_\mu}{\partial u_r}\right) = 0, \quad (r = 1, 2, \ldots, n-1);$$

$$\sum_\mu \left(\frac{\partial Q_\mu}{\partial u_k}\frac{\partial f_\mu}{\partial u_r} - \frac{\partial Q_\mu}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}\right) = 0, \quad (r, k = 1, 2, \ldots, n-1).$$

Applying our condition of orthogonality, we get

$$\frac{\partial X_\lambda}{\partial t} = P_\lambda + t\overline{F}_\lambda + \ldots; \qquad \frac{\partial X_\mu}{\partial t} = P_\mu + t\overline{F}_\mu + \ldots;$$

$$\frac{\partial X_\mu}{\partial u_r} = \frac{\partial f_\mu}{\partial u_r} + t\frac{\partial P_\mu}{\partial u_r} + \ldots; \qquad \frac{\partial X_\mu}{\partial u_k} = \frac{\partial f_\mu}{\partial u_k} + t\frac{\partial P_\mu}{\partial u_k} + \ldots;$$

$$T = \sum_{\lambda\mu} a_{\lambda\mu} P_\lambda P_\mu + t\sum_{\lambda\mu} a_{\lambda\mu}(P_\lambda \overline{F}_\mu + P_\mu \overline{F}_\lambda) + \ldots = 1 + 2t\sum_\mu Q_\mu \overline{F}_\mu + \ldots;$$

$$U_r = \sum_{\lambda\mu} a_{\lambda\mu} P_\lambda \frac{\partial f_\mu}{\partial u_r} + t\sum_{\lambda\mu} a_{\lambda\mu}\left(P_\lambda \frac{\partial P_\mu}{\partial u_r} + \overline{F}_\lambda \frac{\partial f_\mu}{\partial u_r}\right) + \ldots;$$

$$U_r = t\left(\sum_\mu Q_\mu \frac{\partial P_\mu}{\partial u_r} + \sum_{\lambda\mu} a_{\lambda\mu}\overline{F}_\lambda \frac{\partial f_\mu}{\partial u_r}\right) + \ldots;$$

$$U_k = t\left(\sum_\mu Q_\mu \frac{\partial P_\mu}{\partial u_r} + \sum_{\lambda\mu} a_{\lambda\mu}\overline{F}_\lambda \frac{\partial f_\mu}{\partial u_k}\right) + \ldots;$$

$$\frac{\partial U_r}{\partial u_k} = t\left[\sum_\mu \left(Q_\mu \frac{\partial^2 P_\mu}{\partial u_k\,\partial u_k} + \frac{\partial Q_\mu}{\partial u_k}\frac{\partial P_\mu}{\partial u_r}\right) + \sum_{\lambda\mu} a_{\lambda\mu}\overline{F}_\lambda \frac{\partial^2 f_\mu}{\partial u_k\,\partial u_r} + \sum_{\lambda\mu}\frac{\partial(a_{\lambda\mu}\overline{F}_\lambda)}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right] + \ldots;$$

$$\frac{\partial U_k}{\partial u_r} = t\left[\sum_\mu \left(Q_\mu \frac{\partial^2 P_\mu}{\partial u_r\,\partial u_k} + \frac{\partial Q_\mu}{\partial u_r}\frac{\partial P_\mu}{\partial u_k}\right) + \sum_{\lambda\mu} a_{\lambda\mu}\overline{F}_\lambda \frac{\partial^2 f_\mu}{\partial u_r\,\partial u_k} + \sum_{\lambda\mu}\frac{\partial(a_{\lambda\mu}\overline{F}_\lambda)}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}\right] + \ldots.$$

The vanishing of the coefficient of the first power of t in the condition of orthogonality reduces to

$$(91)\ \sum_{\lambda\mu}\left[\frac{\partial(a_{\lambda\mu}\overline{F}_\lambda)}{\partial u_r}\frac{\partial f_\mu}{\partial u_k} - \frac{\partial(a_{\lambda\mu}\overline{F}_\lambda)}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}\right] + \sum_\mu\left[\frac{\partial Q_\mu}{\partial u_r}\frac{\partial P_\mu}{\partial u_k} - \frac{\partial Q_\mu}{\partial u_k}\frac{\partial P_\mu}{\partial u_r}\right] = 0,$$
$$(r, k = 1, 2, \ldots, n-1).$$

Now (37) may be written

$$(92)\qquad \overline{F}_i(u_1, u_2, \dots, u_{n-1}) = F_i(f_1, f_2, \dots, f_n;\ P_1, P_2, \dots, P_n) = F'_i(f_1, f_2, \dots, f_n;\ Q_1, Q_2, \dots, Q_n),$$

where F'_i represents the form which the function F_i takes when the P's are expressed in terms of the Q's, by (87). Then

$$(93)\qquad \begin{aligned} \frac{\partial \overline{F}_\lambda}{\partial u_r} &= \sum_l \frac{\partial F'_\lambda}{\partial x_l}\frac{\partial f_l}{\partial u_r} + \sum_l \frac{\partial F'_\lambda}{\partial q_l}\frac{\partial Q_l}{\partial u_r} \\ \frac{\partial \overline{F}_\lambda}{\partial u_k} &= \sum_l \frac{\partial F'_\lambda}{\partial x_l}\frac{\partial f_l}{\partial u_k} + \sum_l \frac{\partial F'_\lambda}{\partial q_l}\frac{\partial Q_l}{\partial u_k} \end{aligned}$$

with similar expressions for the partial derivatives of $a_{\lambda\mu}$, and (91) then becomes

$$(94)\qquad \sum_{\lambda\mu l}\left[\frac{\partial(a_{\lambda\mu}F'_\lambda)}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k} - \frac{\partial f_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right) + \frac{\partial(a_{\lambda\mu}F'_\lambda)}{\partial q_l}\left(\frac{\partial Q_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k} - \frac{\partial Q_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)\right] + \sum_\mu\left[\frac{\partial Q_\mu}{\partial u_r}\frac{\partial P_\mu}{\partial u_k} - \frac{\partial Q_\mu}{\partial u_k}\frac{\partial P_\mu}{\partial u_r}\right] = 0,$$

$$(r, k = 1, 2, \dots, n-1).$$

These $(n-1)(n-2)/2$ conditions are therefore necessary conditions for a normal hypercongruence. Our problem is to find the forms of the functions F'_i or F_i in order that this condition of orthogonality should hold for *every* hypersurface taken as a base. We may simplify our problem by the following considerations. Since the geodesics in our space form a normal hypercongruence for every hypersurface as base, the equations of the geodesics

$$(18)\qquad \frac{d^2x_\lambda}{ds^2} = -\sum_{ik}\left\{ {ik \atop \lambda} \right\}\frac{dx_i}{ds}\frac{dx_k}{ds}, \qquad (\lambda = 1, 2, \dots, n)$$

or

$$(95)\qquad \frac{d^2x_\lambda}{ds^2} = G_\lambda(f_1, f_2, \dots, f_n;\ P_1, P_2, \dots, P_n) = G'_\lambda(f_1, f_2, \dots f_n;\ Q_1, Q_2, \dots Q_n)$$

must satisfy conditions (94). Substituting G'_λ for F'_λ in (94), subtracting the result from (94) and writing

$$H'_\lambda = F'_\lambda - G'_\lambda \tag{96}$$

we get

$$\sum_{\lambda\mu l} \left[\frac{\partial(a_{\lambda\mu} H'_\lambda)}{\partial x_l} \left(\frac{\partial f_l}{\partial u_r} \frac{\partial f_\mu}{\partial u_k} - \frac{\partial f_l}{\partial u_k} \frac{\partial f_\mu}{\partial u_r} \right) + \frac{\partial(a_{\lambda\mu} H'_\lambda)}{\partial q_l} \left(\frac{\partial Q_l}{\partial u_r} \frac{\partial f_\mu}{\partial u_k} - \frac{\partial Q_l}{\partial u_k} \frac{\partial f_\mu}{\partial u_r} \right) \right] = 0, \quad (r, k = 1, 2, \ldots, n-1). \tag{97}$$

Making the further linear transformation

$$J'_\mu = \sum_\lambda a_{\lambda\mu} H'_\lambda \tag{98}$$

equations (97) take the simple form

$$\sum_{l\mu} \frac{\partial J'_\mu}{\partial x_l} \left(\frac{\partial f_l}{\partial u_r} \frac{\partial f_\mu}{\partial u_k} - \frac{\partial f_l}{\partial u_k} \frac{\partial f_\mu}{\partial u_r} \right) + \sum_{l\mu} \frac{\partial J'_\mu}{\partial q_l} \left(\frac{\partial Q_l}{\partial u_r} \frac{\partial f_\mu}{\partial u_k} - \frac{\partial Q_l}{\partial u_k} \frac{\partial f_\mu}{\partial u_r} \right) = 0, \quad (r, k = 1, 2, \ldots, n-1). \tag{99}$$

Here the form of J' is to be determined if the f's are arbitrary functions, the only relations between the f's and the Q's being the identities

$$\sum_\mu Q_\mu \frac{\partial f_\mu}{\partial u_r} = 0, \quad (r = 1, 2, \ldots, n-1); \tag{89}$$

$$\sum_\mu \left(\frac{\partial Q_\mu}{\partial u_k} \frac{\partial f_\mu}{\partial u_r} - \frac{\partial Q_\mu}{\partial u_r} \frac{\partial f_\mu}{\partial u_r} \right) = 0, \quad (r, k = 1, 2, \ldots, n-1). \tag{90}$$

We note that equations (99) and their accompanying identities (89) and (90) have the same form as equations (41) and their accompanying identities (33) and (34), with i replaced by μ, F by J', and P by Q. The form of F is determined by (80) and (85), and we may therefore immediately write down the form of J', as

$$\frac{J'_1 - \phi_1}{q_1} = \frac{J'_2 - \phi_2}{q_2} = \ldots\ldots = \frac{J'_i - \phi_i}{q_i} = \ldots\ldots = \frac{J'_n - \phi_n}{q_n} \tag{100}$$

where the ϕ's are partial derivatives of a single function $L(x_1, x_2, \ldots x_n)$, or

$$\phi_i = \frac{\partial L}{\partial x_i}, \quad (i = 1, 2, \ldots, n). \tag{101}$$

Letting K stand for the equal ratios in (100), we may write these

$$J'_i = \phi_i + q_i K, \quad (i = 1, 2, \ldots, n)$$

or, by (98)

$$\sum_{\lambda} a_{\lambda i} H'_{\lambda} = \phi_i + q_i K, \quad (i = 1, 2, \ldots n).$$

To solve these for H'_{λ}, we multiply by A_{li} and sum with respect to i, then

$$H'_l = \sum_i A_{li} \phi_i + K \sum_i A_{li} q_i.$$

Hence by (87)

$$H_l = \sum_i A_{li} \phi_i + K p_l.$$

To find the value of K, we now multiply by $a_{\lambda l} p_{\lambda}$, sum with respect to λ and l, and use the relation (88),

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} H_l = \sum_i p_i \phi_i + K \sum_{\lambda l} a_{\lambda l} p_{\lambda} p_l.$$

Now since

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} p_l = 1,$$

then, by total differentiation with respect to s, we may show that

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} F_l + \tfrac{1}{2} \sum_{i\lambda l} \frac{\partial a_{\lambda l}}{\partial x_i} p_i p_{\lambda} p_l = 0$$

or

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} F_l - \sum_{\lambda l} a_{\lambda l} p_{\lambda} G_l = 0 = \sum_{\lambda l} a_{\lambda l} p_{\lambda} H_l.$$

Hence,

$$K = - \sum_i p_i \phi_i$$

and

$$H_l = \sum_i A_{li} \phi_i - p_l \sum_i p_i \phi_i.$$

Finally, replacing H_l by $F_l - G_l$, G_l by its value from (18), and ϕ_i by its value from (101), we get

$$(102) \quad x''_l + \sum_{ik} \left\{ {ik \atop l} \right\} x'_i x'_k = \sum_i \frac{\partial L}{\partial x_i} (A_{li} - x'_l x'_i), \quad (l = 1, 2, \ldots, n)$$

and on comparison with (15) we note that these are the differential equations of our natural family in any space of n dimensions. We may therefore state the converse

THEOREM. *If a system of* $\infty^{2(n-1)}$ *curves (one passing through each point in each direction) in any curved space of* n *dimensions is such that those* ∞^{n-1} *curves of the system which meet on arbitrary hypersurface (space of* n — 1 *dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. Since we have only considered the vanishing of α_1 in the condition (38), we may here, analogous to the case in a euclidean space, state a much stronger converse theorem.

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I. INTRODUCTION.

THE investigation described in this paper and carried on in the Laboratory of Physics of Northwestern University during the years 1914–15–16, is a continuation of measurements on the thermodynamic properties of ammonia, (NH_3), made in the Research Laboratory of Physical Chemistry of the Massachusetts Institute of Technology in 1912–13.

The work at the Massachusetts Institute of Technology was divided into two parts: the equation of state for ammonia was determined by Keyes and Brownlee,[1] while Keyes and the writer made measurements of the specific heat. The mean specific heat capacity of liquid ammonia was determined by a parallel method of mixtures, using water as a reference substance, between 0°– 20° C. and between 20° – 50° C.[2] Realizing the unsuitability of this method for higher temperatures, where the specific heat varies rapidly with temperature, Keyes devised a "point" method which gave the mean specific heat over very small temperature ranges.[3] By this method some rough measurements were made between 25° – 95° C., but the data was not considered of sufficient accuracy to warrant publication.

The investigation described in this paper is an elaboration of this latter method.

II. EXPERIMENTAL METHOD.

The specific heat capacity at constant volume, of a mixture of ammonia, liquid and vapor, (NH_3), has been determined over a temperature range from 30° to 125° C., by an absolute method, which consists of heating the ammonia, by measured amounts of electrical energy, through temperature ranges of about 1° C., from various initial temperatures.

The calorimeter contains the calorimetric fluid, a heater, a thermometer, and a sealed tube enclosing the ammonia. When the calorimeter has been brought to some initial temperature, a measured amount of current is passed through the heater for a certain time, and the change of temperature of the calorimeter observed. The mean heat capacity of the entire apparatus over this temperature interval is obtained by dividing the energy input by the temperature change produced by this energy. Since the rate of change of heat capacity does not vary appreciably during a heating of one degree, the value obtained is taken as the heat capacity at the mean temperature of the experiment. A series of such measurements are made at various temperatures, and the heat capacity, denoted by $\left(\frac{\delta H_1}{\delta\theta}\right)$, is plotted as a function of the temperature.

---

[1] "Thermodynamic Properties of Ammonia," Keyes and Brownlee. John Wiley and Sons. (1916). Also Frederick G. Keyes, Amer. Soc. of Refrig. Eng. Jour., **1**, 9 (1914).

[2] Keyes and Babcock, Amer. Chem. Soc. Jour., p. 1524, 39^2, (1917).

[3] "Thermodynamic Properties of Ammonia," Keyes and Brownlee. John Wiley and Sons. (1916), p. 29.

To correct for the heat capacity of the system exclusive of the ammonia, the liquid container is emptied, and the series of measurements repeated. The heat capacity of the empty apparatus,[4] denoted by $\left(\frac{\delta H_2}{\delta\theta}\right)$, is plotted as a function of the temperature, and this curve constitutes a calibration of the calorimeter. The trend of these $\frac{\delta H}{\delta\theta}$ curves is shown in Figure 1. At any temperature θ_x, the heat capacity of the ammonia is measured by the difference of the ordinates to the two curves, and this difference is denoted by $\frac{\delta Q}{\delta\theta}$.

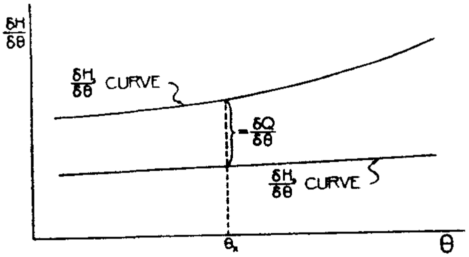
$\frac{\delta H}{\delta\theta}$
$\frac{\delta H_1}{\delta\theta}$ CURVE
$=\frac{\delta Q}{\delta\theta}$
$\frac{\delta H_2}{\delta\theta}$ CURVE
θ_x
θ

FIGURE 1.

To obtain the *specific* heat capacity of the ammonia, it is necessary to divide the values of $\frac{\delta Q}{\delta\theta}$, at each temperature by the mass, M, of ammonia in the container. The volume of this container is sensibly constant, so that the specific heat so obtained is at constant volume.[5] This specific heat is denoted by $c_{v_{12}}$. The subscript $_v$ signifying "at constant volume," and the subscripts $_1$ and $_2$, denoting "of a mixture of liquid and vapor," subscript $_1$ referring to the vapor phase and subscript $_2$ to the liquid phase. To be exact, it is necessary to specify the numerical value of the particular specific volume at which the measurements were made. We have then

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[4] The correction for the heat capacity of the air in the container is negligible.

$$c_{v_{12}} = \frac{1}{M}\frac{\delta Q}{\delta \theta} \tag{1}$$

and for the laboratory equation

$$c_{v_{12}} = \frac{1}{M}\left(\frac{\delta H_1}{\delta \theta} - \frac{\delta H_2}{\delta \theta}\right) \tag{2}$$

in which the terms in the brackets are functions of the temperature, θ. Equation 2 gives the specific heat at any temperature θ, in terms of the mass of ammonia and the heat capacities of the apparatus full and empty, measured at the same temperature θ. The methods of measuring δH_1, δH_2, $\delta \theta$, and M, will be described in detail hereafter.

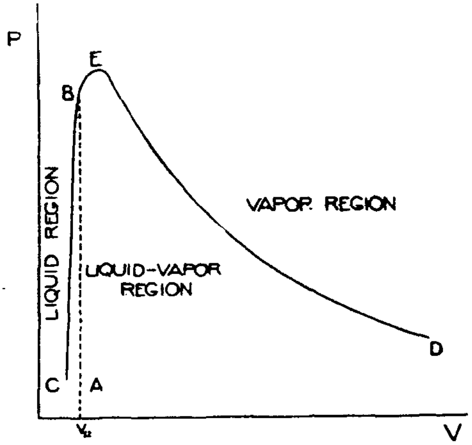

FIGURE 2.

Given values of $c_{v_{12}}$ as a function of the temperature θ, at some specific volume v, it is possible to *calculate* the values of the saturation specific heats of both the liquid and the vapor, over the same temperature range, provided certain thermodynamic quantities for the substance in question are known. This computation will be given later.

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[5] Measurements were made to determine the temperature dilatation of the container, and this quantity was found to be negligible.

Drawing the pressure-volume diagram for a substance, in the neighborhood of the liquid-vapor region, it is possible to indicate graphically the path over which the heating takes place, during a determination of $c_{v_{12}}$. This path is shown by AB in Figure 2. The saturation specific heat of the liquid is the specific heat determined by heating along the liquid line CE; and the saturation specific heat of the vapor is that determined by heating along the vapor line DE.

The measurements described in this paper yield the specific heat along the line AB.

III. Apparatus. Specifications.

1. **Calorimeter.** Figure 3.
   - a. Calorimeter Tube.
     Material steel; inside diameter 62 mm.; wall thickness 1 mm.; length 270 mm. Supported by steel pins from asbestos board cover 7 mm. thick. All metal projections from calorimeter through cover capped with asbestos board.
   - b. Calorimetric Fluid.
     5153 grams of mercury. Advantages: low vapor pressure, negligible vaporization, low heat capacity per unit volume, high heat conductivity.
   - c. Liquid Container. Calorimeter Stirring.
     Constructed from seamless steel tube, outside diameter, 46 mm., wall thickness, 4 mm., length 250 mm. Designed to withstand critical pressure of liquid ammonia: 112.3 atmos. Hexagon nut milled on each end of barrel. Hexagon caps screwed on both ends. Gaskets, thin sheet copper. Joints and barrel tested to 135 atmos. Filling plug screwed into upper cap. Hemispherical joint, set up dry. Six 1.6 mm., stirring fins milled lengthwise of barrel. Container mounted in bearings in calorimeter. Oscillated by stirring shaft and gearing, driven by small motor. Geared to rotate three turns in each direction.

Voltage drop across the armature was kept constant during a measurement by a sliding rheostat. Four bladed copper churn mounted in bearings inside of container to stir ammonia and distribute heat by conduction. Stirring shaft insulated from calorimeter by asbestos board plug. Upward thrust on stirring shaft taken by ball socket thrust bearing at upper end of shaft.

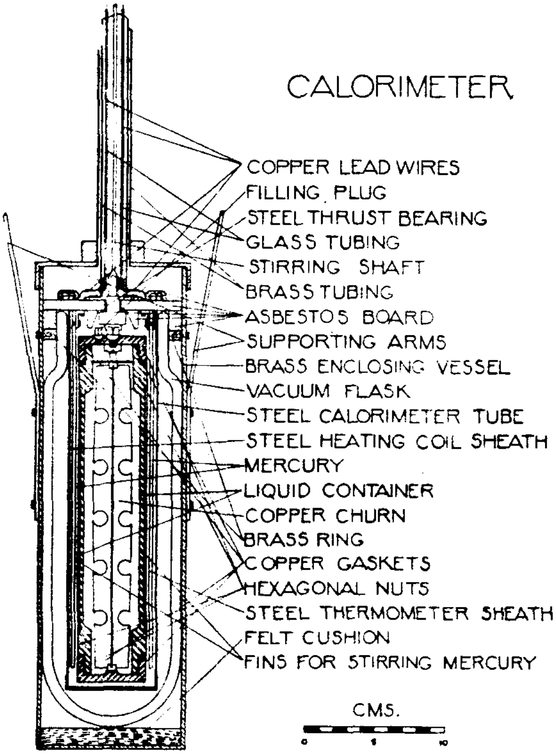

CALORIMETER
COPPER LEAD WIRES
FILLING PLUG
STEEL THRUST BEARING
GLASS TUBING
STIRRING SHAFT
BRASS TUBING
ASBESTOS BOARD
SUPPORTING ARMS
BRASS ENCLOSING VESSEL
VACUUM FLASK
STEEL CALORIMETER TUBE
STEEL HEATING COIL SHEATH
MERCURY
LIQUID CONTAINER
COPPER CHURN
BRASS RING
COPPER GASKETS
HEXAGONAL NUTS
STEEL THERMOMETER SHEATH
FELT CUSHION
FINS FOR STIRRING MERCURY
CMS.
0
5
10

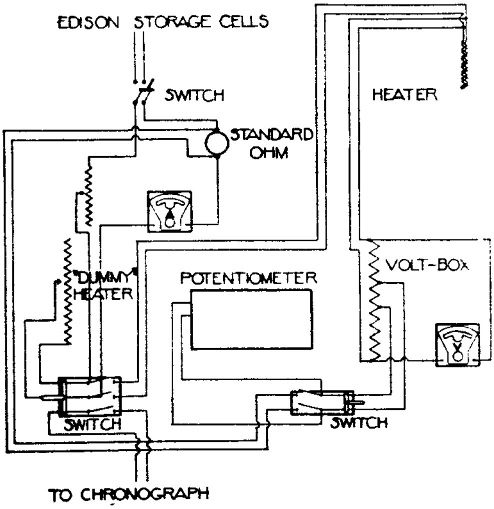
FIGURE 3.

d. Heating Coil.

Manganin resistance coil, built into calorimeter. Enclosed in flattened steel sheath, 3 mm. thick, 8 mm. wide, and 240 mm. long. Heater introduced heat equally at all levels. Supported from asbestos board cover by steel brackets. Described more fully below.

e. Thermometer.
Platinum resistance thermometer, built into calorimeter. Enclosed in flattened steel sheath, similar to heating coil sheath. Extended entire length of calorimeter tube. Averaged differences of temperature at different levels.

f. Vacuum Jacket.
Wide mouthed [6] Dewar flask, enclosing calorimeter tube. Top closed by asbestos board calorimeter cover supported on asbestos board ring carried by enclosing vessel.


EDISON STORAGE CELLS
SWITCH
STANDARD OHM
HEATER
"DUMMY" HEATER
POTENTIOMETER
VOLT-BOX
SWITCH
SWITCH
TO CHRONOGRAPH

POTENTIOMETER CIRCUITS

FIGURE 4.

g. Enclosing Vessel.
Brass cylinder fitted with screwed-on cover, totally immersed in constant temperature oil bath. Cover threads packed with red lead mixed with machine oil. Cover provided with brass tube conveying leads and stirring shaft from the calorimeter to the outside. Leads were insulated by small glass tubes fixed inside the brass tube.

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[6] There is no particular advantage in this contracted neck shown on the Dewar flask. This bottle was at hand, so the rest of the apparatus was built to fit it.

2. **Energy Measuring Apparatus.**
   a. Circuits. Figure 4.
      Arranged to measure current through heating coil, and voltage drop across heating coil, alternately, by potentiometer method. The "dummy" heater, equal in resistance to the heating coil, keeps the line hot and the batteries at the desired voltage while the heater switch is off.
   b. Heating Coil.
      Resistance 12.2 ohms. Manganin wire, laced vertically on thin baked mica strip,[7] enclosed by two baked mica cover strips. Manganin wire used on account of its zero temperature coefficient resulting in no initial excess of current when the heater switch is thrown on. Heating coil enclosed in flattened steel sheath supported in calorimeter.
   c. Potentiometer and Galvanometer.
      Wolff high resistance dial type potentiometer.
      Leeds and Northrup type HN moving coil galvanometer.
   d. Batteries.
      Twenty type B6H, 112.5 amp. hr. Edison Cells, used solely for supplying energy to the calorimeter.
   e. Standard Cell.
      Weston Cadmium Cell, certified by the Buréau of Standards.
   f. Standard Resistance.
      1 ohm. N. B. S. Standard Resistance furnished and certified by Leeds and Northrup.
   g. Chronograph.
      Ordinary pen and drum type, regulated by laboratory clock. Connected to heater switch in such manner as to record automatically length of time heating current was on.

3. **Thermometric Apparatus.**
   a. Standard Thermometer.
      Four lead compensated Callendar open coil type.[8] 2 meters of 0.1 mm. diameter platinum wire wound on

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[7] The vertical lacing was found to be an easy way to overcome the tendency of the stiff manganin wire to twist the delicate mica supporting frame when wound on in the ordinary manner. Holes were punched in the frame with a needle at proper intervals and the wire threaded through these holes.

[8] "Measurement of High Temperatures," Burgess and Le Chatelier. John Wiley and Sons. (1912), p. 202.

dehydrated mica frame enclosed in glass tube. Leads of 0.3 mm. platinum wire are welded to coil terminals. Leads to switch, # 20 B & S guage copper wire are welded to platinum leads and soldered to switch terminals. Resistance at 0°C. 25.87 ohms. Used for calibrating observing thermometer by comparison, and for measuring temperature of oil bath.

b. Observing Thermometer.

Four lead compensated Dickinson & Mueller flat coil type[9] of special construction. 2 meters of 0.1 mm. platinum wire wound on flat mica strip dehydrated by baking in electric furnace at 900° C. Enclosed in flat steel sheath and built into calorimeter. No drying agent was used. Top sealed with china cement. Leads of # 20 B.&S. gauge copper wire, arc welded to coil terminals and soldered to switch terminals. Resistance at 0° C. 25.68 ohms. Used for measuring temperature change in calorimeter.

c. Wheatstone Bridge. Figure 5.

This bridge is a modification of one designed by the Bureau of Standards and built by the Leed's and Northrup Co.,[10] Range of Bureau of Standards instrument is 0° – 115° C. Range of special bridge used in these measurements, 0° – 450° C.

The instrument differs from the Bureau of Standards design in three particulars:

1. It is provided with 9, 11, 7, and 12 ohm extension coils to increase the range of the instrument.
2. A mercury cup contact link is inserted on the thermometer side of the bridge to compensate the resistance of the mercury cup link used to connect in the extension coils.
3. Thirteen instead of ten of the one ohm coils are used.

The extension coils cannot be adjusted to the same accuracy as the 1 ohm coils when the bridge is built, but their value may be determined in terms of the resistance of the slide wire of the bridge, and within the limits of accuracy of the bridge itself by the expedient of making a double setting with a resistance thermometer at a constant temperature and therefore at a constant resistance. With the thermometer in a steam hypsometer, a bridge reading may be made with the link in the 25 ohm position, and then by shifting the link to the 34 ohm position and moving the plug in the one ohm bank back 9 steps, and again balancing with the slide wire, a second reading is obtained which is equal to the first, which gives an equation determining the unknown value of the nominally 9 ohm coil. The bridge is designed for use with thermometers having a resistance at 0° C. of 25.3 ohms and a fundamental interval of about 10.0 ohms. By double settings at the steam point (100°), the napthalene point (218°), the benzophenone point (306°), and the sulphur point (445.7°), the four extension coils may be accurately determined. A thermometer may be calibrated and the extension coils measured at the same time.

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[9] Dickinson and Mueller, Bull. Bur. of Standards, **3** (1907), p. 641.

[10] Leeds and Northrup Bulletin, no. 806.

SPECIAL CALORIMETER BRIDGE

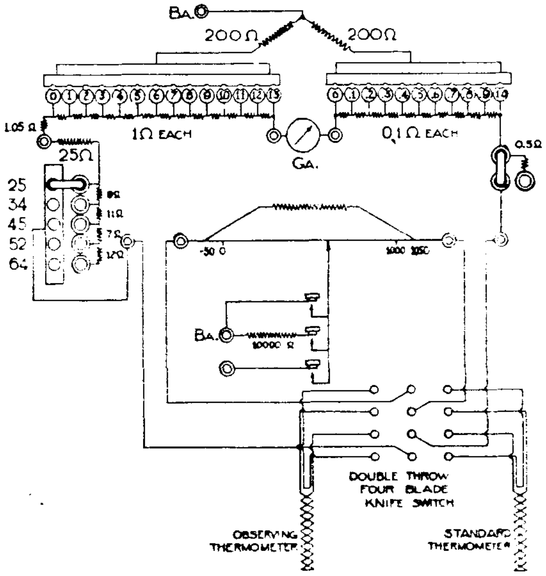
BA.
200Ω
200Ω
0
1
2
3
4
5
6
7
8
9
10
11
12
13
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
14
1.05 Ω
1Ω EACH
0.1 Ω EACH
GA.
0.5Ω
25Ω
25
34
45
52
64
9Ω
11Ω
7Ω
12Ω
-50 0
1000 1050
BA.
10000 Ω
DOUBLE THROW
FOUR BLADE
KNIFE SWITCH
OBSERVING
THERMOMETER
STANDARD
THERMOMETER

FIGURE 5

The measuring current used with this bridge was about 0.004 amperes.

d. Thermometer Switch.

Double throw, four blade, all copper knife switch especially constructed. Used to connect either of the two thermometers to the bridge. Later work showed that mercury cup links are more satisfactory for this purpose.

e. Galvanometer.

High sensitivity, Leeds and Northrup, D'Arsonval instrument,[11] type b: resistance 12 ohms; period 5 secs., sensitivity 5 mm. per micro-volt at a scale distance of 1 meter. Supported on brick wall. Scale observed by telescope at distance of 2.4 meters. Scale divisions were arbitrary but so spaced that a movement of the bridge wire equivalent to 0.00001 ohm produced a deflection of approximately $\frac{1}{10}$ of a scale division. A change in resistance of 0.00001 ohm was equivalent to about 0.0001° C. on the standard thermometer.

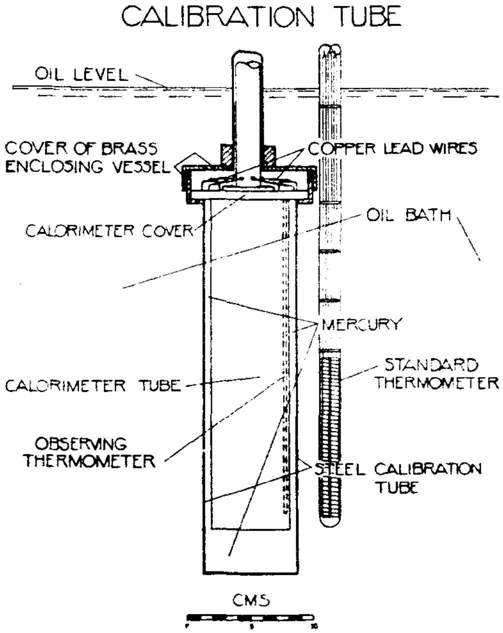
CALIBRATION TUBE
OIL LEVEL
COVER OF BRASS ENCLOSING VESSEL
COPPER LEAD WIRES
CALORIMETER COVER
OIL BATH
MERCURY
STANDARD THERMOMETER
CALORIMETER TUBE
OBSERVING THERMOMETER
STEEL CALIBRATION TUBE
CMS

FIGURE 6.

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[11] Leeds and Northrup Bulletin, no. 228.

f. Apparatus for calibration of Standard Thermometer. Instrument was calibrated at the ice, steam, and naphthalene points. Shaved ice and distilled water mixed to a slush and contained in a thermos bottle were used for the zero point. A steam hypsometer with three outlets and a manometer was used for the steam point. The head of the thermometer was protected from radiation by a square flat tin shield between the head and the hypsometer. A special glass tube with two side condensing tubes was used for the napthalene boiling apparatus. Instrument protected by the usual aluminum radiation and splash shield, and the head protected as in the determination of the steam point.

g. Apparatus for calibration of Observing Thermometer. Assembled calorimeter withdrawn from the vacuum flask and its enclosing brass cylinder, and screwed into the "calibration tube" shown in Figure 6. This calibration tube was filled with mercury. The whole apparatus was then immersed in the oil bath, the observing thermometer being in metallic contact with the oil through the medium of the two tubes of mercury. Comparisons with the standard thermometer were made at three temperatures.

4. Constant Temperature Bath.

a. Tank.
Cubical galvanized iron tank, 2 feet on a side, riveted and soldered at the bottom and lagged with felt. Cover of ½-inch asbestos board with central opening to admit the calorimeter which was supported by arms from this cover. Tank was mounted on castors to facilitate movement in case of emergency.

b. Oil.
Steam engine cylinder oil was used for the bath but owing to its viscosity at low temperatures it is not a particularly desirable substance.

c. Stirring.
Bath was stirred by eight 9-inch propeller blades carried on two vertical shafts, driven by belt from direct current motor whose speed could be controlled by a rheostat.

d. Adjusting heater.
3600 watt heater, wound with nichrome tape on asbestos

paper on steel tubes. Used only for changing the bath temperature from one fixed point to another.

e. Main Heater.
Wound with nichrome wire on asbestos paper on wood. Adjustable by means of external rheostats. Maximum power 550 watts. Used to maintain bath temperature at any desired point. Set so as to just allow the bath to cool.

f. Regulating Heater.
Wound with nichrome wire on asbestos paper on steel tube. Adjustable by means of external rheostats. Maximum power 50 watts. Switched on and off by an automatic relay switch.

g. Temperature Regulator.
U tube of glass 3 cm. in diameter containing 500 cc. of mercury. Stop-cock and reservoir at top of one limb, for coarse adjustment. 1 mm. diameter capillary and reservoir at top of other limb. Contact needle of steel, with well rounded blunt point. Screw adjustment on needle for fine setting.

h. Cooling Coils.
Thirty feet of $\frac{1}{2}$-inch copper pipe coiled in the bottom of tank and connected to the water mains. Needle valve for adjustment. Used for regulation of temperatures below 35° C. and for rapid cooling of the bath from high temperatures.

i. Removable Cover.
Cover arranged to be lifted by means of block and tackle, carrying with it propellers, heating coils, temperature regulator, and calorimeter. This was useful when it was necessary to make repairs.

IV. LABORATORY EQUATIONS AND METHODS OF MEASUREMENT.

1. Laboratory Equation for Specific Heat at Constant Volume of a Mixture of Liquid and Vapor.

This equation, previously given, is

$$c_{v_{12}} = \frac{1}{M}\left(\frac{\delta H_1}{\delta\theta} - \frac{\delta H_2}{\delta\theta}\right) \tag{2}$$

The volume denoted by v_{12}, is the specific volume of the mixture of liquid and vapor, and is given by the relation

$$v = \frac{V}{M} \tag{3}$$

where V = total inside volume of container, and M = total mass of ammonia. Usually we shall write, simply, v with the subscripts $_1$ and $_2$ omitted but understood. However, whenever it is desired to designate the particular specific volume at which these measurements were made, we shall write $v_{2.6593}$, where 2.6593 is understood to be in ccs./gram.

2. **Laboratory Equation for Energy Input to Calorimeter.**

During the time that the heating current is on, alternate readings are made of the voltage drop across the smaller coil of the volt box and of the voltage drop across the standard 1 ohm resistance in series with the heating coil. Between each pair of readings, the e. m. f. of the potentiometer batteries was balanced against that of the standard cell.

For any one measurement, the current through the heating coil is given by

$$I = \frac{\text{average voltage drop across standard ohm}}{\text{resistance of standard ohm at the observed temperature}} \tag{4}$$

and the voltage drop across the heater, is given by,

E = average voltage drop across smaller coil of volt box x ratio of volt box coils (5)

and the energy input to the calorimeter is given by

$$\delta H = EI\,(t_2 - t_1) \tag{6}$$

$(t_2 - t_1)$ being the interval of time during which the heating current is on.

To distinguish between ammonia and calibration measurements, the subscripts $_1$ and $_2$ are used, thus δH_1 = the energy input for an ammonia measurement, δH_2 = the energy input for a calibration measurement. The temperature of the standard ohm was read at the beginning and end of each measurement and the average of these two readings used to compute the resistance required for Equation 4. The temperature of the standard cell was also read at the beginning and end of each measurement.

3. Ratio of Volt Box Coils.

The ratio of the volt box coils, as determined, included the resistance of the lead wires. The average of ten readings was taken to allow for a slightly varying current.

4. Laboratory Equation for the Calorimeter Temperature Change: Cooling Correction.

The temperature change occurring in the calorimeter and measured by the observing thermometer, is not the temperature change actually produced by the introduction of the heat δH_1 or δH_2, but is complicated by several circumstances which must be allowed for in computing the true $\delta\theta$. These circumstances are as follows:

1. The temperature of the mercury in the calorimeter rises more rapidly than that of the container and ammonia, so that the temperature recorded on the thermometer at any instant, while very closely the temperature of the *mercury*, is not the equalized temperature of the *calorimeter*.
2. A heat transfer to the surroundings takes place due to the difference of temperature between the mercury and these surroundings.
3. Heat is added to the mercury and the calorimeter by the friction of stirring.

The method of procedure in correcting for these effects was as follows:

The oil bath was set to regulate at the temperature at which it was desired to make a measurement, and the calorimeter temperature brought to within a degree or so of this bath temperature. No attempt was made to have the calorimeter temperature always bear the same relation to the bath temperature, as it was desired to see how closely the method of computing these temperature data eliminated such variables. The theory takes account of these differences of calorimeter and bath temperature and if the conditions are always the same, the agreement between results is rather a check on the reproducibility of the apparatus than a true indication of the experimental error. A half hour was allowed to elapse before measurements were begun, to permit time for the parts of the calorimeter to equalize their temperature differences. Figure 7 is a typical time-temperature calorimetric curve, and shows the effects described below. Stirring was started and readings made of the temperature at intervals of one minute.[12] It was assumed that the small amount of heat introduced by stirring did not sensibly raise the temperature of the mercury above that of the container and the ammonia. This is, of course, not strictly true. Readings were continued for ten minutes before the heating current was applied, the changes of temperature during this period being due to heat transfer to the surroundings and to the heat of stirring. Exactly at the end of the tenth minute the heater switch was closed and measurements of the energy input begun by an assistant. The temperature of the mercury now began to rise rapidly, and its value was recorded at intervals of one minute. When the container was loaded with ammonia, this heating was continued for

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[12] The readings were made almost exactly on the even minute. To accomplish this, the galvanometer telescope was focused on a watch hung on the

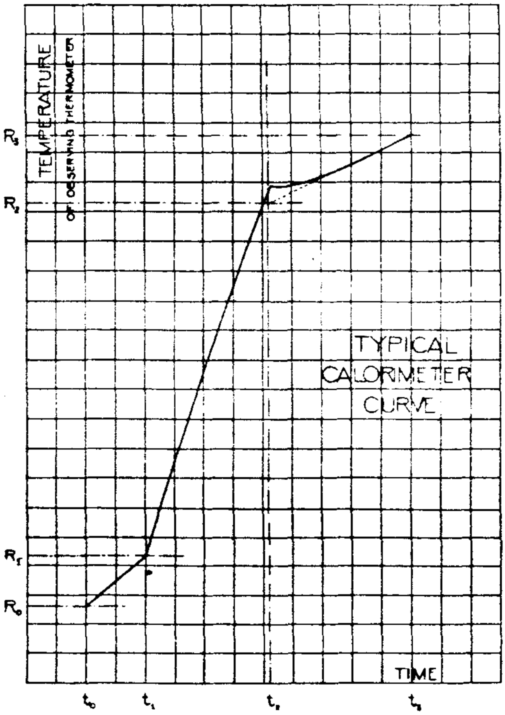

FIGURE 7.

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scale, and so arranged that the cross hair coincided with the center of the second hand. The bridge was kept in balance by changing the slide wire and at the instant the second hand came to the 60 second mark, the movement of the slide wire was stopped and the reading recorded.

twenty minutes, and when the container was empty, for fifteen minutes. At the end of this time the heater was cut off. The temperature of the mercury continued to rise for about twelve seconds and then started to drop. This "coast" of the mercury temperature is due, no doubt, to the lack of equalization of temperature between the thermometer on one side of the calorimeter, and the heater on the other side. The heater is the hottest part of the calorimetric system while the current is on, and some time is required to allow the heat to flow around to the thermometer after the current is shut off. In other words, the temperatures recorded on the thermometer during the heating period are not exactly mercury temperatures, but are slightly below the true equalized mercury temperature. However, the amount of this lag is so small that no correction was made for it.

The drop of temperature occurring after this "coast" takes place, is, however, far from negligible, and is very noticeable. This drop is explained by the fact that the container and its contents are lagging behind the mercury during the heating period, so that after the heater is cut off, the mercury is losing heat both to the surroundings and to the container, which accounts for the rapid drop in its temperature. As the temperature differences between container and mercury get less and less, the curve drops less steeply, and after a time, a point is reached where the curve becomes sensibly a straight line, or at most a line with very slight curvature. Where this occurs, it is assumed that the calorimeter has equalized its temperature differences and that the curve of temperature against time from that point on is due entirely to stirring and heat transfer to the surroundings, as during the initial period. This point of equalization may be located fairly accurately by plotting the curvature, $(d^2\theta/dt^2)$, as a function of the time. After the point of equalization is passed, the curvature is practically zero, while before this it is changing very rapidly. While the slope of the curvature function has no actual discontinuity, a "corner" may be determined with some degree of certainty. The final line may slope either up or down depending on the difference of temperature between bath and calorimeter, and no attempt was made to have this slope the same in each observation. This final line became sensibly straight after about ten minutes in the case of the ammonia measurements, and after about fifteen minutes in the case of the calibration measurements. *This straight line is projected back to the time ordinate corresponding to the instant at which the heater was cut off, and the value of the temperature so obtained is taken to be the equalized temperature of the calorimeter at the instant the switch was cut off.*

The initial value of the equalized temperature is known by the intersection of the initial line with the time ordinate through the point where the heater was switched on, and the above procedure gives the final value; the difference of the two is the "observed temperature change." It remains to correct this observed temperature change for the heat transfer to the surroundings and for the heat due to stirring, during the time of the heating period. Two assumptions are made:

1. That the rate at which the mercury changes temperature due to heat transfer to the surroundings is proportional to the temperature difference. (Newton's Law of Cooling.) This may be written

$$d\theta_r/dt = K\,(\theta_a - \theta)$$

$d\theta_r$ being a change of temperature due to heat transfer alone, K, the modulus of thermal leakage, assumed constant for any one measurement, θ_a, the temperature of the surroundings, assumed constant for any one measurement, and θ, the actual temperature at any instant of the mercury.

2. That the rate of temperature change due to stirring is constant throughout any one measurement. This may be expressed

$$d\theta_s/dt = \lambda = \text{a constant.}$$

where $d\theta_s$ is a change of temperature due to stirring alone.

Since all temperature measurements are made in terms of resistance instead of in degrees, it is convenient to rewrite these two equations in terms of R rather than θ.

$$dR_r/dt = k\,(R_a - R) \qquad (7)$$

$$dR_s/dt = \lambda, \text{ a constant.} \qquad (8)$$

R being the resistance corresponding to the temperature of the mercury at any instant. The sum of these two rates, given by (7) and (8), $\left(\frac{dR_r}{dt} + \frac{dR_s}{dt}\right)$, represents the rate at which the temperature is changing due to heat transfer and stirring alone. Calling this sum, $\left(\frac{dR_c}{dt}\right)$, we have

$$\left(\frac{dR_c}{dt}\right) = (kR_a + \lambda) - kR \qquad (9)$$

which is the differential equation of both the initial and final lines, and moreover gives the "cooling rate" during the heating period. This equation may be integrated to give the "cooling correction." To obtain the value of the constant quantity $k\,(R_a + \lambda)$, we make use of the fact that the initial and final lines are sensibly straight, which amounts to saying that the change in R of Equation 9 for short intervals of time, is not enough to change the slope, $\frac{dR_c}{dt}$. Giving R its mean value for the initial line which is, $\frac{R_1 + R_0}{2}$, where R_0 is the first point on that line and R_1 the last, Equation 9 becomes

$$\frac{dR_c}{dt} = \frac{R_1 - R_0}{t_1 - t_0} = (kR_a + \lambda) - \frac{k}{2}(R_1 + R_0) \tag{10}$$

and giving R its mean value for the final line, $\frac{R_3 + R_2}{2}$, where R_2 is the value of the resistance obtained by projecting back the final line to the time ordinate through the point at which the heater was cut off, and R_3 is the last point on the final line, Equation 9 becomes

$$\frac{dR_c}{dt} = \frac{R_3 - R_2}{t_3 - t_2} = (kR_a + \lambda) - \frac{k}{2}(R_3 + R_2) \tag{11}$$

(10) gives us the value of $(kR_a + \lambda)$ and eliminating $(kR_a + \lambda)$ between (10) and (11) gives us the value of k, as follows

$$(kR_a + \lambda) = \frac{R_1 - R_0}{t_1 - t_0} + \frac{k}{2}(R_1 + R_0) \tag{12}$$

$$\frac{k}{2} = \frac{\dfrac{R_1 - R_0}{t_1 - t_0} - \dfrac{R_3 - R_2}{t_3 - t_2}}{(R_3 + R_2) - (R_1 + R_0)} \tag{13}$$

Integrating equation (9) from t_1 to t_2 to obtain the cooling correction to be applied to the observed temperature change, and substituting (12) in the expression,

$$\int_{t_1}^{t_2} \frac{dR_c}{dt}\,dt = \frac{R_1 - R_0}{t_1 - t_0}(t_2 - t_1) + \frac{k}{2}(R_1 + R_0)(t_2 - t_1) - k\int_{t_1}^{t_2} R\,dt \tag{14}$$

The mean value of R between R_1 and R_2 may be called R_m so

$$\int_{t_1}^{t_2} R\,dt = R_m\,(t_2 - t_1) \tag{15}$$

and substituting this in (14)

$$\int_{t_1}^{t_2} \frac{dR_c}{dt} dt = \frac{R_1 - R_0}{t_1 - t_0} (t_2 - t_1) - \frac{k}{2} (2R_m - R_1 - R_0) (t_2 - t_1) \qquad (16)$$

Now the observed temperature change is $(R_2 - R_1)$ where R_2 and R_1 have been obtained graphically as explained above, and the cooling correction is given by (16). Calling δR the corrected value of the change of resistance corresponding to the change of temperature in the calorimeter

$$\delta R = R_2 - R_1 - \int_{t_1}^{t_2} \frac{dR_c}{dt} dt =$$

$$R_2 - R_1 + \frac{k}{2} (2R_m - R_1 - R_0) (t_2 - t_1) - \frac{R_1 - R_0}{t_1 - t_0} (t_2 - t_1) \qquad (17)$$

where the value of $k/2$ is given by (13).

Thus δR is given in terms of four points, $(R_0 t_0)$, $(R_1 t_1)$, $(R_2 t_2)$, and $(R_3 t_3)$, and in terms of R_m. The four points are obtained by plotting the initial and final lines, and R_m is obtained by averaging the observed values of R during the heating period.

To convert δR into degrees C, use is made of a formula given by Dickinson and Mueller.[13] Starting with the Callendar-Griffiths equations for the resistance of platinum as a function of the temperature, they obtain.

$$\delta\theta = \frac{\dfrac{100}{R_{100°} - R_{0°}}}{\left(1 + \dfrac{\delta_c}{100} - \dfrac{2\delta_c}{100^2}\,\theta_m\right)} \delta R \qquad (18)$$

where $R_{100°}$, $R_{0°}$, and δ_c, are the constants of the Callendar-Griffiths equation for the observing thermometer, and θ_m is the mean temperature, $\theta + \delta\theta/2$. θ_m need be known only approximately, and in this work was taken to be the temperature corresponding to R_m.

5. Callendar-Griffiths Equation for Resistance of Platinum as a Function of the Temperature.[14]

$$\theta - \theta_p = \delta_c \left(\frac{\theta}{100} - 1\right) \frac{\theta}{100} \qquad (19)$$

$$\theta_p = 100 \frac{R - R_{0°}}{R_{100°} - R_{0°}} \qquad (20)$$

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[13] Bull. Bur. of Standards, **9**, (1913), p. 483.
[14] Reference 8, p. 201.

where θ = temperature in degrees Centigrade
θ_p = "platinum temperature" defined by (20)
R = resistance of thermometer at temperature θ.
R_{100° = resistance at 100°
R_{0° = resistance at 0°
δ_c = a third constant whose value depends on the purity of the platinum.

6. Calibration of Standard Thermometer.

Resistance at 0° C. Thermometer was deeply immersed in a thermos bottle containing commercial distilled water ice, finely shaved and soaked with distilled water. Preliminary measurements showed no appreciable difference between commercial ice and distilled water frozen in the laboratory. The average of six different determinations was taken.

Resistance at 100° C. Thermometer in a Regnault hypsometer equipped with manometer; hypsometer filled with distilled water. Alternate readings made on bridge and barometer. Barometer instrumental error determined by comparison with a standard 25 mm. bore barometer. Boiling points taken from Landolt and Bornstein's *Tabellen.* The average of four different determinations was taken.

Resistance at 218° C. Thermometer immersed in a naphthalene boiling apparatus, similar to the usual sulphur boiling tube, but equipped with two side condensing tubes. The purity of the naphthalene was tested, as usual, by measuring its melting point. In these measurements the thermometer tube was protected by an aluminium radiation and splash shield. Alternate readings were made on the bridge and the barometer. Boiling points were taken from Burgess and LeChatelier "Measurement of High Temperatures," p. 450. The average of three different determinations was taken.

7. Calibration of Observing Thermometer.

The observing thermometer in its calibration tube, and the standard thermometer, were both immersed in the constant temperature bath, which was set to regulate at some definite temperature. This bath temperature was not constant but fluctuated between narrow limits due to the regulation. Both thermometers followed these fluctuations but the observing thermometer lagged behind the standard. Comparisons were made by averaging the readings over at least one complete temperature wave for both thermometers. Seven comparisons for the two thermometers were thus made, three at temperatures slightly above room temperature, two in the neighborhood of 60° C., and two in the neighborhood of 115° C. It requires comparison at three different temperatures to determine the constants of the Callendar-Griffiths equation. Of the three room temperature comparisons, two were at practically the same temperature, so these were averaged to give a single point, making two sets of comparisons of three each. From each of these sets the constants for the observing thermometer were computed, and then these two sets of constants were averaged to obtain the final accepted values.

8. **Mass of Ammonia.**

The mass of the empty container was determined by the average of three weighings, and the mass when loaded with ammonia, was determined by the average of six weighings. Weighings were made on a 2 Kilo Bunge Balance, using the Laboratory Standard weights.

9. **Volume of Ammonia Container.**

The volume of the ammonia container was determined by weighing the quantity of distilled water required to just fill it at room temperature. The specific volume of water at the observed temperature was taken from Landolt and Bornstein's *Tabellen*. The average of three determinations was taken as the value of the volume. Weighings were made on the 2 Kilo Bunge Balance using the Laboratory Standard Weights. The volume was again determined at 78° C. by the same method and no change was detected.

V. EXPERIMENTAL OBSERVATIONS AND DATA.

In this section are given all of the original observations and such other data as were used in the computation of the results, or were necessary to determine the precision of these measurements.

1. **Measurements of the Heat Capacity of the Apparatus:**

(a) When filled with Ammonia. (Ammonia Measurements.)
(b) When Empty. (Calibration Measurements.)

Explanation of the tables.— The first two columns give the bridge readings of the resistance of the observing thermometer as a function of the time. These readings were made at the end of each minute. The notations "Switch on" and "Switch off," indicate that at those instants the heater switch was thrown on and off, respectively. This heater switch was thrown as nearly as possible at the exact end of the minute indicated in the first column. The temperature of the standard ohm, the standard cell, and the oil bath, were recorded at the beginning and end of each measurement. The potential drop readings given in the third and fourth columns, were made as rapidly as possible during the heating period, but not at definite time intervals. The values of R_0, R_1, R_2, and R_3, were obtained by plotting column 2 against column 1, and drawing the best initial and final lines. The values given were then read from the plot. The time intervals given in connection with these resistance readings, are the elapsed times between successive pairs of readings, as is indicated by their location in the table.

Ammonia Measurement 1 at 32°.9 C.

20 May, 1916

Std. Ohm Temp. Start: 23°.6 C. End: 23°.6 C.
Std. Cell Temp. Start: 21°.3 C. End: 21°.4 C.
Oil Bath Temp. Start: 4.21075 End: 4.21070
4.21060 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	2.86457				
1	6621				
2	6795				
3	6967				
4	7146				
5	7305				
6	7480				
7	7650				
8	7820				
9	8000				
10	8168				
11	8338	Switch On.			
12	9083		0.40924	1.10819	1261.00
13	9740		0.40929	1.10824	
14	2.90361		0.40928	1.10822	
15	1005		0.40927	1.10800	
16	1619		0.40927	1.10797	
17	2225		0.40923	1.10787	
18	2825		0.40921	1.10785	
19	3436		0.40920	1.10770	
20	4017		0.40914	1.10762	
21	4600		0.40915	1.10755	
22	5191		0.40914	1.10730	
23	5757		0.40910	1.10731	
24	6325		0.40909	1.10726	
25	6896		0.40908	1.10728	
26	7469		0.40909	1.10723	
27	8035		0.40908	1.10721	
28	8585		0.40906	1.10722	
29	9140				
30	9677				
31	3.00221				
32	———	Switch Off.			
33	0701				
34	0749				
35	0807				
36	0870				
37	0939				
38	1009				
39	1083				
40	1156				
41	3.01235				
42	1317				
43	1397				
44	1480				
45	1562				
46	1642				
47	1740				
48	1833				
49	1919				
50	2015				
51	2108				
52	2208				
53	2312				
54	2403				
55	2497				
56	2589				

R_0: 2.86625
10 Min.
R_1: 2.88338
21 Min.
R_2: 3.00300
24 Min.
R_3: 3.02592

AMMONIA MEASUREMENT 2 AT 39°.6 C.

13 May, 1916.

Std. Ohm Temp. Start: 23°.2 C. End: 23°.3 C.
Std. Cell Temp. Start: 20°5. C. End: 21°.0 C.
Oil Bath Temp. Start: 4.85440 End: 4.85488
4.85475 4.85456
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	3.33529				
1	3671				
2	3815				
3	3975				
4	4129				
5	4278				
6	4434				
7	4585				
8	4740				
9	4903				
10	5056	Switch On.			
11	5756		0.40630	1.11249	1202.25
12	6379		0.40617	1.11233	±1.0
13	6992		0.40616	1.11225	
14	7572		0.40612	1.11202	
15	8161		0.40603	1.11200	
16	8735		0.40596	1.11188	
17	9304		0.40592	1.11169	
18	9863		0.40584	1.11164	
19	3.40402		0.40583	1.11157	
20	0954		0.40577	1.11140	
21	1505		0.40574		
22	2060				
23	2615				
24	3170				
25	3717				
26	4253				
27	4802				
28	5340				
29	5878				
30	6417	Switch Off.			
31	6392				
32	6448				
33	6517				
34	6605				
35	6697				
36	6785				
37	6887				
38	6991				
39	7096				
40	7197				
41	7300				
42	7402				
43	7504				
44	7606				
45	——				
46	7828				
47	7942				
48	8057				
49	8173				
50	8283				
51	8404				
52	8513				
53	8628				
54	8745				
55	8859				

R_0: 3.33513
10 Min.
R_1: 3.35056
20 Min.
R_2: 3.46166
15 Min.
R_3: 3.47710

AMMONIA MEASUREMENT 3 AT 40°.6 C.

20 May, 1916.

Std. Ohm Temp. Start: 23°.1 C. End: 23°.0 C.
Std. Cell Temp. Start: 20°.9 C. End: 20°.5 C.
Oil Bath Temp. Start: 4.97432 End:
4.97446 4.97540
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	3.38607				
1	8720				
2	8835				
3	8960				
4	9092				
5	9221				
6	9347				
7	9482				
8	9612				
9	9749				
10	9891	Switch On.			
11	——		0.40850	1.10521	1139.10
12	3.41173		0.40823	1.10430	
13	1775		0.40784	1.10343	
14	2361		0.40747	1.10225	
15	2943		0.40713	1.10156	
16	3510		0.40686	1.10073	
17	4066		0.40661	1.10021	
18	4620		0.40623	1.09887	
19	5163		0.40595	1.09797	
20	5703		0.40571	1.09750	
21	6245		0.40553	1.09669	
22	6773		0.40525	1.09628	
23	7315				
24	7818				
25	8340				
26	8873				
27	9374				
28	9881				
29	3.50390	Switch Off.			
30	0357				
31	——				
32	0453				
33	——				
34	0568				
35	0642				
36	0724				
37	0802				
38	0891				
39	0985				
40	1073				
41	1170				
42	1270				
43	1362				
44	1458				
45	1546				
46	1635				
47	1719				
48	1804				
49	1890				
50	1971				
51	2057				
52	——				
53	2225				
54	——				
55	2396				

R_0: 3.38538
10 Min.
R_1: 3.39891
19 Min.
R_2: 3.50198
25 Min.
R_3: 3.52309

AMMONIA MEASUREMENT 4 AT 51°.9 C.

14 May, 1916.

Std. Ohm Temp. Start: 23°.1 C. End: 22°.9 C.
Std. Cell Temp. Start: 20°.3 C. End: 20°.0 C.
Oil Bath Temp. Start: 6.16097 End: 6.16071
6.16100 6.16072
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	4.17268				
1	7414				
2	7556				
3	7709				
4	7850				
5	7994				
6	8152				
7	8305				
8	8452				
9	8601				
10	8756	Switch On.			
11	9442	5 Sec.	0.40635	1.11171	1196.05
12	4.20065	late.	0.40623	1.11170	
13	0645		0.40623	1.11148	
14	1240		0.40622	1.11160	
15	1822		0.40616	1.11142	
16	2396		0.40613	1.11132	
17	2965		0.40610	1.11132	
18	3525		0.40602	1.11130	
19	4082		0.40600	1.11112	
20	4638		0.40577	1.11100	
21	5199			1.11070	
22	5741				
23	6281				
24	6823				
25	7361				
26	7895				
27	8422				
28	8949				
29	9481				
30	4.30004	Switch Off.			
31	4.29948				
32	4.30004				
33	0065				
34	0140				
35	0222				
36	0304				
37	0382				
38	0476				
39	0570				
40	0655				
41	0752				
42	0847				
43	0942				
44	1035				
45	1131				
46	1223				
47	1318				
48	1408				
49	1504				
50	1598				
51	1691				
52	1794				
53	1886				
54	1980				
55	2080				

R_0: 4.17272
10 Min.
R_1: 4.18764
20 Min.
R_2: 4.29700
25 Min.
R_3: 4.32073

AMMONIA MEASUREMENT 5 AT 62°.8 C.

15 May, 1916.

Std. Ohm Temp. Start: 24°.2 C. End: 24°.1 C.
Std. Cell Temp. Start: 21°.5 C. End: 21°.5 C.
Oil Bath Temp. Start: 7.21045 End: 7.21059
7.21085 7.21067
7.21090 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	4.92585				
1	2685				
2	2793				
3	2907				
4	3025				
5	3136				
6	3256				
7	3374				
8	3491				
9	3612				
10	3732				
11	3850	Switch On.			
12	4545		0.40997	1.11939	1199.40
13	5137		0.40993	1.11938	
14	5702		0.40986	1.11923	
15	6265		0.40978	1.11916	
16	6806		0.40974	1.11903	
17	7346		0.40967	1.11880	
18	7881		0.40964	1.11873	
19	8415		0.40958	1.11870	
20	8931		0.40953	1.11852	
21	9460		0.40946	1.11851	
22	9986		0.40944	1.11832	
23	5.00486		0.40939	1.11824	
24	0982			1.11820	
25	1490			1.11812	
26	1990				
27	2496				
28	2991				
29	3474				
30	3967				
31	4461	Switch Off.			
32	4368				
33	4385				
34	4416				
35	4456				
36	4510				
37	4557				
38	4612				
39	4667				
40	4731				
41	4794				
42	4853				
43	4918				
44	4983				
45	5050				
46	5115				
47	5183				
48	5253				
49	5315				
50	5379				
51	5447				
52	5515				
53	5578				
54	5647				
55	5711				
56	5776				

R_0: 4.92666
10 Min.
R_1: 4.93850
20 Min.
R_2: 5.04129
25 Min.
R_3: 5.05776

AMMONIA MEASUREMENT 6 AT 72°.7 C.

15 May, 1916.

		Start	End
Std. Ohm	Temp.	26°.3 C.	26°.7 C.
Std. Cell	Temp.	24°.7 C.	24°.7 C.
Oil Bath	Temp.	8.15018 8.15013	8.15175 8.15165 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	5.60004				
1	0069				
2	0131				
3	0202				
4	0280				
5	0355				
6	0432				
7	0507				
8	0586				
9	0666				
10	0745	Switch On.			
11	1406		0.40996	1.11932	1200.05
12	1943		0.40955	1.11806	
13	2475		0.40922	1.11800	
14	2967		0.40931	1.11821	
15	3468		0.40921	1.11772	
16	3946		0.40903	1.11736	
17	4431		0.40905	1.11722	
18	4912		0.40891	1.11712	
19	5382		0.40886	1.11701	
20	5841		0.40945	1.11852	
21	6294		0.40943	1.11852	
22	6752		0.40930		
23	7205				
24	7645				
25	8107				
26	8557				
27	8989				
28	9439				
29	9875				
30	5.70327	Switch Off.			
31	0177				
32	0151				
33	0142				
34	0137				
35	0147				
36	0150				
37	0155				
38	0169				
39	0190				
40	0204				
41	0228				
42	0246				
43	0273				
44	0301				
45	0327				
46	0357				
47	0379				
48	0404				
49	0443				
50	0442	Shifted 0.1 Ohm Plug.			
51	0468				
52	0487				
53	0505				
54	0533				
55	0556				

R_0: 5.59985

10 Min.

R_1: 5.60737

20 Min.

R_2: 5.69975

25 Min.

R_3: 5.70556

AMMONIA MEASUREMENT 7 AT 83°.2 C.

15 May, 1916.

Std. Ohm Temp. Start: 25°.6 C. End: 25°.5 C.
Std. Cell Temp. Start: 23°.0 C. End: 22°.8 C.
Oil Bath Temp. Start: 9.23486 End: 9.23515
9.23495 9.23510
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	6.30702				
1	0788				
2	0870				
3	0955				
4	1040				
5	1135				
6	1226				
7	1316				
8	1407				
9	1500				
10	1590	Switch On.			
11	2245		0.40842	1.11511	1200.10
12	2805		0.40841	1.11516	
13	3330		0.40840	1.11510	
14	3856		0.40840	1.11503	
15	4362		0.40838	1.11480	
16	4850		0.40836	1.11468	
17	5333		0.40843	1.11472	
18	5815		0.40835	1.11443	
19	6293		0.40832	1.11440	
20	6770		0.40825	1.11420	
21	7242		0.40823	1.11414	
22	7706		0.40822	1.11403	
23	8178		0.40822	1.11400	
24	8640		0.40820	1.11386	
25	9099				
26	9561		R_0: 6.30685		
27	6.40011			10 Min.	
28	0457	Shifted Plug.	R_1: 6.31590		
29	0900			20 Min.	
30	1348	Switch Off.	R_2: 6.41007		
31	1240			25 Min.	
32	1226		R_3: 6.41989		
33	1240				
34	1252				
35	1269				
36	1296				
37	1321				
38	1353				
39	1382				
40	1411				
41	1442				
42	1483				
43	1517				
44	1550				
45	1586				
46	1629				
47	1674				
48	1709				
49	1753				
50	1798				
51	1838				
52	1874				
53	1910				
54	1946				
55	1985				

AMMONIA MEASUREMENT 8 AT 92°.7 C.

16 May, 1916.

Std. Ohm Temp. Start: 26°.2 C. End: 26°.7 C.
Std. Cell Temp. Start: 25°.0 C. End: 25°.0 C.
Oil Bath Temp. Start: 10.26575 End: 10.26520
10.26560 10.26487
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds
0	6.95560				
1	5652				
2	5758				
3	5851				
4	5940				
5	6040				
6	6134				
7	6226				
8	6322				
9	6407				
10	6503	Switch On.			
11	7160		0.40896	1.11252	1200.35
12	7702		0.40890	1.11235	
13	8221		0.40881	1.11230	
14	8719		0.40876	1.11220	
15	9218		0.40872	1.11202	
16	9715		0.40867	1.11192	
17	7.00203		0.40865	1.11182	
18	0656		0.40862	1.11167	
19	1133		0.40855	1.11153	
20	1602		0.40852	1.11144	
21	2072		0.40847	1.11136	
22	2534		0.40844	1.11128	
23	2996		0.40838	1.11116	
24	3450		0.40835	1.11105	
25	3895				
26	4347				
27	4786				
28	5241				
29	5677				
30	6117	Switch Off.			
31	5995				
32	5987				
33	5991				
34	6005				
35	6030				
36	6055				
37	6088				
38	6114				
39	6147				
40	6182				
41	6211				
42	6245				
43	6284				
44	6313				
45	6348				
46	6381				
47	6416				
48	6452				
49	6485				
50	6518				
51	6558				
52	6595				
53	6632				
54	6666				
55	6705				

R_0: 6.95570
10 Min.
R_1: 6.96510
20 Min.
R_2: 7.05834
25 Min.
R_3: 7.06697

AMMONIA MEASUREMENT 9 AT 102°.9 C.

16 May, 1916.

Std. Ohm Temp. Start: 27°.8 C. End: 27°.3 C.
Std. Cell Temp. Start: 24°.6 C. End: 24°.0 C.
Oil Bath Temp. Start: 11.24065 End: 11.24784
11.24062 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	7.64368				
1	4423				
2	4473				
3	4524				
4	4585				
5	4647				
6	4700				
7	4766				
8	4834				
9	4895				
10	4962	Switch On.			
11	5563		0.40884	1.11224	1200.40
12	6070		0.40865	1.11160	
13	6563		0.40860	1.11133	
14	7035		0.40847	1.11121	
15	7496		0.40836	1.11101	
16	7957		0.40827	1.11053	
17	——		0.40815	1.11044	
18	8843		0.40826	1.11069	
19	9291		0.40784	1.11000	
20	9728		0.40800	1.11002	
21	7.70151		0.40803	1.10972	
22	0580		0.40790	1.10940	
23	1019				
24	1442				
25	1860				
26	2270				
27	2687				
28	3111				
29	3514				
30	3930	Switch Off.			
31	3792				
32	3757				
33	3743				
34	3745				
35	3750				
36	3753				
37	3760				
38	3777				
39	3795				
40	3806				
41	3831				
42	3846				
43	3871				
44	3881				
45	3903				
46	3930				
47	3955				
48	3967				
49	3993				
50	4020				
51	4038				
52	4065				
53	4090				
54	4110				
55	4138				

R_0: 7.64339
10 Min.
R_1: 7.64957
20 Min.
R_2: 7.73574
25 Min.
R_3: 7.74135

AMMONIA MEASUREMENT 10 AT 102°.9 C.

17 May, 1916.

Std. Ohm Temp. Start: 25°.3 C. End: 25°.2 C.
Std. Cell Temp. Start: 22°.5 C. End: 22°.4 C.
Oil Bath Temp. Start: 11.27404 End: 11.27417
11.27431 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	7.63535				
1	3635				
2	3736				
3	3835				
4	3936				
5	4052				
6	4150				
7	4242				
8	4336				
9	4445				
10	4545	Switch On.			
11	5180		0.40956	1.10734	1199.50
12	5729		0.40902	1.10642	
13	6246		0.40884	1.10402	
14	6735		0.40797	1.10370	
15	7241		0.40787	1.10307	
16	7717		0.40776	1.10279	
17	8195		0.40767	1.10263	
18	8666		0.40756	1.10256	
19	9150		0.40748	1.10224	
20	9614		0.40736	1.10217	
21	7.70083		0.40732	1.10178	
22	0530		0.40728	1.10173	
23	0970		0.40727	1.10158	
24	1423				
25	1885				
26	2325				
27	2768				
28	3210				
29	3650				
30	4079	Switch Off.			
31	3976				
32	3977				
33	3985				
34	4018				
35	4045				
36	4077				
37	4115				
38	4154				
39	4200				
40	4238				
41	4286				
42	4335				
43	4377				
44	4423				
45	4465				
46	4513				
47	4563				
48	4608				
49	4652				
50	4685				
51	4742				
52	4790				
53	4840				
54	4890				
55	4941				

R_0: 7.63534
10 Min.
R_1: 7.64545
20 Min.
R_2: 7.73778
25 Min.
R_3: 7.74934

Ammonia Measurement 11 at 113°.0 C.

16 May, 1916.

Std. Ohm Temp. Start: 25°.3 C. End: 25°.1 C.
Std. Cell Temp. Start: 22°.6 C. End: 22°.5 C.
Oil Bath Temp. Start: 12.28816 End: 12.28762
12.28725 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	8.31905				
1	1979				
2	2060				
3	2130				
4	2202				
5	2273				
6	——				
7	2390				
8	2463				
9	2550				
10	2623	Switch On.			
11	3235		0.40502	1.09762	1200.15
12	3730		0.40501	1.09722	
13	4197		0.40500	1.09712	
14	4659		0.40495	1.09715	
15	5107		0.40494	1.09714	
16	5569		0.40492	1.09703	
17	6023		0.40490	1.09690	
18	6460		0.40485	1.09690	
19	6891		0.40485	1.09686	
20	7317		0.40483	1.09675	
21	7737		0.40479	1.09673	
22	8162		0.40479	1.09662	
23	8570		0.40476	1.09655	
24	8990				
25	9402				
26	9816				
27	8.40223				
28	0607				
29	1031				
30	1429	Switch Off.			
31	1303				
32	1279				
33	1276				
34	1280				
35	1294				
36	1305				
37	1325				
38	1345				
39	1365				
40	1388				
41	1414				
42	1446				
43	1468				
44	1506				
45	1529				
46	1565				
47	1593				
48	1627				
49	1659				
50	1690				
51	1718				
52	1752				
53	1784				
54	1814				
55	1847				

R_0: 8.31908
10 Min.
R_1: 8.32622
20 Min.
R_2: 8.41067
25 Min.
R_3: 8.41845

Ammonia Measurement 12 at 122°.5 C.

18 May, 1916.

Std. Ohm Temp. Start: 26°.1 C. End: 25°.2 C.
Std. Cell Temp. Start: 22°.0 C. End: 21°.8 C.
Oil Bath Temp. Start: 13.23410 End: 13.22465
13.23402 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds
0	8.94585				
1	4664				
2	4765				
3	4841				
4	4928				
5	5021				
6	5104				
7	5200				
8	5285				
9	5377				
10	5463	Switch On.			
11	6072		0.41029	1.10759	1200.30
12	6566		0.41025	1.10759	
13	7059		0.41023	1.10749	
14	7512		0.41020	1.10742	
15	7977		0.41012	1.10726	
16	8440		0.40983	1.10645	
17	8900		0.40983	1.10645	
18	9335		0.40983	1.10659	
19	9776		0.40985	1.10657	
20	9.00201		0.40984	1.10654	
21	0626		0.40983	1.10655	
22	1066		0.40987	1.10656	
23	1503		0.40988	1.10669	
24	1925		0.40990	1.10680	
25	2351		0.40993	1.10680	
26	2772		0.40992		
27	3200				
28	3613				
29	4034				
30	4465	Switch Off.			
31	4338				
32	4317				
33	4337				
34	4356				
35	4376				
36	4407				
37	4437				
38	4470				
39	4506				
40	4541				
41	4577				
42	4620				
43	4657				
44	4706				
45	4745				
46	4790				
47	4830				
48	4877				
49	4910				
50	4968				
51	——				
52	5046				
53	5082				
54	5123				
55	5168				

R_0: 8.94573
10 Min.
R_1: 8.95464
20 Min.
R_2: 9.04112
25 Min.
R_3: 9.05167

CALIBRATION MEASUREMENT 1 AT 23°.2 C.

8 June, 1916.

Std. Ohm	Temp. Start:	26° 8 C.	End:	26°.8 C.
Std. Cell	Temp. Start:	24°.4 C.	End:	24°.3 C.
Oil Bath	Temp. Start:	3.10262	End:	3.10415
		3.10248		3.10381
				Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.	
0	2.21430	
1	1555	
2	1710	
3	1823	
4	1963	
5	——	Made Reading of Bath Temperature.
6	1987	
7	2072	
8	2213	
9	2350	
10	2489	Switch On.
11	——	
12	4367	
13	5170	
14	5930	
15	6670	
16	7430	
17	8138	
18	8847	
19	9575	
20	2.30242	
21	0911	
22	1583	
23	2250	
24	——	
25	3563	
26	4213	Switch Off.
27	4001	
28	3892	
29	3870	
30	3869	
31	3885	
32	3921	
33	3965	
34	4000	
35	4045	
36	4090	
37	4147	
38	——	
39	4264	
40	4315	
41	4385	
42	4440	
43	4508	
44	4575	
45	4641	
46	4712	
47	4780	
48	4850	
49	4915	
50	4995	
51	5065	

P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0.40889	1.10820	958.40
0.40880	1.10803	
0.40876	1.10786	
0.40871	1.10770	
0.40865	1.10753	
0.40860	1.10730	
0.40852	1.10714	

R_0: 2.21098

10 Min.

R_1: 2.22489

16 Min.

R_2: 2.33392

25 Min.

R_3: 2.35042

Calibration Measurement 2 at 26°.9 C.

26 May, 1916.

Std. Ohm Temp. Start: 29°.0 C. End: 29°.0 C.
Std. Cell Temp. Start: 26° 5 C. End: 26°.5 C.
Oil Bath Temp. Start: 3.52491 End: 3.52006
3.52524 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	2.48690				
1	8797				
2	8912				
3	9026				
4	9141				
5	9265				
6	9380				
7	9502				
8	9625				
9	9741				
10	9868	Switch On.			
11	2.50830	(?)	0.40920	1.11094	599.33
12	1710		0.40916	1.11088	
13	2500		0.40913	1.11074	
14	3250		0.40907	1.11061	
15	3987		0.40904	1.11053	
16	4703		0.40899	1.11043	
17	5406		0.40894	1.11030	
18	6100		0.40890	1.11019	
19	6785				
20	7467	Switch Off.			
21	7285				
22	7212				
23	7205				
24	7234				
25	7274				
26	7326				
27	7387				
28	7452				
29	7521				
30	7595				
31	7673				
32	7756				
33	7839				
34	7919				
35	8010				
36	8102				
37	8192				
38	8281				
39	8373				
40	8445				
41	8561				
42	8651				
43	8746				
44	8838				
45	8932				

R_0: 2.48658
10 Min.
R_1: 2.49867
10 Min.
R_2: 2.56631
25 Min.
R_3: 2.58931

CALIBRATION MEASUREMENT 3 AT 39°.8 C.

9 June, 1916.

Std. Ohm Temp. Start: 24°.6 C. End: 24°.6 C.
Std. Cell Temp. Start: 23°.0 C. End: 23°.0 C.
Oil Bath Temp. Start: 4.86603 End: 4.86585
4.86599 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	3.37685				
1	7750				
2	7817				
3	7887				
4	7965				
5	8036				
6	8114				
7	8198				
8	8280				
9	8357				
10	8436	Switch On.			
11	9005		0.30946	0.83889	900.05
12	9510		0.30941	0.83884	
13	9958		0.30940	0.83883	
14	3.40403	?	0.30940	0.83885	
15	0812		0.30939	0.83882	
16	1230		0.30938	0.83881	
17	1640		0.30936	0.83880	
18	2050		0.30936	0.83880	
19	2445		0.30933	0.83874	
20	2842		0.30933	0.83870	
21	3229				
22	3611				
23	3992				
24	4371				
25	4759	Switch Off.			
26	4663				
27	4594				
28	4585				
29	4595				
30	4613				
31	4637				
32	4660				
33	4696				
34	4733				
35	4772				
36	4810				
37	4846				
38	4889				
39	4930				
40	4983				
41	5023				
42	5069				
43	5116				
44	5162				
45	5196				
46	5247				
47	5295				
48	5340				
49	5387				
50	5432				

R_0: 3.37639
10 Min.
R_1: 3.38436
15 Min.
R_2: 3.44303
25 Min.
R_3: 3.45432

CALIBRATION MEASUREMENT 4 AT 39°.8 C.

10 June, 1916.

Std. Ohm Temp. Start: 23°.2 C. End: 23°.7 C.
Std. Cell Temp. Start: 23°.1 C. End: 23°.1 C.
Oil Bath Temp. Start: 4.86571 End: 4.86577
4.86632 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds
0	3.37685				
1	7777				
2	7872				
3	7973				
4	8075				
5	8173				
6	8272				
7	8365				
8	8468				
9	8565				
10	8659	Switch On.			
11	9303		0.32564	0.88242	900.10
12	9871		0.32537	0.88189	
13	3.40386		0.32512	0.88121	
14	0852		0.32500	0.88060	
15	1332		0.32478	0.88035	
16	1787		0.32454	0.87953	
17	2252		0.32436	0.87911	
18	2692		0.32418	0.87863	
19	3144		0.32398	0.87800	
20	3570				
21	4015				
22	4462				
23	4883				
24	5304				
25	5712	Switch Off.			
26	5598				
27	5541				
28	5535				
29	5547				
30	5571				
31	5603				
32	5635				
33	5677				
34	5715				
35	5758				
36	5801				
37	5849				
38	5900				
39	5944				
40	5997				
41	6042				
42	6099				
43	6150				
44	6199				
45	6260				
46	6312				
47	6365				
48	6412				
49	6470				
50	6522				

R_0: 3.37691
10 Min.
R_1: 3.38659
15 Min.
R_2: 3.45202
25 Min.
R_3: 3.46524

CALIBRATION MEASUREMENT 5 AT 40°.2 C.

10 June, 1916.

Std. Ohm Temp. Start: 24°.2 C. End: 24°.2 C.
Std. Cell Temp. Start: 22°.3 C. End: 22°.1 C.
Oil Bath Temp. Start: 4.86612 End: 4.86620
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds
0	3.41111				
1	1191				
2	1268				
3	1347				
4	1425				
5	1503				
6	1585				
7	1664				
8	1748				
9	1825				
10	1908	Switch On.			
11	2464		0.30826	0.83583	900.34
12	2969		0.30824	0.83580	
13	3410		0.30822	0.83572	
14	3844		0.30821	0.83572	
15	4263		0.30820	0.83567	
16	4670		0.30819	0.83562	
17	5065		0.30817	0.83556	
18	5468		0.30814	0.83552	
19	5857		0.30816	0.83550	
20	6253		0.30812	0.83545	
21	6632		0.30813	0.83545	
22	7008				
23	7398				
24	7770				
25	8154	Switch Off			
26	8037				
27	7973				
28	7962				
29	7969				
30	7982				
31	8001				
32	8028				
33	8049				
34	8081				
35	8109				
36	8143				
37	8175				
38	8217				
39	8250				
40	8287				
41	8319				
42	8362				
43	8404				
44	8443				
45	8482				
46	8518				
47	8567				
48	8611				
49	8653				
50	8696				

R_0: 3.41101
10 Min.
R_1: 3.41908
15 Min.
R_2: 3.47679
25 Min.
R_3: 3.48693

CALIBRATION MEASUREMENT 6 AT 40°.2 C.

10 June, 1916.

Std. Ohm Temp. Start: 24°.0 C. End: 23°.9 C.
Std. Cell Temp. Start: 22°.1 C. End: 22°.0 C.
Oil Bath Temp. Start: 4.86600 End: 4.86680
Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts	P. D. across Small Coil of Volt Box in Volts	Heating Period ($t_2 - t_1$) in Seconds.
0	3.39803				
1	9921				
2	3.40042				
3	0162				
4	0288				
5	0408				
6	0530				
7	0653				
8	0775				
9	0890				
10	1013	Switch On.			
11	1605		0.30474	0.82624	900.15
12	2134		0.30474	0.82623	
13	2623		0.30473	0.82620	
14	3080		0.30472	0.82629	
15	3530		0.30475	0.82626	
16	3978		0.30473	0.82620	
17	4413		0.30469	0.82613	
18	4845		0.30470	0.82617	
19	5265		0.30470	0.82617	
20	5685		0.30471	0.82616	
21	6103		0.30470	0.82614	
22	6514				
23	6920				
24	7321				
25	7725	Switch Off.			
26	7650				
27	7642				
28	7668				
29	7718				
30	7771				
31	7840				
32	7900				
33	7972				
34	8043				
35	8115				
36	8193				
37	8271				
38	8341				
39	8421				
40	8495				
41	8577				
42	8656				
43	8734				
44	8817				
45	8895				
46	8970				
47	9052				
48	9130				
49	9213				
50	9288				

R_0: 3.39803
10 Min.
R_1: 3.41013
15 Min.
R_2: 3.47308
25 Min.
R_3: 3.49288

Calibration Measurement 7 at 40°.2 C.

27 May, 1916.

Std. Ohm Temp. Start: 26°.8 C. End: 27° 2 C.
Std. Cell Temp. Start: 24°.7 C. End: 25°.0 C.
Oil Bath Temp. Start: 4.96933 End: 4.96923
4.97008 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds.
0	3.37349				
1	7494				
2	7656				
3	7806				
4	7975				
5	8133				
6	8303				
7	8460				
8	8618				
9	8787				
10	8947	Switch On.			
11	9980		0.40935	1.11095	901.10
12	3.40855		0.40955	1.11092	
13	1691		0.40952	1.11082	
14	2469		0.40945	1.11070	
15	3241		0.40944	1.11065	
16	3988		0.40939	1.11050	
17	4723		0.40934	1.11036	
18	5444		0.40928	1.11029	
19	6173		0.40927	1.11013	
20	6872		0.40919	1.11000	
21	7574		0.40915	1.10978	
22	8277		0.40907		
23	8966				
24	9636				
25	3.50331	Switch Off.			
26	0172				
27	0091				
28	0095				
29	0120				
30	0171				
31	0229				
32	0290				
33	0360				
34	0414				
35	0495				
36	0578				
37	0660				
38	0749				
39	0827				
40	0927				
41	1018				
42	1110				
43	1200				
44	1290				
45	1385				
46	1500				
47	1593				
48	1683				
49	1773				
50	1867				

R_0: 3.37316
10 Min.
R_1: 3.38949
15 Min.
R_2: 3.49522
25 Min.
R_3: 3.51870

CALIBRATION MEASUREMENT 8 AT 61°.0 C.

27 May, 1916.

Std. Ohm Temp. Start: 27°.8 C. End: 27°.6 C.
Std. Cell Temp. Start: 25°.5 C. End: 25°.0 C.
Oil Bath Temp. Start: 7.04726 End: 7.04743
7.04722 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	4.78922				
1	9053				
2	9186				
3	9318				
4	9444				
5	9565				
6	9697				
7	9832				
8	9957				
9	4.80087				
10	0199	Switch On.			
11	1172		0.40922	1.10692	900.30
12	2011		0.40872	1.10609	
13	2781		0.40861	1.10587	
14	3500		0.40851	1.10560	
15	4216		0.40843	1.10556	
16	4900		0.40837	1.10532	
17	5586		0.40831	1.10530	
18	6245		0.40834	1.10532	
19	6903		0.40831	1.10513	
20	7560		0.40826	1.10502	
21	8182		0.40822	1.10495	
22	8820				
23	9446				
24	4.90060				
25	0671	Switch Off.			
26	0443				
27	0303				
28	0252				
29	0237				
30	0240				
31	0252				
32	0272				
33	0293				
34	0323				
35	0355				
36	0388				
37	0425				
38	0465				
39	0502				
40	0545				
41	0581				
42	0629				
43	0673				
44	0714				
45	0747				
46	0796				
47	0841				
48	0891				
49	0934				
50	0984				

R_0: 4.78922
10 Min.
R_1: 4.80215
15 Min.
R_2: 4.89891
25 Min.
R_3: 4.90977

CALIBRATION MEASUREMENT 9 AT 79°.8 C.

27 May, 1916.

Std. Ohm Temp. Start: 27°.9 C. End: 27°.8 C.
Std. Cell Temp. Start: 25°.3 C. End: 25°.5 C.
Oil Bath Temp. Start: 9.04247 End: 9.04215
9.04306 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period $(t_2 - t_1)$ in Seconds
0	6.07542				
1	7660				
2	7790				
3	7920				
4	8043				
5	8169				
6	8302				
7	8435				
8	8567				
9	8695				
10	8824	Switch On.			
11	9781		0.41134	1.11312	900.00
12	6.10645		0.41124	1.11295	
13	1425		0.41121	1.11286	
14	2149		0.41118	1.11276	
15	2853		0.41114	1.11266	
16	3555		0.41112	1.11257	
17	4234		0.41107	1.11248	
18	4905		0.41105	1.11240	
19	5574		0.41100	1.11231	
20	6233		0.41096	1.11226	
21	6876		0.41096	1.11218	
22	7518		0.41093	1.11211	
23	8150				
24	8793				
25	9432	Switch Off.			
26	9212				
27	9091				
28	9065				
29	9079				
30	9106				
31	9136				
32	9179				
33	9233				
34	9286				
35	9348				
36	9423				
37	9482				
38	9545				
39	9615				
40	9677				
41	9750				
42	9820				
43	9895				
44	9981				
45	6.20057				
46	0130				
47	0208				
48	0282				
49	0357				
50	0430				

R_0: 6.07508
10 Min.
R_1: 6.08824
15 Min.
R_2: 6.18566
25 Min.
R_3: 6.20430

CALIBRATION MEASUREMENT 10 AT 82°.7 C.

28 May, 1916

Std. Ohm Temp. Start: 27°.3 C. End: 26°.8 C.
Std. Cell Temp. Start: 24°.5 C. End: 24°.5 C.
Oil Bath Temp. Start: 9.18862 End: 9.18855
9.18879 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	6.27210				
1	7301				
2	7391				
3	7491				
4	7580				
5	7670				
6	7760				
7	7849				
8	7935				
9	8023				
10	8106	Switch On.			
11	9038		0.40937	1.10812	899.18
12	9820		0.40929	1.10781	
13	6.30539		0.40924	1.10770	
14	1236		0.40916	1.10749	
15	1886		0.40911	1.10734	
16	2525		0.40906	1.10729	
17	3162		0.40903	1.10705	
18	3781		0.40894	1.10687	
19	4403		0.40887	1.10667	
20	5000		0.40880	1.10652	
21	5593		0.40875	1.10638	
22	6165		0.40868	1.10623	
23	6760				
24	7315				
25	7900	Switch Off.			
26	7630				
27	7461				
28	7389				
29	7342				
30	7325				
31	7305				
32	7291				
33	7290				
34	7291				
35	7296				
36	7287				
37	7306				
38	7311				
39	7330				
40	7345				
41	7362				
42	7384				
43	7403				
44	7417				
45	7435				
46	7457				
47	7482				
48	7505				
49	7520				
50	7537				

R_0: 6.27225
10 Min.
R_1: 6.28112
15 Min.
R_2: 6.37053
25 Min.
R_3: 6.37540

Calibration Measurement 11 at 100°.6 C.

29 May, 1916.

Std. Ohm Temp. Start: 26°.0 C. End: 25°.2 C.
Std. Cell Temp. Start: 24°.2 C. End: 24°.4 C.
Oil Bath Temp. Start: 11.03763 End: 11.03691
11.03764 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	7.48176				
1	8247				
2	8330				
3	8415				
4	8487				
5	8573				
6	8655				
7	8748				
8	8841				
9	8928				
10	9017	Switch On.			
11	9936		0.40794	1.10202	899.20
12	——		0.40787	1.10171	
13	7.51430		0.40767	1.10100	
14	2100		0.40745	1.10043	
15	2741		0.40715	1.10013	
16	3365		0.40705	1.09935	
17	3994		0.40684	1.09900	
18	4590		0.40673	1.09883	
19	5193		0.40665	1.09844	
20	5776		0.40648	1.09792	
21	6362		0.40631	1.09751	
22	6945				
23	7505				
24	8072				
25	8616	Switch Off.			
26	8352				
27	8195				
28	8125				
29	8087				
30	8063				
31	8051				
32	8040				
33	8043				
34	8051				
35	8055				
36	8065				
37	8083				
38	8085				
39	8104				
40	8124				
41	8144				
42	8161				
43	8182				
44	8204				
45	8217				
46	8247				
47	8265				
48	8285				
49	8314				
50	8332				

R_0: 7.48128
10 Min.
R_1: 7.49017
15 Min.
R_2: 7.57826
25 Min.
R_3: 7.58326

CALIBRATION MEASUREMENT 12 AT 121°.9 C.

29 May, 1916.

Std. Ohm Temp. Start: 25°.7 C. End: 25°.6 C.
Std. Cell Temp. Start: 23°.5 C. End: 23°.3 C.
Oil Bath δTemp. Start: 13.08932 End: 13.08922
13.08925 Ohms Std. Therm.

Time in Min.	Ohms Obs. Therm.		P. D. across Standard Ohm in Volts.	P. D. across Small Coil of Volt Box in Volts.	Heating Period ($t_2 - t_1$) in Seconds.
0	8.91155				
1	1217				
2	1275				
3	1333				
4	1400				
5	1460				
6	1516				
7	1572				
8	1624				
9	1680				
10	1745	Switch On.			
11	2625		0.40919	1.10484	900.10
12	3383		0.40918	1.10479	
13	4061		0.40917	1.10476	
14	4679		0.40916	1.10465	
15	5302		0.40915	1.10465	
16	5884		0.40913	1.10459	
17	6446		0.40913	1.10458	
18	7018		0.40912	1.10456	
19	7571		0.40911	1.10454	
20	8118		0.40910	1.10453	
21	8672		0.40909	1.10452	
22	9205		0.40906	1.10444	
23	9750		0.40905	1.10441	
24	9.00268				
25	0781	Switch Off.			
26	0478				
27	0285				
28	0167				
29	0096				
30	0041				
31	0001				
32	8.99962				
33	9918				
34	9890				
35	9879				
36	9860				
37	9847				
38	9835				
39	9818				
40	9805				
41	9801				
42	9789				
43	9775				
44	9771				
45	9762				
46	9745				
47	9740				

R_0: 8.91157
10 Min.
R_1: 8.91745
15 Min.
R_2: 8.99948
25 Min.
R_3: 8.99709

2. **Ratio of Volt Box Coils.** See Part IV, Sec. 3.

Column 4 gives the potential drop across the large coil of the volt box including the leads to the heating coil; column 5 gives the potential drop across the small coil of the volt box. The drop across the small coil was read, in each case, immediately after the drop across the large coil. The temperature readings are room temperatures in the neighborhood of the coils.

Determination of Ratio of Volt Box Coils.
10 May 1916.

Wolff Potentiometer #2868.
Weston Cadmium Cell #1458.

No.	Time Mins.	Temperature °C.	Potentiometer Readings	
1	0	23.0	1.25451	0.25020
2	3		1.25426	0.25019
3	5		1.25410	0.25014
4	7	23.1	1.25393	0.25010
5	10		1.25374	0.25003
6	13		1.25359	0.25000
7	15		1.25349	0.25000
8	17	23.1	1.25340	0.24997
9	21		1.25323	0.24993
10	22	23.0	1.25312	0.24991

Average Ratio of Coils: 5.014005

3. **Calibration Data for Standard Thermometer.** See Part IV, Sec. 6.

Resistance at 0° C. (Extension Coil link in 25 ohm position.)

1. 3 April 1916.

Time Hrs.:Mins.	Resistance Ohms.
2:18	0.87280
2:19	0.87276
2:20	0.87275
2:21	0.87274
2:22	0.87277
2:24	0.87273
2:27	0.87263

Time Hrs.:Mins.	Resistance Ohms
2:29	0.87268
2:30	0.87259
2:31	0.87258
2:33	0.87258
2:34	0.87260

$R_{0^\circ} = 0.87258$

2. 8 April 1916.

1:02	0.87262
1:04	0.87268
1:06	0.87263
1:08	0.87268
1:12	0.87253
1:18	0.87249
1:21	0.87249
1 24	0.87256
1:27	0.87246
:28	0.87248

$\mathrm{R}_{0^\circ} = 0.87248$

3. 8 April 1916.

10:28	0.87323
10:30	0.87324
10:31	0.87323
10:33	0.87323
10:36	0.87328
10:42	0.87330

R_{0° = Ice Heating, Measurement not used.

4. 8 April 1916.

12:00	0.87349
12:02	0.87349
12:04	0.87343
12:07	0.87356
12:12	0.87355

R_{0° = Ice Heating, Measurement not used.

5. 9 April 1916.

12:15	0.87467
12:16	0.87467

Time Hrs.:Mins.	Resistance Ohms
12:22	0.87465
12:25	0.87466
12:31	0.87466
2:48	0.87442

$R_{0°}$ = Ice Cooling, Measurement not used.

6. 9 April 1916.

3:21	0.87428
3:25	0.87430
3:35	0.87430
3:50	0.87430
3:56	0.87437

$R_{0°}$ = Ice Heating, Measurement not Used.

7. 10 April 1916.

3:46	0.87241
3:48	0.87239
3:53	0.87231
3:55	0.87231
3:58	0.87231
5:02	0.87242

$R_{0°} = 0.87231$

8. 10 April 1916.

5:30	0.87260
5:32	0.87259
5:37	0.87259
5:43	0.87263

$R_{0°} = 0.87259$

9. 10 April 1916.

6:02	0.87268
6:03	0.87265
6:05	0.87265
6:09	0.87266

$R_{0°} = 0.87265$

10. 10 April 1916.

7:59	0.87303
8:02	0.87303

Time Hrs.:Mins.	Resistance Ohms
8:04	0.87304
8:05	0.87303
8:21	0.87303

$R_{0^\circ} = 87303$

Average Value of $R_{0^\circ} = 0.87261$ ohms ± 0.00016 ohms.

Resistance at *100°* C. *and* E*valuation of* 9 ohm extension coil.

1. 5 April 1916.

Time Mins.	Position of Extension Coil.	Resistance Ohms	Barometer Inches	Temp. of Bar. 0° F.
0	25	10.90825	29.519	73.0
1	25	10.90829		
2	25	10.90828		
3	25	10.90841	29.521	73.0
5	25	10.90833	29.519	73.5
7	34	1.90607		
8	25	10.90835	29.519	74.0
10	34	1.90607		
11	25	10.90832		
12	34	1.90613	29.517	74.5
13	25	10.90837		
15	34	1.90614	29.519	74.5
16	25	10.90857		
17	34	1.90628	29.518	74.5

Barometer Instrumental Correction = + 0.023 inches.
$R = 10.90835$ at 99.532 °C.
Fundamental Interval = 10.08292 ohms.
9 ohm extension coil = 9.00211 ohms.

2. 8 April 1916.

0	25	10.90333	29.464	72.0
1	25	10.90343		
3	25	10.90361	29.463	72.0
4	25	10.90352		
6	25	10.90349		
7	25	10.90352		
9	25	10.90359	29.470	72.5

Time Mins.	Position of Extension Coil.	Resistance Ohms	Barometer Inches	Temp. of Bar. 0° F.
10	34	1.90139	29.470	72.5
12	25	10.90359	29.470	72.5
14	34	1.90141		
16	25	10.90371		
17	34	1.90144		
18	25	10.90371	29.468	72.0
19	34	1.90133		
21	25	10.90362		
22	34	1.90120		
23	25	10.90355		
24	34	1.90136	29.482	74.5

R = 10.90356 at 99.488° C.
Fundamental Interval = 10.08257 ohms.
9 ohm extension coil = 9.00220 ohms.

3. 10 April 1916.

0	25	10.90821		
1	25	10.90814	29.503	67.0
4	25	10.90824	29.506	67.0
5	34	1.90616	29.500	67.0
6	25	10.90818	29.496	68.0
8	25	10.90803	29.506	68.0
9	34	1.90586	29.498	68.0
10	25	10.90809		
11	34	1.90592		
12	25	10.90803	29.490	68.0
13	34	1.90592	29.490	68.0
15	34	1.90592	29.498	68.0
17	25	10.90825	29.506	68.0
20	34	1.90596	29.504	68.0
22	25	10.90796	29.500	68.0
23	34	1.90588	29.494	68.0
25	25	10.90812		

R = 10.90813 at 99.528 °C.
Fundamental Interval = 10.08311 ohms.
9 ohm extension coil = 9.00218 ohms.

4. 10 April 1916.

Time Mins.	Position of Extension Coil.	Resistance Ohms	Barometer Inches	Temp. of Bar. 0° F.
0	25	10.90748	29.481	67.0
1	34	1.90530	29.481	67.0
3	25	10.90742		
4	25	10.90715		
6	25	10.90684		
7	25	10.90674	29.470	67.0
8	25	10.90676		
9	25	10.90670	29.472	67.0
11	25	10.90670	29.474	67.0
12	25	10.90667		
13	25	10.90667	29.475	67.0
14	25	10.90665	29.472	67.5
15	34	1.90450		
17	25	10.90661	29.476	67.5
19	34	1.90455		

R = 10.90687 at 99.506 °C.
Fundamental Interval = 10.08407 ohms.
9 ohm extension coil = 9.00209 ohms.

Average Value of $R_{100°}$ = 10.95578 ohms ± 0.00050 ohms.
Average value of 9 ohms extension coil = 9.00214 ohms ± 0.00004 ohms.

Resistance at 218.0 °C. (Extension Coil Link in 34 ohm position.)

1. 6 April 1916.

Time Mins.	Resistance Ohms.	Barometer Inches	Bar. Temp. 0° F.
0	13.42348	29.732	72.5
1	13.42331	29.724	72.5
3	13.42323		
4	13.42322	29.726	72.5
6	13.42352		
7	13.42376		
9	13.42400	29.730	72.5
10	13.42405		
11	13.42429	29.734	72.5
12	13.42431		
14	13.42464	29.738	72.5

R = 13.42380 ohms at 217.58 °C.

2. 8 April 1916.

Time Mins.	Resistance Ohms	Barometer Inches	Bar. Temp. 0° F.
0	13.38990		
1	13.39001	29.483	73.5
2	13.39001	29.484	73.5
3	13.39035	29.484	73.5
4	13.39046	29.485	73.5
5	13.39102	29.485	74.0
6	13.39090	29.483	74.0
7	13.39115		
8	13.39136		
9	13.39175	29.490	74.0
11	13.39197	29.492	74.0
13	13.39219	29.490	74.0

Using first seven readings only.
R = 13.39038 ohms at 217.21 °C.

3. 8 April 1916.

0	13.39655		
1	13.39666	29.510	67.5
3	13.39693	29.510	67.5
4	13.39696	29.510	67.5
5	13.39705	29.512	67.5
6	13.39745	29.510	67.5
7	13.39743	29.508	67.5
8	13.39749	29.509	67.5
9	13.39773	29.512	67.5
10	13.39773	29.511	67.5
11	13.39811	29.522	67.5
12	13.39841	29.520	67.0
15	13.39861	29.520	67.0

Using first ten readings only.
R = 13.39720 ohms at 217.28 °C.

4. 8 April 1916.

0	13.39964		
1	13.39983		
2	13.39931	29.513	65.0
3	13.39849	29.510	65.0

Time Mins.	Resistance Ohms	Barometer Inches	Bar. Temp. 0° F.
4	13.39849	29.511	65.0
5	13.39890	29.510	65.0
6	13.39900	29.508	65.5
7	13.39893	29.508	65.5
8	13.39909	29.510	65.5
10	13.39938	29.524	65.5

$R = 13.39911$ ohms at 217.28 °C.

Computed values of δ_c:

1. 1.532
2. 1.488 The average of Nos. 2, 3, and 4 was taken as the
3. 1.487 value of δ_c.
4. 1.479 $\delta_c = 1.485 \pm 0.004$

4. **Calibration Data for Observing Thermometer.** See Part IV, Sec. 7.

Column 2 gives the resistance of the Standard Thermometer, and Column 3 gives the resistance of the Observing Thermometer, the units being ohms. The Extension Coil Link was in the 25 ohm position. The readings in Column 3 were not made simultaneously with those in Column 2 but were made as soon after as the bridge could be reset.

1. 6 May 1916. Oil Bath at 63.2 °C.

Time Mins.	Standard Thermometer Ohms	Observing Thermometer Ohms
0	7.27945	5.01562
2	7.27955	5.01578
4	7.27923	5.01575
7	7.27937	5.01561
9	7.27917	5.01566
12	7.27970	5.01550
14	7.27952	5.01572
17	7.27911	5.01563
19	7.28018	5.01562
22	7.27973	5.01583
Averages	7.27950	5.01567

2. 7 May 1916. Oil Bath at 63.2 °C.

Time Mins.	Standing Thermometer Ohms	Observing Thermometer Ohms
0	7.27958	5.01609
2	7.27932	5.01589
4	7.28011	5.01593
6	7.27958	5.01605
9	7.27935	5.01592
11	7.27991	5.01594
13	7.27929	5.01604
16	7.27959	5.01580
Averages	7.27959	5.01596

3. 7 May 1916. Oil Bath at 114.8 °C.

0	12.42866	8.49468
2	12.42795	8.49478
5	12.42763	8.49459
7	12.42819	8.49466
9	12.42813	8.49460
11	12.42898	8.49475
13	12.42861	8.49488
16	12.42773	8.49488
18	12.42888	8.49463
21	12.42876	8.49500
Averages	12.42835	8.49475

4. 7 May 1916. Oil Bath at 25.1 °C.

0	3.43125	2.41286
2	3.43121	2.41288
4	3.43135	2.41277
6	3.43155	2.41290
9	3.43159	2.41299
11	3.43172	2.41306
14	3.43188	2.41306
17	3.43165	2.41313
19	3.43153	2.41318
21	3.43136	2.41309
Averages	3.43151	2.41299

5. 8 May 1916. Oil Bath at 24.3 °C.

Time Mins.	Standing Thermometer Ohms	Observing Thermometer Ohms
0	3.34948	2.35780
2	3.34917	2.35761
4	3.34906	2.35738
5	3.34913	2.35728
8	3.34942	2.35718
10	3.34946	2.35718
11	3.34957	2.35723
13	3.34954	2.35731
15	3.34967	2.35739
17	3.34997	2.35742
Averages	3.34945	2.35738

6. 8 May 1916. Oil Bath at 62.7 °C.

0	7.23368	4.98518
2	7.23407	4.98496
5	7.23430	4.98506
7	7.23404	4.98498
11	7.23357	4.98495
14	7.23414	4.98490
18	7.23410	4.98494
20	7.23368	4.98478
24	7.23336	4.98456
26	7.23420	4.98473
Averages	7.23391	4.98490

7. 9 May 1916. Oil Bath at 113.0 °C.

0	12.24023	8.36802
3	12.24053	8.36785
6	12.24212	8.36840
8	12.24130	8.36870
11	12.24036	8.36805
13	12.24197	8.36855
15	12.24086	8.36840
17	12.24203	8.36842
19	12.24084	8.36836
22	12.24127	8.36799
Averages	12.24115	8.36827

Average Results: $R_{100^\circ} - R_{0^\circ} = 6.81925$ ohms.
$R_{0^\circ} = 0.68084$ ohms.
$\delta_c = 1.600$

5. **Mass of Ammonia.** See Part IV, Sec. 8.

1. 3 April 1916. 8:20 P. M.

Apparent Mass of Container Empty	1163.6746 grams.
Temperature in Balance Case	20.4 °C.
Barometer	29.431 inches.
Temperature of Barometer	68.0 °F.

2. 3 April 1916. 9:08 P.M.

Apparent Mass of Container Empty	1163.6741 grams.
Temperature in Balance Case	19.8 °C.
Barometer	29.412 inches.
Temperature of Barometer	67.0 °F.

3. 3 April 1916. 10:50 P. M.

Apparent Mass of Container Empty	1163.6720 grams.
Temperature in Balance Case	18.7 °C.
Barometer	29.400 inches.
Temperature of Barometer	65.0 °F.

Inside volume of container (see Part V, Sec. 6) approx.	273 cc.
Volume of steel of container, approx.	149 cc.

Weights, brass; small weights, quartz.

Fortin Barometer, brass scale; instrumental correction + 0.023 in.

Reduction to Vacuo:

1. 1164.0094 grams.
2. 1164.0097 grams.
3. 1164.0088 grams.

Average 1164.0093 = True Mass of Container Empty.

---

1. 4 April 1916. 9:10 P. M.

Apparent Mass of Container Filled with NH_3	1266.3938 grams.
Temperature in Balance Case	21.2 °C.
Barometer	29.530 inches.
Temperature of Barometer	71.0 °F.

2. 4 April 1916. 10:15 P. M.

Apparent Mass of Container Filled with NH_3	1266.3918 grams.
Temperature in Balance Case	21.2 °C.
Barometer	29.576 inches.
Temperature of Barometer	70.5 °F.

3. 4 April 1916. 11:00 P. M.

Apparent Mass of Container Filled with NH_3	1266.3905 grams.
Temperature in Balance Case	21.2 °C.
Barometer	29.576 inches.
Temperature of Barometer	70.5 °F.

4. 5 April 1916. 11:30 A. M.

Apparent Mass of Container Filled with NH_3	1266.3883 grams.
Temperature in Balance Case	20.5 °C.
Barometer	29.560 inches.
Temperature of Barometer	71.0 °F.

5. 5 April 1916. 2:20 P. M.

Apparent Mass of Container Filled with NH_3	1266.3902 grams.
Temperature in Balance Case	21.4 °C.
Barometer	29.518 inches.
Temperature of Barometer	72.0 °F.

6. 5 April 1916. 3:00 P. M.

Apparent Mass of Container Filled with NH_3	1266.3905 grams.
Temperature in Balance Case	21.6 °C.
Barometer	29.516 inches.
Temperature of Barometer	73.0 °F.

Reduction to Vacuo:

1. 1266.7136 grams.
2. 1266.7121 grams.
3. 1266.7108 grams.
4. 1266.7095 grams.
5. 1266.7097 grams.
6. 1266.7100 grams.

Average 1266.7110 grams = True Mass of Container filled with NH_3.

Mass of Ammonia enclosed in container = 102.7017 grams

6. **Volume of Ammonia Container.**

1. 31 March 1916.

Apparent Mass of Container Empty	1163.6765 grams.
True Mass of Container Empty (in Vacuo)	1164.0175 grams.
Apparent Mass of Contianer filled with distilled water at the temperature 22.5° C.	1436.7602 grams.
True Mass of Container filled (in Vacuo)	1437.0619 grams.

V = 273.6779 cc.

2. 1 April 1916.

Apparent Mass of Container Empty	1163.7653 grams.
True Mass of Container Empty (in Vacuo)	1164.1063 grams.
Apparent Mass of Container filled with distilled water at the temperature 23.5 °C.	1435.7703 grams.
True Mass of Container filled (in Vacuo)	1436.0718 grams.

V = 272.6615 cc.

3. 1 April 1916.

Apparent Mass of Container Empty	1163.8013 grams.
True Mass of Container Empty (in Vacuo)	1164.1423 grams.
Apparent Mass of Container filled with distilled water at the temperature 21.5 °C.	1436.5200 grams.
True Mass of Container filled (in Vacuo)	1436.8217 grams.

V = 272.9766 cc.

Average value of V = 273.11 cc. ± 0.38 cc.

v = 2.6593 ccs./gram.

---

4. 11 July 1919

Apparent Mass of Container Empty	1167.6036 grams.
True Mass of Container empty (in Vacuo)	1167.9352 grams.
Apparent Mass of Container filled with distilled water at the temperature 26.60 °C.	1440.8404 grams.
True Mass of Container filled (in Vacuo)	1441.1329 grams.
Apparent Mass of Container filled with distilled water at the temperature 78.0 °C.	1434.3345 grams.

True Mass of Container filled (in Vacuo)	1434.6228 grams.
Volume of Container at 26.60 °C.	274.113 cc.
Volume of Container at 78.0° C.	274.054 cc.

This change in V is within the experimental error and the conclusion is that δV is negligible.

7. E. M. F. of Standard Cell.

Certified by the U. S. National Bureau of Standards, Oct. 18, 1915.

"Unsaturated Weston Standard Cell # 1458.

Electromotive Force at 22.1° C. = 1.0180_8 international volts. The above value is correct to 0.01 percent."

8. Resistance of Standard Ohm.

Certified by the Leeds & Northrup Co. Oct. 11, 1915.

"No. 4020 Standard Resistance, Serial # 27659

1.000_3 International Ohms at 20° C.

Its temperature coefficient between 15° and 30° is

0.000007 per degree Centigrade.

The error in the above comparison is not greater than 1/100th%."

9. Quality of Ammonia Used.

The ammonia used in these experiments was commercial anhydrous ammonia, NH_3, obtained from Armour & Co., Chicago, in February 1916, and shipped in a fifty pound drum.

As to the purity of the ammonia, Mr. J. R. Powell, Chief Chemist, Armour Ammonia Works, may be quoted:

"As to the impurities that might be expected present, there will probably be traces of water, carbon dioxide, and possibly methyl cyanide, but the total of these impurities should not be over one or two hundredths of one per cent. Likewise there may be bare traces of foreign gas dissolved in the ammonia, these gases being principally air. Anhydrous ammonia, as manufactured today is practically a chemically pure product."

10. Data on Method of Filling Container with Ammonia.

The ammonia container was connected to the ammonia tank by means of a copper pipe of 5 mm. outside diameter. In this copper pipe was inserted a glass Tee, the branch of which led to a vacuum pump equipped with a gauge. The copper-glass joints were made with Khotinsky cement.

The ammonia valve was opened slightly, before connecting up, to clear the opening of any foreign matter which might be present in the valve, and then closed. The line was pumped down to a pressure of about 2 mm. of mercury, and then ammonia was admitted to the system and allowed to stand for about twenty minutes. The line was again pumped down to about 2 mm. of mercury and the glass tube leading to the pump sealed off. A thermos bottle had been previously filled with solid carbon dioxide and the ammonia container was now immersed in this CO_2 snow and allowed to cool for fifteen minutes. The ammonia valve was again opened and ammonia admitted to the system until no more flowed. The pipe line was then unscrewed and the filling plug screwed in place, there being no appreciable escape of ammonia from the container during this operation.

After the container had warmed up to room temperature it was weighed, and the ammonia allowed to escape until the mass of ammonia in the container had been reduced to slightly over 100 grams. The exact mass of ammonia has been given above.

It was desired to use as large a mass of ammonia as possible in the container but to so load it that all the vapor would not be condensed at the highest temperature contemplated for these measurements. Using Keyes and Brownlee's data on the specific volume of the liquid, it was determined that with a filling of 100 grams the vapor would be condensed at a temperature of about 123° C. which put an upper limit on the measurements, for that particular filling. It was originally intended to let out about fifty grams of ammonia, after these measurements had been made up to 123° C. and go on up to the critical point, 132.9° C. but time was not available to do this.

11. **Data on Calorimetric Lag.**

In Part IV, Section 4, the various calorimetric lags are discussed. With the assumptions and methods there outlined, the following data was obtained from the observed calorimeter curves.

The length of time taken for the mercury in the calorimeter to equalize its differences of temperature after the heater was switched off, was, on the average, about 12 secs., varying from 10 to 14 secs. The rise of temperature of the mercury after the heater was switched off was, on the average, about 0.00030 ohms on the observing thermometer, or about 0°.0044 C. The time taken to equalize the differences of temperature in the entire calorimeter after the heater was switched off, was different for the ammonia measurements and for the calibration measurements, averaging 11 minutes for the former, and 15 minutes for the latter. In the case of the ammonia measurements, the temperature of the mercury at the instant the heater was switched off, was, on the average, about 0°.046 C. hotter than the equalized temperature of the calorimeter at the same instant. In the case of the calibration measurements, it was 0°.12 C. hotter than the equalized temperature.

12. **Data on Accuracy of Chronograph.**

The chronograph was controlled by a laboratory pendulum clock which was compared with standard time for several days before the final measurements were made, and was found to be correct to within a few seconds a day.

The distance between second marks made by the pen on the chronograph sheet, were approximately 1 cm., and the record marks could be located to within $\frac{1}{2}$ mm., giving an error at each end of the time interval of not more than 1/20th sec. The time record was made automatically by throwing the heater switch on and off.

13. **Data on Regulation of Constant Temperature Bath.**

The thermostat maintained the bath temperature constant within a range of 0°.02 C. and in most measurements, within a range of 0°.01 C. In Ammonia Measurement 9, the bath got out of control and the temperature shows a variation of 0°.07 C. This variation is in a direction to give too high a value of $\frac{\delta H_1}{\delta \theta}$.

VI. EXPERIMENTAL RESULTS.

1. ***Computed and Faired Values of Heat Capacity of Apparatus:***

(a) **When Filled with Ammonia.** (b) **When Empty.**

Equations (6) and (18) are the laboratory equations by which the values of δH and $\delta \theta$ are computed. Equation (13) gives the value of $k/2$.

These experimental results are summarized in the following table, whose explanation is as follows:

Column 1.	Number of the measurement, for reference to the tables from which the values were computed.
Column 2.	Temperature of the measurement, in °C., taken as the temperature equivalent to R_m for that measurement.
Column 3.	Experimental values of $\frac{\delta H_1}{\delta \theta}$ and $\frac{\delta H_2}{\delta \theta}$, in Joules /°C., computed by equations (6) and (18) from the tabulated data.

Column 4. Faired values of $\frac{\delta H_1}{\delta \theta}$ and $\frac{\delta H_2}{\delta \theta}$, obtained by drawing the best representative curve through the data in Column 3, plotted as a function of temperatures given in Column 2. Units: Joules /°C.

Column 5. Percentage deviation of the values of Column 3, from those of Column 4.

Column 6. Values of $k/2$ in minutes $^{-1}$.

EXPERIMENTAL RESULTS. TABLE I.

No.	θ_m	Ammonia Measurements. $\frac{\delta H_1}{\delta \theta}$ Observed.	 $\frac{\delta H_1}{\delta \theta}$ Curve.	% Diff.	$k/2$
1	32.9	2135.8	2136.0	0.00	0.00271
2	39.6	2174.4	2151.0	+1.09	0.00203
3	40.3	2125.4	2153.0	−1.28	0.00211
4	51.9	2169.0	2179.0	−0.46	0.00211
5	62.9	2212.6	2206.0	+0.30	0.00225
6	72.7	2245.4	2231.0	+0.65	0.00262
7	83.2	2265.5	2258.7	+0.30	0.00247
8	92.8	2275.3	2287.5	−0.53	0.00291
9	102.9	2332.2	2331.0	+0.09	0.00214
10	102.9	2327.9	2331.0	−0.13	0.00266
11	113.0	2396.3	2391.0	+0.22	0.00219
12	122.5	2469.9	2478.7	−0.35	0.00244
		Calibration Measurements. $\frac{\delta H_2}{\delta \theta}$	 $\frac{\delta H_2}{\delta \theta}$		
1	23.3	1604.2	1602.9	+0.08	0.00288
2	26.9	1641.9	1606.8	+2.18	0.00170
3	39.8	1617.3	1619.0	−0.10	0.00253
4	39.8	1618.8	1619.0	0.00	0.00286
5	40.2	1625.5	1619.5	+0.37	0.00300
6	40.2	1616.6	1619.5	−0.18	0.00265
7	40.2	1616.9	1619.5	−0.16	0.00276
8	60.9	1635.2	1641.5	−0.38	0.00395
9	79.9	1691.6	1661.0	+1.84	0.00252
10	82.7	1675.9	1664.0	+0.71	0.00359
11	100.6	1675.5	1684.2	−0.52	0.00362
12	121.9	1708.7	1708.7	0.00	0.00406

Average Deviation of $\frac{\delta H_1}{\delta\theta}$ without regard to sign, $= 0.45\%$.

Average Deviation of $\frac{\delta H_2}{\delta\theta}$ without regard to sign $= 0.54\%$.

Figure 8 is a plot of the values of the heat capacity of the apparatus both full of ammonia and empty, and shows the computed values and the faired curves drawn through these points.

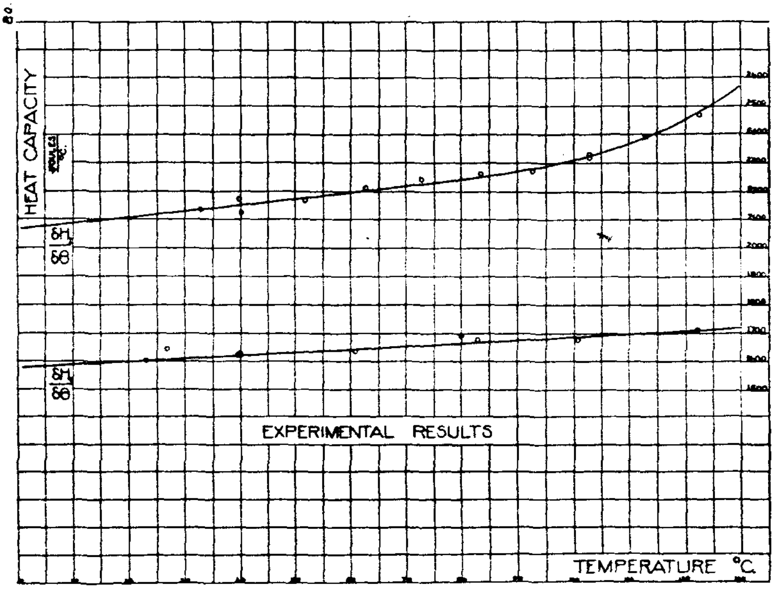

Figure 8.

2. **Specific Heat at Constant Specific Volume of Mixture of Liquid and Vapor.**

The specific heat of the mixture of liquid and vapor at the constant specific volume 2.6593 ccs./gram, was computed from Equation 2, using the faired values of $\frac{\delta H_1}{\delta\theta}$ and $\frac{\delta H_2}{\delta\theta}$ given in Table I and the value of M previously given.

These values are given for every five degrees Centigrade in the following table.

EXPERIMENTAL RESULTS. TABLE II.

θ in °C.	$c_{v_2 \cdot 6593}$ in Joules/Gm.-degree.	θ in °C.	$c_{v_2 \cdot 6593}$ in Joules/Gm.-degree.
30	5.075	80	5.737
35	5.132	85	5.815
40	5.190	90	5.907
45	5.251	95	6.021
50	5.313	100	6.165
55	5.378	105	6.344
60	5.446	110	6.570
65	5.515	115	6.878
70	5.587	120	7.262
75	5.660	125	7.802

These values are plotted in Figure 9.

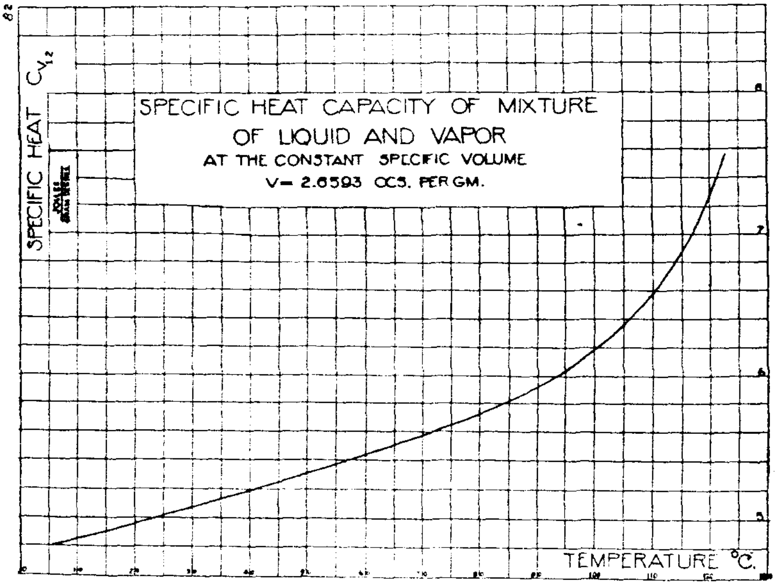

FIGURE 9.

VII. THE SATURATION SPECIFIC HEAT OF THE LIQUID c_{s_2}, AND THE SATURATION SPECIFIC HEAT OF THE VAPOR c_{s_1}, IN TERMS OF $c_{v_{12}}$.

1. **Derivation of Equations.**

Let the total mass of substance in the liquid container be M, and since this is constant

$$\delta M = 0 \tag{21}$$

Let M_1 and M_2 be the masses of the vapor and liquid phases respectively. These are variables and functions of the temperature but related by the equation

$$M_1 + M_2 = M \tag{22}$$

Let V = the volume of the container, a constant,
and call $V/M = v$.
Let v_1 = the specific volume of the saturated vapor, and
v_2 = the specific volume of the saturated liquid.
Then

$$v_1 M_1 + v_2 M_2 = V = Mv \tag{23}$$

If $\delta\theta$ = the temperature change of the substance due to the heat δQ added to that substance, and
L = the total heat of vaporization of the liquid phase, then

$$\delta Q = c_{s_2} M_2\,\delta\theta + c_{s_1} M_1\,\delta\theta + L\,\delta M_1 \tag{24}$$

By means of (21), (22) and (23), M_2, M_1, and δM_1, are eliminated from (24). Rearranging terms after this elimination,

$$\frac{1}{M}\frac{\delta Q}{\delta\theta} = c_{s_2}\frac{v_1 - v}{v_1 - v_2} + c_{s_1}\frac{v - v_2}{v_1 - v_2} - \frac{L}{v_1 - v_2}\left\{\left(1 - \frac{v - v_2}{v_1 - v_2}\right)\frac{\delta v_2}{\delta\theta} - \left(\frac{v - v_2}{v_1 - v_2}\right)\left(-\frac{\delta v_1}{\delta\theta}\right)\right\} \tag{25}$$

By Equation 1

$$c_{v_{12}} = \frac{1}{M}\frac{\delta Q}{\delta\theta} \tag{1}$$

and a general thermodynamic relation [15] gives

$$c_{s_1} = c_{s_2} - \left(\frac{L}{T} - \frac{dL}{dT}\right) \tag{26}$$

---

[15] Reference 3, p.

where T denotes absolute temperature, as distinguished from θ which is used to denote the usual Centigrade scale.
Substituting (1) and (26) in (25) and solving for c_{s_2},

$$c_{s_2} = c_{v_{12}} + \left(\frac{v - v_2}{v_1 - v_2}\right)\left(\frac{L}{T} - \frac{dL}{dT}\right) - \frac{L}{v_1 - v_2}\left\{\left(\frac{v - v_2}{v_1 - v_2}\right)\left(-\frac{\delta v_1}{\delta\theta}\right) - \left(1 - \frac{v - v_2}{v_1 - v_2}\right)\frac{\delta v_2}{\delta\theta}\right\} \tag{27}$$

By use of Equations (26) and (27), the specific heats along the boundary line of the liquid vapor region shown in Figure 2, can be computed, when the experimentally determined values of $c_{v_{12}}$ (Table II), and the thermodynamic quantities, L, v_1 and v_2, are given as functions of the temperature.

In Equation 27, v denotes the specific volume at which $c_{v_{12}}$ was measured. But from values of $c_{v_{12}}$ as a function of the temperature at some one particular specific volume, the values of $c_{v_{12}}$ at any other specific volume can be computed. Let the volume at which the measurements are made be denoted v_a instead of v, and let the specific heat at this specific volume be c_{va} where the subscripts $_1$ and $_2$ are understood. Similarly, let c_{vb} be the specific heat of the liquid-vapor mixture at the specific volume v_b. To obtain an equation connecting c_{va} and c_{vb}, write Equation 27 twice, once using c_{va} and v_a, and again using c_{vb} and v_b. Subtracting these two and rearranging

$$c_{va} - c_{vb} = \left(\frac{v_b - v_a}{v_1 - v_2}\right)\left\{\frac{L}{T} - \frac{dL}{dT} - \frac{L}{v_1 - v_2}\left(-\frac{\delta v_1}{\delta\theta} + \frac{\delta v_2}{\delta\theta}\right)\right\} \tag{28}$$

Further, the specific heat along any path whatsoever,[16] may be written

$$c = c_v + T\left(\frac{\partial p}{\partial T}\right)_v \frac{dv}{dT} \tag{29}$$

where p is the pressure and $\frac{dv}{dT}$ depends on the path.

These equations, (26), (27), (28) and (29), show that the specific heat along any path in the liquid-vapor region, and along the boundary of that region, may be computed from values of the specific heat along some one constant specific volume line in that region. Such values of c_v at the specific volume 2.6593 ccs./gram, have been given in Table II. over the temperature range 30° C. to 125° C.

---

[16] Reference (3) p. 5. Modification of Eq. 10.

2. **Keyes-Brownlee Data for Ammonia.**

To compute the numerical values of c_{s_2} and c_{s_1} from these values of c_v, by means of (26) and (27) the data of Keyes and Brownlee is used.

For the heat of vaporization, Keyes,[17] using 1 cal.: = 4.182 Joules, gives

$$\log_{10} L = 1.56817 - 2.822 \cdot 10^{-5} (406.0 - T) + 0.43387 \log_{10} (406.0 - T) \qquad (30)$$

where L is in calories per gram; and T is in degrees absolute on the Centigrade scale.

From this equation we obtain

$$\left(\frac{dL}{dT} - \frac{L}{T}\right) = \mathrm{L} \left\{ 6.49788 \cdot 10^{-5} - \frac{0.43387}{406.0 - T} - \frac{1}{T} \right\} \qquad (31)$$

In the following table are given the values of L and of $\left(\frac{L}{T} - \frac{dL}{dT}\right)$ for each 10°C. computed from equations (30) and (31). The first column gives the temperature in degrees Centigrade; the units of column two and three, are Joules per gram, and Joules per gram-degree, respectively.

θ	L	$\frac{L}{T} - \frac{dL}{dT}$
0	1279.8	8.7807
10	1237.9	8.6616
20	1194.2	8.5863
30	1147.8	8.5511
40	1098.7	8.5688
50	1046.4	8.6464
60	990.3	8.8018
70	929.5	9.0608
80	862.8	9.4632
90	788.3	10.0918
100	703.0	11.1088
110	601.0	12.9161
120	468.9	16.9329
130	245.5	36.9541

---

[17] Reference (3) p. 24. Eq. 34.

The specific volume of the vapor, v_1, was obtained by solving Keyes' equation of state for ammonia, simultaneously with the equation for the vapor pressure as a function of the temperature. The equation of state is:[18]

$$p = \frac{R_k T}{v - \delta_k} - \frac{a}{(v - l)^2} \tag{32}$$

$R_k = 4.8177$ $\log_{10}\delta_k = 0.98130 - \frac{3.08}{v}$

$a = 34610.1$ $l = -1.173$

p in atmos. T in degrees absolute on Centigrade scale.

v in ccs./gram.

The vapor pressure equation[19] is:

$$\log_{10}p = -1969.65\,T^{-1} + 16.19785 - 0.0323858\,T + 5.4131\cdot10^{-5}\,T^2 - 3.2715\cdot10^{-8}\,T^3 \tag{33}$$

where p is pressure in mm. of mercury, and T temperature on the absolute scale in degrees centigrade.

The method of solving these two equations is to substitute values of v in (32) thus reducing (32) to an equation giving p as a linear function of T for any given volume. Equation 33 was solved for p for each 5° C. and these values plotted to a large scale. The straight lines connecting p and T were then drawn on the same diagram, and from the point of intersection of each line with the curve, the values of T were read off. Since each straight line corresponds to a value of v, this process gives v_1 as a function of T and therefore as a function of θ.

The following tables give the computed values of p and v_1 as a function of θ. θ is in °C., p in atmos. and v_1 in ccs./gm.

θ	p	θ	p
10	6.128	75	36.697
15	7.256	80	40.937
20	8.535	85	45.567
25	9.979	90	50.553
30	11.605	95	55.973
35	13.419	100	61.797
40	15.438	105	68.053

---

[18] Reference (3) p. 20. Eq. 33.
[19] Reference (3) p. 13. Eq. 30.

θ	p	θ	p
45	17.681	110	74.782
50	20.164	115	82.036
55	22.901	120	89.753
60	25.897	125	98.061
65	29.186	130	106.883
70	32.778	132.9	112.281

v_1	θ	v_1	θ
16	104.33	104	32.03
18	99.76	115	28.64
20	95.50	126	25.66
30	78.86	137	22.79
40	67.39	148	20.24
51	57.99	160	17.80
62	50.67	172	15.48
72	45.21	184	13.39
82	40.45	196	11.31
93	35.92	208	9.73

The equation of state (32) with the constants there given does not hold very far beyond 100° C., so that values of the saturation specific heats cannot be computed in the range 100° C. – 125° C., until further experimental data are available.

The following empirical equation was set up by the writer to represent v_1 as a function of θ,

$$v_1 = 338.990 - 172.7327\,\theta + 46.2976\,\theta^2 - 7.6262\,\theta^3 + 0.73653\,\theta^4 - 0.035417\,\theta^5 + 0.0005278\,\theta^6 \qquad (34)$$

This equation readily yields the values of $\frac{dv_1}{d\theta}$,[20] required by Equation 27. The following table gives $\left(-\frac{dv_1}{d\theta}\right)$ in ccs./gm.-degrees, for each 10° C. as computed from Equation 34.

---

[20] The quantities $\frac{\delta v_1}{\delta \theta}$ and $\frac{\delta v_2}{\delta \theta}$ are here replaced by the differential coefficients $\frac{dv_1}{d\theta}$ and $\frac{dv_2}{d\theta}$, the error introduced being negligible.

θ	$-\dfrac{dv_1}{d\theta}$
0	9.6202
10	6.6829
20	4.6430
30	3.2561
40	2.3255
50	1.6988
60	1.2645
70	0.9487
80	0.7120
90	0.5456
100	0.4687

Keyes gives a table of values of the specific volume of the liquid,[21] v_2, at various temperatures as shown in the table below; θ in °C. and v_2 in ccs./gram.

θ	v_2
0	1.5657
20	1.6387
40	1.7256
60	1.8331
80	1.9747
100	2.1836

The following empirical equation was set up by the writer to represent v_2 as a function of θ.

$$v_2 = 1.49380 + 0.079723\tau - 0.014021\,\tau^2 + 0.0074209\tau^3 - 0.00132917\,\tau^4 + 0.0001075\,\tau^5 \quad \text{where } \tau = (\theta/20 + 1) \tag{35}$$

By differentiation, the values of $\dfrac{dv_2}{d\theta}$ are obtained. These values of $\dfrac{dv_2}{d\theta}$ in ccs./gram-degree for each 10° C. as computed from (35) are given below.

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[21] Reference (3) p. 15.

θ	$\frac{dv_2}{d\theta}$
0	0.002458
10	0.003626
20	0.003938
30	0.004334
40	0.004797
50	0.005350
60	0.006055
70	0.007014
80	0.008372
90	0.010311
100	0.013056

All of the quantities occurring in the right hand members of Equations (26) and (27) are now computed, and the values of c_{s_2} and c_{s_1} may be obtained by substitution in those two equations.

3. Computed Values of the Saturation Specific Heats.

Table III below, gives the values of the saturation specific heats for each 5° C. from 30° C. to 100° C. The first column gives the

TABLE III.

θ	$c_{s_2 .6593}$	Corr. Term	c_{s_2}	c_{s_1}
30	5.075	−0.187	4.888	−3.663
35	5.132		4.937	−3.616
40	5.190	−0.199	4.991	−3.578
45	5.251		5.048	−3.550
50	5.313	−0.202	5.111	−3.535
55	5.378		5.179	−3.534
60	5.446	−0.194	5.252	−3.550
65	5.515		5.335	−3.583
70	5.587	−0.159	5.428	−3.633
75	5.660		5.538	−3.703
80	5.737	−0.072	5.665	−3.798
85	5.815		5.816	−3.930
90	5.907	+0.079	5.986	−4.106
95	6.021		6.181	−4.342
100	6.165	+0.268	6.433	−4.676

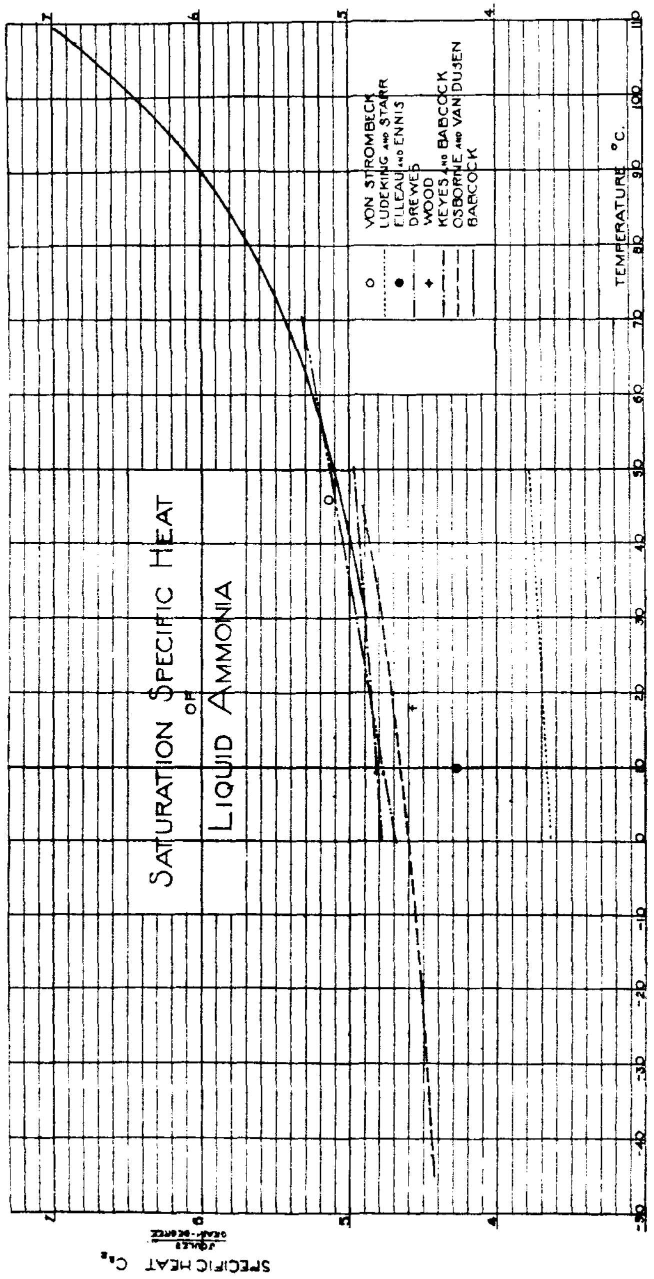
SATURATION SPECIFIC HEAT OF LIQUID AMMONIA
○ VON STROMBECK
● LUDEKING AND STARR
+ FILLEAU AND ENNIS
DREWES
WOOD
KEYES AND BABCOCK
OSBORNE AND VAN DUSEN
BABCOCK
TEMPERATURE °C.
110
100
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
.7
.6
.5
.4
SPECIFIC HEAT C_s
JOULES / GRAM-DEGREE

FIGURE 10.

temperature in °C. and the second, c_v from Table II. In the third column, for every 10° C. is given the correction term to be added to c_v to give c_{s_2}. The fourth column gives c_{s_2} and the fifth gives c_{s_1}, whose values, it will be noted, are negative. All of the specific heats are expressed in Joules per gram degree. The values of the saturation specific heats given for 35°, 45°, 55°, etc., were obtained by interpolation, not computed.

The saturation specific heat of the liquid is shown by the heavy curve in Figure 10, and the saturation specific heat of the vapor is shown in Figure 11.

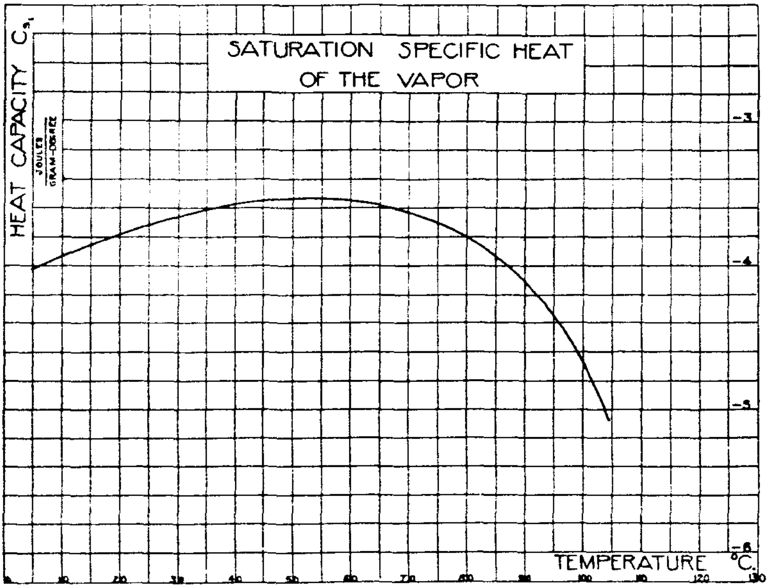

FIGURE 11.

4. Comparison with Previous Results.

Osborne and Van Dusen,[22] in their recent work, have discussed briefly the measurements of the specific heat of ammonia made previously to their own investigation. There is little to add to this except perhaps to mention a little more in detail the work of Drewes, reported by Dieterici.[23]

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[22] Osborne & Van Dusen, Bull. Bur. of Standards, **14**, p. 397.

Drewes used the Bunsen Ice Calorimeter in the range 0°–70° C. He placed 0.5 grams of ammonia in a glass vessel whose volume was 1.2 cc. His results were expressed in terms of a calorie which appears to have the value 4.187 Joules. He assumed that the specific heat at constant volume was a linear function of the temperature, and gives the following equation for c_v.

$$c_v = 1.118 + 0.00208\,\theta \qquad 0° < \theta < 70° \tag{36}$$

where c_v is expressed in calories, and θ is in degrees C. Dieterici corrects only for the external work of liquid expansion in computing the saturation specific heat of the liquid, and gives the following equation for c_{s_2},:[24]

$$c_{s_2} = 1.118 + 0.00220\,\theta \qquad 0° < \theta < 70° \tag{37}$$

Dieterici gives only the approximate values of the volume of the container and the mass of ammonia enclosed therein, so it is not possible to apply Equation (27) to his equation for c_v. The specific volume at which his measurements were carried out was approximately 2.4 ccs./gram and hence the correction for the heat capacity of the vapor, the heat of vaporization, and the expansion of the liquid would appear to make his c_{s_2} *less* than c_v in the temperature region in which he operated, rather than larger, as he states, so that it is probable that his quoted values of c_{s_2} are too high, though how much too high it is not possible to determine.

The most accurate measurements of the specific heat of ammonia

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[23] Dieterici, Zeit. f. d. Gesamte Kalte-Industrie. (1904), pp. 21, 47.

[24] The values given for the specific heat of ammonia by Landolt & Bornstein, Physikalisch-Chemische Tabellen, 1905 Edition, are as follows:

θ in °C.	c in cal./gm.-degree
0	0.876
10	1.140
20	1.190
30	1.218
40	1.231
50	1.239
60	1.240
70	1.233

These values do not appear in the article by Dieterici, Reference 22, and the writer has no explanation for their appearance in the Tabellen.

which have yet appeared are those of Osborne and Van Dusen.[25] Using an aneroid calorimeter of special construction, the specific heat both at constant volume and at constant pressure was determined in the temperature region which is most useful in the refrigeration industry, − 45° C. to + 45° C.

Especial attention was paid to the elimination of heat transfer to the surroundings.

The final results of their measurements are expressed by the equation,

$$c_{s_2} = 3.1365 - 0.00057\,\theta + \frac{16.842}{(133-\theta)^{\frac{1}{2}}} \tag{38}$$

where θ is in degrees C., and the saturation specific heat of the liquid is in Joules per gram degree C.

These experimenters put the precision of their results at 1 part in 1000.

The work of the present paper overlaps by about 15° the range of Osborne and Van Dusen's measurements, and in this overlapping region, differs from their results by about two and a quarter per cent. The difference between the values of v_1, v_2, and L, used by Osborne and Van Dusen, and the writer, is too slight to account for the difference in the saturation specific heats.

In Table IV. are given the values of the saturation specific heat of liquid ammonia, expressed in Joules per gram degree, for comparison. Figure 10 shows graphically the various reported values of the saturation specific heat of the liquid.

VIII. Critical Survey.

The experimental method and apparatus described in this paper were designed for the measurement of the specific heat of ammonia as a function of temperature up to the critical point. From the outset, the conditions met with at the higher temperatures determined the design of the apparatus and placed a limit on the accuracy attainable. At the highest temperature reached, 125° C., the pressure in the container was about 100 atmospheres, which necessitated a rather massive container, which in turn gave rise to temperature lag difficulties already discussed. Furthermore, to obtain measurements of the specific heat in the region near the critical temperature, where it is

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[25] Osborne & Van Dusen, Bull. Bur. of Standards, **14**, p. 397.

TABLE IV.

COMPARISON OF VALUES OF THE SATURATION SPECIFIC HEAT OF THE LIQUID.

θ	Von Strombeck 1890	Ludeking and Starr 1893	Elleau and Ennis 1898	Drewes 1904	Wood 1912	Keyes and Babcock 1912	Osborne and Van Dusen 1916	Bab-cock 1916
−40	.....	.....	.....	.....	.....	.....	4.442	.....
−30	.....	.....	.....	.....	.....	.....	4.476	.....
−20	.....	.....	.....	.....	.....	.....	4.509	.....
−10	.....	.....	.....	.....	.....	.....	4.551	.....
0	.....	.....	.....	4.681	.....	.....	4.597	.....
+10	.....	.....	4.270	4.773	.....	4.818	4.651	.....
+13	.....	3.672	.....	4.799	.....	.....	4.668	.....
+18	.....	.....	.....	4.844	4.576	.....	4.698	.....
+20	.....	.....	.....	4.861	.....	.....	4.710	.....
+30	.....	.....	.....	4.945	.....	.....	4.777	4.888
+35	.....	.....	.....	4.991	.....	4.901	4.819	4.939
+36	.....	3.739	.....	5.000	.....	.....	4.827	4.949
+40	.....	.....	.....	5.037	.....	.....	4.861	4.991
+46	5.140	.....	.....	5.092	.....	.....	4.919	5.066
+50	.....	.....	.....	5.129	.....	.....	.....	5.111
+60	.....	.....	.....	5.221	.....	.....	.....	5.252
+70	.....	.....	.....	5.313	.....	.....	.....	5.428
+80	.....	.....	.....	.....	.....	.....	.....	5.665
+90	.....	.....	.....	.....	.....	.....	.....	5.986
+100	.....	.....	.....	.....	.....	.....	.....	6.433

known that the specific heat is increasing very rapidly with increasing temperature, it was imperative that the ammonia be heated only over small temperature ranges so that the mean value of the specific heat so determined, would be very closely the actual value of the specific heat at the mean temperature of the experiment. These two requirements, (1) a massive container, and, (2) heating over small temperature ranges, seriously affect the precision attainable with this method, though it is believed that it is the only one so far proposed which is adapted to measurements of the specific heat in the region under investigation.

Probably the principal source of error lies in the fact that the parts of the calorimetric apparatus do not change temperature at the same rate during the heating, and that considerable time must elapse after the heating current is cut off before these temperature differences are equalized, which makes the equalized temperature at the time the switch is opened, difficult to determine.

Some time was devoted to an experimental study of the law of cooling of this calorimeter for temperature differences of 4° or 5° between the calorimeter and the bath, and these experiments showed, what is well known, that Newton's Law of Cooling is not exactly obeyed. The actual law was complicated and no simple expression was derived for it. The error made by the assumption of Newton's Law, is however, much smaller than the error made by the uncertainty in the final equalized temperature. Its use was therefore considered to be legitimate under these conditions. The assumption of linearity of the initial and final lines is theoretically unjustifiable, but practically, no error is made by using them as such. A very slightly higher value of R_2 and therefore a lower value of c_v would result from taking account of the curvature of the final line.

The values of k in Newton's Law of Cooling, varied from one measurement to another, which is accounted for by the fact that the environment of the calorimeter was not absolutely reproducible. In a vacuum-jacketed calorimeter, a major part of the heat transfer to the surroundings takes place through the cover; and in this apparatus, the stirring shaft and leads served to conduct a considerable quantity of heat from the cover directly to the room, which in most measurements was considerably below the temperature of the calorimeter, and furthermore, was not controllable as was the oil bath. The most accurate method, therefore, of dealing with the heat transfer, seemed to be: to compute k for each measurement from the differences of slope and temperature of the initial and final lines.

The constancy of the stirring energy input is hardly open to question, because during all the measurements, readings were taken at one minute intervals of the voltage drop across the armature of the stirring motor, and in no case did this voltage change enough to give a measurable change of slope in the temperature curve. This effect was separately investigated by changing the motor voltage and observing the effect on the temperature-time curve. A change of 10% in the stirring motor voltage was found to be necessary before the slope of the temperature curve was changed by a measurable amount. During the measurements the voltage variations were very much less than this.

As the measurements were being made and computed, it was realized that the accuracy attainable by this method was less than had been hoped for, the average deviation of $\frac{\delta H_1}{\delta \theta}$ and $\frac{\delta H_2}{\delta \theta}$ from the curves taken to represent them, being of the order of 0.5%. No considerable amount of time was spent in checking these values, it being considered more profitable to spend the time improving the method so as to avoid the principal sources of error, and at the same time retain the essential idea, which is the measurement of the specific heat *at* a point.

These results are now published because there have hitherto been no data above 70° C., and the values of Drewes are not wholly reliable even to that temperature.

It is unfortunate that while these measurements of c_v extend to 125° C. or within 8° of the critical point, the values of c_{s_2} and c_{s_1}, can only be computed up to 100° C. owing to the lack of data on v_1 above that temperature.

ACKNOWLEDGMENTS.

The writer is under great obligations to William H. Gran, and Frederick L. Brown for their careful and conscientious work in taking observations during the final measurements.

To Mr. F. Küng, Departmental Mechanician, is due the mechanical excellence of the apparatus. The writer owes much to his continued interest in the investigation.

Throughout the course of the investigation, Dr. George V. McCauley has offered many suggestions of great value, and to him the writer's best thanks are due for his very material assistance.

To Professor Henry Crew, under whose direction the work was carried on, the writer offers his sincere thanks for his advice and assistance, as well as for the generous manner in which the facilities of the laboratory have been placed at his disposal, and special apparatus and equipment purchased.

Acknowledgment is made of a generous grant from the Rumford Fund of the American Academy of Arts and Sciences, which made possible the purchase of the Calorimetric Wheatstone Bridge.

IX. APPENDIX.

1. SUMMARY OF NOTATION.

δH	Any increment of energy added to the calorimetric system through the heating coil.
δH_1	An increment of energy added to the calorimetric system through the heating coil, container filled with ammonia.

δH_2	An increment of energy added to the calorimetric system through the heating coil, container empty.
$\delta\theta$	Change of temperature, on the Centigrade scale, of the calorimetric system due solely to the heat δH.
$\frac{\delta H}{\delta\theta}$	General expression for the heat capacity of the calorimetric system.
$\frac{\delta H_1}{\delta\theta}$	Heat capacity of the calorimetric system when the container is filled with ammonia.
$\frac{\delta H_2}{\delta\theta}$	Heat capacity of the calorimetric system when the container is empty.
δQ	That portion of δH_1 absorbed by the ammonia.
$\frac{\delta Q}{\delta\theta}$	Heat capacity of the ammonia.
θ	In general, temperature in ° C.; also used to denote the temperature of the observing thermometer at any instant.
θ_x	Any particular temperature θ.
θ_a	Temperature of the surroundings of the calorimeter.
θ_m	Mean temperature during a calorimetric temperature change.
θ_p	Platinum temperature defined by $\theta_p = \frac{100\,(R - R_{0^\circ})}{R_{100^\circ} - R_{0^\circ}}$
$\frac{d\theta_r}{dt}$	Time rate of change of temperature in the calorimeter due solely to heat transfer to the surroundings.
$\frac{d\theta_s}{dt}$	Time rate of change of temperature in the calorimeter due solely to stirring.
T	Absolute temperature on the Centigrade scale.
c	Specific heat capacity.
c_{s_1}	Saturation specific heat capacity of the vapor.
c_{s_2}	Saturation specific heat capacity of the liquid.
c_v	Specific heat capacity at constant specific volume.
$c_{v_{12}}$	Specific heat capacity at constant specific volume, of a mixture of liquid and vapor.
c_{v_a}	Specific heat capacity at the constant specific volume a.
c_{v_b}	Specific heat capacity at the constant specific volume b.
$c_{v_{2.6593}}$	Specific heat capacity at the constant specific volume 2.6593 ccs/gm.
v	Specific volume.
v_1	Specific volume of saturated vapor.
v_2	Specific volume of saturated liquid.
v_{12}	Specific volume of mixture of liquid and vapor.

v_a	The particular specific volume a.
v_b	The particular specific volume b.
$v_{2.6593}$	The particular specific volume 2.6593 ccs/gm.
V	Total volume enclosed by the container.
E	Average voltage drop across heating coil during a heating period.
I	Average current through heating coil during a heating period.
M	Total mass of ammonia in container.
M_1	Mass of ammonia vapor in container.
M_2	Mass of liquid ammonia in container.
p	Pressure.
L	Total heat of vaporization of the liquid.
K	Modulus in Newton's Law of Cooling, using Centigrade scale.
k	Modulus in Newton's Law of Cooling, using resistance scale.
R	Resistance of thermometer at temperature θ.
R_k	Constant in Keyes equation of state. (32)
R_0	Resistance of thermometer corresponding to the temperature at beginning of an initial line.
R_1	Resistance of thermometer corresponding to the temperature at end of an initial line.
R_2	Resistance of thermometer corresponding to the temperature at end of a heating period.
R_3	Resistance of thermometer corresponding to the temperature at end of a final line.
R_a	Resistance of thermometer corresponding to θ_a.
R_m	Resistance of thermometer corresponding to θ_m.
$R_{0°}$	Resistance of thermometer at 0 °C.
$R_{100°}$	Resistance of thermometer at 100 °C.
δR	Change of resistance of thermometer corresponding to $\delta\theta$.
$\frac{dR_r}{dt}$	The equivalent on the resistance scale of $\frac{d\theta_r}{dt}$
$\frac{dR_s}{dt}$	The equivalent on the resistance scale of $\frac{d\theta_s}{dt}$
$\frac{dR_c}{dt}$	The equivalent on the resistance scale of $\left(\frac{d\theta_r}{dt} + \frac{d\theta_s}{dt}\right)$
t	Time.
t_0	Time at beginning of an initial line.
t_1	Time at end of an initial line.
t_2	Time at end of a heating period.
t_3	Time at end of a final line.

λ	A constant denoting the rate of temperature change due to stirring alone.
τ	A parameter used in Equation 35.
δ	Used as a symbol for an increment.
δ_c	Constant in the Callendar-Griffith's equations (19) and (20).
δ_k	A function in Keyes equation of state, (32).
a	A constant in Keyes equation of state, (32).
l	A constant in Keyes equation of state, (32).

ABSTRACT.

The specific heat-capacity at the constant specific volume 2.6593 ccm./gm. of a mixture of the liquid and vapor phases of ammonia (NH_3), has been determined in the temperature range 30° – 125° C. The method employed consists of transferring to a calorimeter measured amounts of electrical energy, each sufficient to raise the temperature of the system approximately one degree. These changes of temperature were measured by means of a platinum resistance thermometer. The heat capacity of the system was measured first when the apparatus was loaded with ammonia, and again when the apparatus was empty. At any particular temperature, the difference of these heat capacities is the heat capacity of the ammonia, and knowing the mass of ammonia present, the specific heat capacity is computed.

The calorimeter was a steel tube containing mercury, in which were immersed the ammonia container, a thermometer and a heating coil, the whole being jacketed in a Dewar vacuum flask, enclosed in a vessel and totally submerged in an oil bath whose temperature could be controlled. A special calorimetric Wheatstone Bridge was used in connection with the platinum resistance thermometer.

A special method of computing the calorimetric curves and the cooling correction was developed.

The experimental results consist of two curves, giving the heat capacities of the apparatus when loaded with ammonia, and when empty, as a function of the temperature. The values of the specific heat of the particular mixture of liquid and vapor mentioned above, are computed and the results given in Table II.

Equations are given which permit the computation of any other specific heat in the liquid-vapor region, or on the boundary of that region, from the experimentally determined values of the specific heat at constant specific volume.

Using the Keyes-Brownlee data for ammonia, the saturation specific heats of the liquid and vapor phases have been computed as far as 100° C. and these results are given in Table III.

The principal sources of error of this method are discussed.

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A MONOGRAPH OF THE AMERICAN FROGS OF THE GENUS *RANA*.

By G. A. Boulenger, LL.D., D.Sc., F.R.S., &c.

A MONOGRAPH OF THE AMERICAN FROGS OF THE GENUS *RANA*.

By G. A. Boulenger, LL.D., D.Sc., F.R.S., &c.

Presented by Thomas Barbour. Received December 1, 1919.

I have lately been able to make a thorough revision of the numerous species, about 195, constituting the genus *Rana* in the broad sense in which I take it.[1]

For purposes of convenience, and also in order to facilitate its publication, I have divided the work into four parts, based on the geographical distribution. The first part, dealing with the South Asian, Papuan, Malanasian and Australian species, will be published, thanks to the kind interest of Dr. Annandale, in the Records of the Indian Museum, and is nearly entirely printed. The part dealing with America, comprising a comparatively small number of species, is now offered for publication.

The latest comprehensive accounts of the North American species are contained in the works of Cope, Batrachia of North America, (Bull. U. S. Nat. Mus. No. 34, 1889), and of Miss Dickerson, whose Frog Book (1906) is so excellently illustrated and swarms with interesting information on the life histories. The Central American species have been dealt with by Günther in the Biologia Centrali-Americana (1900).

From a systematic point of view, the descriptions in these works are not so precise and comparative as I should wish, and do not convey that information on individual variations in the proportions that are of essential importance in a group in which the absence of scales deprives us of so many characters which facilitate the identification and study of reptiles. That is why I have thought it desirable to prepare tables of measurements of a large number of individuals to accompany the descriptions, drawn up on a strictly comparative system and taking into consideration all departures from the normal.

These descriptions and tables will enable the student to form a correct idea of the material in the British Museum, material which, though less exhaustive than that in some institutions in the United States, is yet of great extent, especially as concerns Mexico and Central America. It has been recently increased through the generous help of Miss Dickerson, Dr. Stejneger and Dr. Barbour to whom I beg to express my best thanks. I have also been able to examine the types of the Central American species preserved in the Paris Museum and assign them their place in the synonymy.

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[1] Cf. Bull. Soc. Zool. France (1918), p. 111.

A few words of explanation are necessary concerning the method of taking measurements.

The length of the head and of the snout are measured along the axis, and the posterior extremity of the head corresponds to the articulation of the skull with the vertebral column, which can be approximately ascertained, in the flesh, by feeling with the points of the compasses. The width of the head is the greatest width at the commissures of the jaws.

The tympanum, of which the greatest diameter is given, is compared with the eye, measured along the side of the head.

The limbs are measured fully stretched out, the hind limb from the middle line of the thigh where it joins the body. By foot is meant the foot without the tarsus; it is measured from the tarso-metatarsal articulation. Each finger or toe is measured from the point at which it joins its fellows, and if this is not the same on both sides of it, as in the case of the fourth toe, from the point most remote from the tip. The length of the first toe is reckoned from the distal extremity of the base of the inner metatarsal tubercle.

Synopsis of the Species.

I. Toes pointed or with slightly swollen tips.

A. Glandular dorso-lateral fold absent or flat and ill-defined, or, if very prominent, not extending to the hip; tympanum at least $\frac{2}{3}$ diameter of eye, usually much larger, especially in males; tibia $2\frac{1}{2}$ to 4 times as long as broad; toes $\frac{3}{4}$ to entirely webbed; outer metatarsals separated nearly to the base; nasal bones in contact with each other or narrowly separated.

1. Male with internal vocal sacs; dorso-lateral fold absent or flat and very indistinct.

Vomerine teeth between (rarely just behind) the choanæ; first finger as long as or a little longer than second; tibio-tarsal articulation reaching tympanum or eye; heels meeting or slightly overlapping; tibia $2\frac{1}{10}$ to $2\frac{1}{2}$ times in length from snout to vent; tip of fourth toe free; no dorso-lateral fold........................*R. catesbiana*, Shaw.

Vomerine teeth between the choanæ; first finger a little shorter than second; tibia-tarsal articulation reaching tympanum or eye; heels meeting or narrowly separated; tibia $2\frac{1}{4}$ to $2\frac{1}{3}$ times in length from snout to vent; web extending to tip of fourth toe; no dorso-lateral fold . . . . *R. grylio*, Stejn.

Vomerine teeth on a level with posterior borders of choanæ, or just behind them; first and second fingers equal; tibio-tarsal articulation reaching eye; heels meeting or slightly overlapping; tibia 2 to $2\frac{1}{3}$ times in length from snout to vent; one or two phalanges of fourth toe free; dorso-lateral fold present or absent . . *R. septentrionalis*, Baird.

2. Male with internal vocal sacs; dorso-lateral fold usually very distinct; vomerine teeth between the choanæ or just behind the level of their posterior borders; first finger as long as or a little longer than second; tibio-tarsal articulation reaching tympanum, eye, or between eye and tip of snout; heels meeting or overlapping; tibia 3 to 4 times as long as broad, $1\frac{8}{9}$ to $2\frac{1}{3}$ times in length from snout to vent; one or two phalanges of fourth toe free.

Dorso-lateral fold not extending beyond the sacral region.
R. clamitans, Daud.

Dorso-lateral fold extending beyond the sacral region . . *R. onca*, Cope.

3. Male with external vocal sacs; tympanum not larger than eye; first finger as long as or slightly longer or shorter than second; heels meeting or not.

Head as long as broad; tibio-tarsal articulation reaching shoulder or tympanum; tibia $2\frac{2}{5}$ to $2\frac{2}{3}$ times in length from snout to vent; two phalanges of fourth toe free; no dorso-lateral fold . . *R. virgatipes*, Cope.

Head broader than long; tibio-tarsal articulation reaching tympanum or eye; tibia 2 to $2\frac{1}{3}$ times in length from snout to vent; fourth toe webbed to the tip, or last phalanx free; usually a more or less distinct dorso-lateral fold . . . . *R. montezumæ*, Baird.

B. Glandular dorso-lateral fold very distinct, extending to the hip (exceptionally interrupted posteriorly); nasal bones widely separated from each other.

1. Outer metatarsals separated nearly to the base; toes obtusely pointed, $\frac{2}{3}$ to nearly entirely webbed; tibio-tarsal articulation reaching eye, tip of snout, or a little beyond; tibia $3\frac{1}{2}$ to $5\frac{1}{2}$ times as long as broad, $1\frac{2}{3}$ to $2\frac{1}{10}$ times in length from snout to vent.

a. Male with vocal sacs.

Head as long as broad or a little broader or a little longer; interorbital space much narrower than upper eyelid; dorso-lateral folds narrow or moderately broad, usually with interrupted folds or elongate warts between them; outer metatarsal tubercle absent or very indistinct; vocal sacs external or internal....................*R. halecina*, L.

Head as long as broad; interorbital space much narrower than upper eyelid; dorso-lateral folds very broad, with a pair of similar folds between them; outer metatarsal tubercle usually present; vocal sacs internal....................................*R. palustris*, Leconte.

b. Male without vocal sacs.

Head broader than long; interorbital space as broad as or narrower than upper eyelid; tympanum $\frac{2}{3}$ to once diameter of eye; outer metatarsal tubercle absent or very indistinct.........*R. draytonii*, B. & G.

Head as long as broad or slightly broader than long; interorbital space narrower than upper eyelid; tympanum $\frac{2}{5}$ to $\frac{1}{2}$ diameter of eye; more or less distinct outer metatarsal tubercle....*R. aurora*, B. & G.

2. Web not penetrating beyond basal third or half of outer metatarsals.

a. Glandular dorso-lateral fold narrow or moderately broad; head moderately large; vomerine teeth on a level with or behind posterior borders of choanæ.

Tibio-tarsal articulation reaching tympanum or eye; tibia 3 to $4\frac{1}{2}$ times as long as broad, 2 to $2\frac{1}{5}$ times in length from snout to vent; toes $\frac{3}{4}$ to nearly entirely webbed; inner metatarsal tubercle $\frac{1}{4}$ to $\frac{3}{4}$ length of inner toe; male without vocal sacs......*R. pretiosa*, B. & G.

Tibio-tarsal articulation reaching tympanum or eye; tibia 3 to 4 times as long as broad, 2 to $2\frac{1}{2}$ times in length from snout to vent; toes $\frac{1}{2}$ to $\frac{2}{3}$ webbed; inner metatarsal tubercle $\frac{1}{2}$ to $\frac{2}{3}$ length of inner toe; male with internal vocal sacs........*R. cantabrigensis*, Baird.

Tibio-tarsal articulation reaching beyond eye; tibia 4 to 5 times as long as broad, $1\frac{3}{5}$ to 2 times in length from snout to vent; toes $\frac{2}{3}$ to $\frac{3}{4}$ webbed; inner metatarsal tubercle $\frac{2}{5}$ to $\frac{3}{5}$ length of inner toe; male with internal vocal sacs......................*R. silvatica*, Leconte.

b. Glandular dorso-lateral fold broad; vomerine teeth between the choanæ.

Head moderate; loreal region fully oblique; tibio-tarsal articulation reaching eye; tibia $4\frac{1}{2}$ to 5 times as long as broad, $2\frac{1}{7}$ to $2\frac{1}{5}$ times in length from snout to vent; toes $\frac{2}{3}$ webbed; male without vocal sacs. *R. godmani*, Stur.

Head very large; loreal region very oblique; tibio-tarsal articulation reaching tympanum or eye, or a little beyond; tibia 3 to 4 times as long as broad, $1\frac{3}{4}$ to nearly $2\frac{1}{4}$ times in length from snout to vent; toes $\frac{1}{2}$ webbed; male with external vocal sacs....*R. areolata*, B. & S.

II. Toes ending in very small discs; outer metatarsals separated nearly to the base; interorbital space equal to or a little less the width of upper eyelid; nasal bones widely separated from each other.

A. Loreal region moderately oblique; toes entirely webbed or two phalanges of fourth free.

1. Tips of fingers swollen; vomerine teeth behind level of choanæ; tympanum $\frac{2}{5}$ to $\frac{3}{5}$ diameter of eye; tibio-tarsal articulation reaching tip of snout or beyond; head broader than long.

No dorso-lateral fold; tympanum distinct; heels not overlapping; tibia $1\frac{5}{6}$ to 2 times in length from snout to vent; no outer metatarsal tubercle; male without vocal sacs...........*R. tarahumaræ*, Blgr.

Dorso-lateral fold, if distinct, very broad and flat and restricted to the anterior half of the body; tympanum feebly distinct, ill-defined; heels overlapping; tibia $1\frac{3}{5}$ to $1\frac{4}{5}$ times in length from snout to vent; an outer metatarsal tubercle; male with internal vocal sacs.

R. boylii, Baird.

A moderately prominent dorso-lateral fold, extending to the hip, its distance from its fellow, on the back, $3\frac{1}{2}$ times in length from snout to vent; tympanum very distinct; heels overlapping; tibia $1\frac{3}{4}$ times in length from snout to vent; no outer metatarsal tubercle.

R. pustulosa, Blgr.

2. Tips of fingers obtuse or rather pointed; vomerine teeth between choanæ; tympanum $\frac{1}{2}$ to $\frac{5}{6}$ diameter of eye; tibio-tarsal articulation reaching eye or tip of snout; tibia $1\frac{3}{4}$ to $2\frac{1}{4}$ times in length from snout to vent; dorso-lateral fold prominent, its distance from its fellow, on the back, 4 to $5\frac{1}{2}$ times in length from snout to vent; no outer metatarsal tubercle; head as long as broad or a little broader than long; male with internal vocal sacs............*R. palmipes*, Spix.

B. Loreal region vertical or nearly so; toes $\frac{2}{3}$ to $\frac{3}{4}$ webbed; tips of fingers swollen; tympanum $\frac{1}{2}$ to $\frac{2}{3}$ diameter of eye; tibio-tarsal articulation reaching eye or between eye and tip of snout; heels overlapping; tibia 5 to 6 times as long as broad, $1\frac{2}{3}$ to 2 times in length from snout to vent; dorso-lateral fold prominent, its distance from its fellow, on the back, 5 to 6 times in length from snout to vent; no outer metatarsal tubercle; head as long as broad or a little longer than broad; male without vocal sacs........................*R. cæruleopunctata*, Stdr.

The American frogs all belong to the subgenus *Rana*, agreeing with the type-species, *R. temporaria*, L., in the structure of the pectoral arch (strong horizontal clavicles, omosternal style not forked at the base). I conceive the most primitive type as with large nasal bones in contact with each other and with the frontoparietals entirely covering the ethmoid; pointed, fully webbed toes with the outer metatarsals separated by web to the base; a distinct tympanum; no glandular dorso-lateral fold. I therefore regard the species grouped together in division I. A. of the above synopsis as nearest to this prototype; from this group I. B. 1. and II. seem to be directly and independently derived, probably also I. B. 2. *b.*; whilst I. B. 2. *a.* is obviously connected with I. B. 1. The species under Division II. are furthest removed from the prototype; I see no reason for regarding *R. boylii* as nearly allied to the *Ranæ temporariæ*, and it is connected with *R. palmipes* by *R. pustulosa*.

1. **Rana catesbiana.**

Rana boans (non Linn.), LACEP., Hist. Quadr. Ov. I, Syn. Meth., and p. 541 (1788).

Rana pipiens, part. (Daud.) LATR., Hist. Rept. II, p. 153 (1801); DAUD., Hist. Rain. Gren. Crap. (1803), p. 53,[2] and Hist. Rept. VIII, p. 113 (1803).

Rana grunniens, part., LATR., t. c., p. 155.

Rana catesbeiana SHAW, Gen. Zool., III, p. 103, Pl. XXXIII (1802); LECONTE, Proc. Ac. Philad., 1855, p. 423; BAIRD, Rep. U. S. Expl. Surv., XII, ii, Pl. XXIX, fig. 5 (1860); BOULENG., Cat. Batr. Ecaud., p. 36 (1882); HINCKLEY, Proc. Bost. Soc. N. H. XXI, 1882, p. 311, Pl. v, fig. 1, 2, 7; H. GARM., Bull. Illin., Lab. III, 1892, p. 328; RHOADS, Proc. Ac. Philad., 1895, p. 300; STEJNEG. Proc. U. S. Nat. Mus., XXIV, 1901, p. 212; DITMARS, Amer. Mus. Journ. V, 1905, p. 203, fig. 1; DICKERSON, Frog Book, p. 227, Pls. LXXXVII–XLVI (1906); WRIGHT, Publ. Carnegie Instit. No. 197, 1914, p. 77, Pl. XIX; BOULENG., Ann. and Mag. N. H. (9) III, 1919, p. 408.

Rana mugiens MERR., Tent. Syst. Amph., p. 175 (1820); DUM. AND BIBR., Exp. Gén. VIII, p. 370 (1841); GÜNTH., Cat. Batr. Sal., p. 36 (1858); WIED, N. Acta Ac. Leop. Carol., XXII, 1865, p. 106.

Rana scapularis HARL., Am. Journ. Sc., X, 1825, p. 59, and Journ. Ac. Philad., V, 1825, p. 338.

Rana pipiens HARL., tt. cc. pp. 62, 335; HOLBR., N. Am. Herp. IV, p. 77, Pl. XVIII (1842); DEKAY, N. Y. Faun., Rept., p. 60, Pl. XIX, fig. 48 (1842).

Rana conspersa LECONTE, Proc. Ac. Philad., 1855, p. 425.

Rana catesbiana, part., COPE, Batr. N. Am., p. 424, fig. (1889).

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[2] This figure is a puzzle as no aquatic American *Rana* is known ever to have a light vertebral stripe. It perhaps represents the Indian *R. hexadactyla*.

Vomerine teeth in small groups or short oblique series between the choanæ (rarely just behind them), a little nearer to each other than to the latter, or close together.

Head broader than long, much depressed; snout rounded, fully projecting beyond the mouth, as long as the eye or a little longer; canthus rostralis very indistinct; loreal region very oblique, slightly concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the former; distance between the nostrils equal to or a little greater than the interorbital width, which is $\frac{1}{2}$ to nearly once that of the upper eyelid; tympanum very distinct, nearly as large as the eye in females, larger in males (up to $1\frac{2}{5}$ times the diameter of the eye), close to the eye or narrowly separated from it in males, 2 to 3 times its distance from the latter in females.

Fingers moderate or rather long, sometimes pointed, sometimes obtuse, first as long as or a little longer than the second, third longer than the snout; a narrow round fold sometimes present along the sides of the fingers; subarticular tubercles small, feebly prominent.

Hind limb moderately long, the tibio-tarsal articulation reaching the tympanum or the eye, the heels meeting or feebly overlapping when the limbs are folded at right angles to the body; tibia 3 to $3\frac{1}{2}$ times as long as broad, $2\frac{1}{10}$ to $2\frac{1}{2}$ times in length from snout to vent, shorter than the fore limb or than the foot. Toes rather long, obtuse or with slightly swollen tips, fully webbed, but the tip of the fourth free; outer metatarsals separated to the base; subarticular tubercles small; feebly prominent; no tarsal fold; inner metatarsal tubercle elliptic, feebly prominent, $\frac{1}{3}$ to $\frac{2}{3}$ the length of the inner toe; no outer tubercle.

Skin smooth on back with small warts; a strong glandular fold from the eye to the shoulder.

Brown or olive above, rarely green, uniform or spotted or marbled with dark brown or, very nearly, uniform blackish; limbs with or without dark cross-bands; hinder side of thighs often marbled black and yellow. Lower parts white, sometimes marbled with brown, or trout brown, spotted with white (females), or yellow (males); sometimes a black band along the posterior part of the thighs.

Male with internal vocal sacs and a retracting pad on the inner side of the first finger.

Skeleton very similar to that of *R. esculenta.* Nasal bones moderately large, in contact with each other or narrowly separated, separated from the ethmoid, a small part of the upper surface of which is exposed; frontoparietals grooved along the middle and compressed behind.

Rana catesbiana.

Measurements in Millimeters.

	1. ♂	2. ♂	3. ♂	4. ♀	5. ♀	6. ♀	7. ♀	8. ♂	9. ♀	10. ♀	11. ♀	12. ♂	13. ♀	14. ♀	15. ♂
From snout to vent	132	127	103	160	120	112	95	130	157	78	125	120	122	200	158
Head	47	39	35	47	40	36	21	47	50	28	41	43	40	57	49
Width of head	54	47	39	56	44	41	36	53	59	30	47	51	49	75	59
Snout	16	13	12	16	13	12	11	16	17	10	13	14	13	19	16
Eye	15	12	12	13	13	12	11	14	15	9	12	13	13	16	15
Interorbital width	5	4	4	6	4	5	4	5	6	3	5	4	4	3	5
Tympanum	21	19	13	14	14	11	10	18	15	7	12	18	11	19	20
Fore limb	77	70	58	78	73	63	53	70	85	43	66	69	68	102	83
1st finger	18	17	16	18	16	16	12	16	20	10	17	16	17	25	20
2nd "	17	17	15	17	16	15	11	15	19	10	16	16	16	25	16
3rd "	22	21	18	23	20	18	15	20	32	13	19	20	20	29	22
4th "	16	15	12	17	15	12	10	14	16	8	13	14	14	20	16
Hind limb	192	182	153	220	190	174	149	194	232	120	181	183	179	270	217
Tibia	55	50	47	70	55	51	43	60	72	37	57	55	56	85	64
Foot	71	63	59	80	64	58	50	64	79	42	63	65	61	93	76
3rd toe	43	36	32	45	36	32	30	38	50	25	35	39	35	56	45
4th "	62	55	51	70	56	51	45	57	69	36	55	57	54	82	65
5th "	46	42	37	55	42	37	33	42	51	27	42	42	38	65	49

1–3. Lucknow, Ontario.— 4. Oshkosh, Wisconsin.— 5–7. Cambridge, Mass.—8. Ashokan, Ulster Co., N. Y.— 9. New Jersey.— 10. Bloomington, Indiana.— 11. Marshall, N. Carolina.— 12–13. N. Carolina.— 14. Garnet, S. Carolina.— 15. N. America.

The tadpole is also very similar to that of *R. esculenta* and reaches the size of that of *Pelobates fuscus*. Mouth small; back narrowly edged with black; having white in a long marginal upper series with a very short one (rarely 2) on each side and 3 lower series, the innermost narrowly interrupted in the middle. Upper parts often dotted with black.

The eggs are very small, as in *R. esculenta*.

Habitat.—North America east of the Rocky Mountains, from Canada (Quebec, Ontario) to Florida and Texas.

Measurements on p. 420.

2. **Rana grylio.**

Rana catesbiana, part., Cope, Batr. N. Am., p. 424 (1889).

Rana grylio Stejneg., Proc. U. S. Nat. Mus., XXIV, 1901, p. 212; Dickerson, Frog Book, p. 226, Pls. LXXXV, LXXXVI (1906); Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 409.

Vomerine teeth in small groups close together between the choanæ.

Head nearly as long as broad or a little broader than long, much depressed; snout rounded or obtusely pointed, feebly projecting beyond the mouth; canthus rostralis very indistinct; loreal region very oblique, concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the latter; distance between the nostrils greater than the interorbital width, which equals ⅔ to ½ that of the upper eyelid; tympanum very distinct, as large as the eye in females, 1⅓ to nearly twice its diameter in males, close to the eye in males, 3 times its distance from the eye in females.

Fingers rather long, pointed, first a little shorter than the second, third longer than the snout, second and third with a more or less distinct dermal margin; subarticular tubercles small, feebly prominent.

Hind limb moderately long, the tibiotarsal articulation reaching the tympanum or the eye, the heels meeting or narrowly separated when the limbs are folded at right angles to the body; tibia 3 to 3½ times as long as broad, 2¼ to 2⅓ times in length from snout to vent, shorter than the fore limb or than the foot. Toes long, obtuse, third longer in proportion to the fourth than in *R. catesbiana*,[3] all webbed to the very tips; outer metatarsals separated nearly to the base; subarticular tubercles small, feebly prominent; no tarsal fold; inner metatarsal tubercle elliptic, feebly prominent, $\frac{1}{3}$ to $\frac{2}{5}$ the length of the inner toe; no outer tubercle.

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[3] Third meaning, according to Stejneger, 80 to 84 per cent of fourth, as against 70 to 76 (both measured from the anterior edge of the metatarsal tubercle).

Skin smooth; a strong glandular fold from the eye to the shoulder. The coloration is thus described by Miss Dickerson:—"Head and shoulders usually vivid green; olive posteriorly, with many irregular black spots; the whole frog may be olive or rich dark brown; ear orange-brown, with green center; middle and posterior back may have four longitudinal bands of bright orange-brown, alternating with bands of olive; small black spots on legs; under parts light, unspotted, except posteriorly; throat of male bright yellow; under surface of legs may be mottled and reticulated in coarse pattern with black and yellow." The male from New Orleans in the Museum has a light longitudinal band between two black ones on the back of the thighs.

Male with internal vocal sacs.

Measurements in Millimeters.

	1.	2.	3.
	♂	♀	♂
From snout to vent	83	124	126
Head	30	40	41
Width of head	31	40	46
Snout	9	13	13
Eye	9	13	11
Interorbital width	2	3.5	3.5
Tympanum	12	13	19
Fore limb	43	61	70
1st finger	9	14	16
2nd "	10	15	17
3rd "	13	19	21
4th "	8	14	15
Hind limb	115	182	183
Tibia	36	55	56
Foot	42	62	63
3rd toe	26	35	39
4th "	35	53	57
5th "	26	42	46

1–2. New Orleans.—3. Placquamine, Louisiana.

3. Rana septentrionalis.

Rana septentrionalis Baird, Proc. Ac. Philad., 1855, p. 51 and Rep. U. S. Expl. Surv., XII, ii, Pl. xxix, fig. 2 (1860); Yarrow, Rep. Explor. Surv. w. of 100th Mer., Batr. Rept., p. 528 (1875); Bouleng., Cat. Batr. Ecaud., p. 37 (1882) and Ann. and Mag. N. H. (5) XI, 1883, p. 16; Garnier, Amer. Nat., XVII, 1883, p. 945; Cope, Batr. N. Am., p. 416, Pl. lxxxvi (1889); Werner, Jahrnb. Nat. Ver. Magdeb., 1894, p. 135; Dickerson, Frog Book, p. 224 (1906); Bouleng., Ann. and Mag. N. H. (9), III, 1919, p. 409.

Rana sinuata Baird, l. c.

Vomerine teeth in small groups close together, on a level with the posterior border of the choanæ or just behind the latter.

Head a little broader than long, much depressed; snout rounded, scarcely projecting beyond the mouth, as long as or slightly longer than the eye; canthus rostralis very indistinct; loreal region very oblique, slightly concave; nostril equidistant from the eye and from the tip of the snout, or a little nearer the former; distance between the nostrils greater than the interorbital width, which is about ½ that of the upper eyelid; tympanum very distinct, ⅔ to ⅚ the diameter of the eye in females, as large as or larger than the eye in males, close to the eye or narrowly separated from it.

Fingers pointed, first and second equal, third much longer than the snout; subarticular tubercles small, moderately or feebly prominent.

Hind limb moderately long, the tibio-tarsal articulation reaching the eye, the heels meeting or slightly overlapping when the limbs are folded at right angles to the body; tibia 3 to 4 times as long as broad, 2 to 2⅓ times in length from snout to vent, shorter than the fore limb or than the foot. Toes rather pointed, broadly webbed, the web reaching the tips of the third and fifth but leaving one or two phalanges of the fourth free; outer metatarsals separated nearly to the base; subarticular tubercles small, moderately or feebly prominent; tarsal fold absent or merely indicated; inner metatarsal tubercle elliptical, feebly prominent, ⅓ to ⅖ the length of the inner toe; no outer tubercle.

Skin smooth; glandular dorso-lateral fold, from above the tympanum, absent or flat and very indistinct, and not extending posteriorly beyond the sacral region; a fold from the eye to the shoulder.

Brownish olive or purplish brown above, spotted or marbled with black, or blackish with sinuous whitish lines or with light vermiculations; limbs without angular cross-bands; hinder side of thighs marbled black and white. Lower parts white, throat yellowish.

Male with internal vocal sacs; fore limbs thickened; and large pad on the inner side of the first finger.

Nasal bones moderately large, in contact with each other and with the ethmoid, which is largely uncovered in front.

Measurements in Millimeters

	1.	2.	3.	4.	5.	6.	7.	8.
	♂	♂	♂	♀	♀	♀	♀	♂
From snout to vent	56	55	55	76	63	63	60	58
Head	19	19	19	24	20	18	19	20
Width of head	21	21	20	27	21	19	21	21
Snout	6	6.5	6	7	6	6	6.5	6
Eye	6	6.5	7	8	7	6	6.5	6
Interorbital space	2.5	2	2	3	2	2	2	2
Tympanum	6	8	8	6	5	5	5	8
Fore limb	33	35	35	44	36	36	35	32
1st finger	6.5	7	6	8	7	7	7	7
2nd "	6.5	7	6	8	7	7	7	7
3rd "	10	10	9	13	9	9	10	9
4th "	6	6	5	8	6	6	6	6
Hind limb	91	91	91	121	94	93	98	91
Tibia	26	26	26	38	27	27	28	27
Foot	32	34	33	40	34	34	34	33
3rd toe	19	20	20	22	20	20	20	18
4th "	28	29	27	33	29	29	30	26
5th "	20	22	20	23	20	20	22	19

1–7. Lucknow, Ontario.— 8. Eustice, Maine.

Habitat.— Southern Canada and New York to Montana and Utah.

This species is very closely allied to *R. catesbiana,* and I have failed to find a single distinctive character that will apply to all specimens, regardless of size. The two species are thus contrasted by Cope in his key:—

> Dorsal fold, size of tympanum, and extent of web variable; length not exceeding 6 centimeters; large dark spots on the back. *R. septentrionalis.*
>
> No dorso-lateral dermal fold; web generally leaving one phalange of fourth digit free; length reaching 20 centimeters. . . *R. catesbiana.*

Considering that our largest specimen of *R. septentrionalis* measures 76 mm. and that one smallest nearly adult *R. catesbiana* measures 78 mm., whilst large dark spots are frequent in the latter species, this definition is very unsatisfactory. I trust the characters given in my synopsis will answer better for purposes of identification without regard to size, but they may fail in exceptional cases.

I cannot agree with Cope when he says, "Taking all its characters together, this species occupies a position intermediate between nearly all the American species of the genus," and he expresses this view in a diagram of the chain of affinities (p. 397); nor do I think him justified in adding that, "from such form it might be supposed that all the *Rana* of the northern hemisphere have been derived," a statement I would rather apply to *R. catesbiana* and *R. grylio*, regarding *R. septentrionalis* as connecting *R. catesbiana* with *R. clamitans*.

4. **Rana clamitans.**

Rana clamitans (Daud.), Latr., Hist. Rept. II, p. 157 (1801); Merr., Tent. Syst. Amph., p. 175 (1820); Holbr., N. Am. Herp., IV, p. 85, Pl. xx (1842); H. Garm., Bull. Illin. Lab., III, 1892, p. 327; Dickerson, Frog Book, p. 198, Pl. xiii, fig. 1, and Pls. lxxv and lxxvi (1906); Bouleng., Ann. and Mag. N. H. (9), III, 1919, p. 409.

Rana clamata Daud., Hist. Rain. Gren. Crap., p. 54, Pl. lii, fig. 2 (1802), and Hist. Rept., VIII, p. 104 (1803); Harlan, Amer. Journ. Sc., X, 1825, p. 63, and Journ. Ac. Philad., V, 1825, p. 335; Dum. and Bibr., Exp. Gén., VIII, p. 373 (1841); Günth., Cat. Batr. Sal., p. 14 (1858); Bouleng., Cat. Batr. Ecaud., p. 36 (1882); Cope, Batr. N. Am., p. 419, figs. (1889); Werner, Jahresb. Nat. Ver. Magdeb., 1894, p. 135; Ditmars, Amer. Mus. Journ., V, 1905, p. 201, fig.; Wright, Publ. Carnegie Instit., No. 197, 1914, p. 70, Pl. xvii.

?*Ranaria melanota* Rafin., Ann. of Nature, p. 5 (1820).

Rana fontinalis Leconte, Ann. Lyc. N. Y., I, 1825, p. 282, and Proc. Ac. Philad., 1835, p. 424; Holbr. t. c., p. 87, Pl. xxi; Dekay, N. Y. Faun., Rept. p. 62, Pl. xxi, fig. 54 (1842); Hinckley, Proc. Bost. Soc. N. H., XX, 1882, p. 311, Pl. v, fig. 8.

Rana flaviviridis Harl., tt. cc., pp. 58, 338, and Med. Phys. Res. p. 103 (1835).

Rana horiconensis Holbr., t. c., p. 83, Pl. xix; Dekay, op. cit., p. 61, Pl. xxii, fig. 62; Günth., op. cit., p. 131.

Rana nigricans Agass., Lake Super., p. 379, Pl. vi, figs. 4, 5 (1850).

Rana nigrescens Leconte, Proc. Ac. Philad., 1855, p. 20.

Rana clamator Leconte, l. c.

Rana clamitans melanota Rhoads, Proc. Ac. Philad., 1895, p. 394.

Vomerine teeth in small groups close together between the choanæ or just behind the level of their posterior borders.

Head as long as broad or a little broader than long, moderately or rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as the eye or slightly longer or shorter; canthus rostralis obtuse; loreal region oblique, concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the former; distance between the nostrils greater than the interorbital width, which is about $\frac{1}{2}$ that of the upper eyelid or a little more; tympanum very distinct, $\frac{3}{4}$ to once the diameter of the eye in females, 1 to $1\frac{1}{2}$ times in males, close to the latter or separated from it by a space not exceeding $\frac{1}{3}$ its diameter.

Fingers moderate or rather long, usually pointed, sometimes rather obtuse, first as long as or a little longer than the second, third longer than the snout; a narrow dorsal fold sometimes present along the sides of the fingers; subarticular tubercles moderate or rather small, moderately prominent.

Hind limb moderately long, the tibio-tarsal articulation reaching the eye or between the eye and the tip of the snout, the heels overlapping when the limbs are folded at right angles to the body; tibia 3 to 4 times as long as broad, $1\frac{8}{9}$ to $2\frac{1}{4}$ times in length from snout to vent, shorter than the fore limb, as long as or shorter than the foot. Toes obtuse or rather pointed, $\frac{3}{4}$ to nearly entirely webbed, 1 or 2 phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles rather small, feebly prominent; tarsal fold absent or very feeble; inner metatarsal tubercle elliptic, feebly prominent, $\frac{1}{4}$ to $\frac{2}{7}$ the length of the inner toe; no outer tubercle.

Skin smooth or granular above, on back with numerous small warts; a glandular dorso-lateral fold, usually narrow and prominent, sometimes broken up or very indistinct, from above the tympanum to the sacral region or, rarely, a little beyond; the distance between the folds $4\frac{1}{2}$ to $5\frac{1}{2}$ times in the length from snout to vent.

Head and anterior part of body usually bright green, and posterior part of body and limbs olive or brown; body sometimes entirely green or brown; back usually with more or less numerous black spots or marblings; tympanum often reddish brown; dark cross-bands on the limbs present or absent. Lower parts white, the throat often yellow, immaculate or marbled with gray.

Male with internal vocal sacs; fore limbs thickened; a moderately large pad on the inner side of the first finger.

Skeleton as in *R. esculenta.* Nasal bones in contact or narrowly separated from each other, sometimes in contact with the frontoparietals. No character of any importance distinguishes the tadpole from that of *R. catesbiana,* and the eggs are likewise very small.

Rana clamitans.

Measurements in Millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.
	♂	♂	♀	♀	♀	♀	♂	♂	♂	♂	♂	♀	♀	♀	♂	♀	♂	♀	♂	♂	♂	♀	♀
From snout to vent	82	74	83	77	70	66	71	77	71	76	60	98	65	65	68	73	69	71	65	70	82	58	70
Head	28	26	27	24	23	22	25	26	24	28	23	30	22	22	24	24	24	24	23	24	27	20	22
Width of head	30	29	29	25	24	23	27	28	26	28	23	33	23	22	26	24	26	25	24	26	30	22	23
Snout	9	8	9	8	8	7	8	8	8	8	7	11	7	7	8	8	8	9	7	8	9	6	8
Eye	9	8	9	8	8	7	8	8	8	8	7	10	7	8	8	8	8	7	7	8	9	7	8
Interorbital	3.5	2.5	3	2.5	2.5	2	3	3	2.5	3	2	3	2	3	3	3	3	3	3	2.5	3	2	3
Tympanum	11	12	8	7	8	7	11	12	9	11	8	9	7	6	12	7	11	7	8	10	10	6	7
Fore limb	45	40	47	45	40	37	39	43	38	43	34	53	36	38	38	40	44	42	37	40	45	31	37
1st finger	10	9	11	10	9	9	9	9	7	9	6.5	11	9	8	9	9	11	10	7	9	10	7	9
2nd "	10	8	10	9	9	8	8	8	6	8	6.5	11	8	7	8	8	10	9	7	8	9	6	8
3rd "	12.5	11	13	12	12	11	11	11	9	12	9	14	10	9	10	11	12	12	9	11	12	9	11
4th "	8	7	8	7	7	6	6	7	5	7	5	10	6	6	6	6	7	7	5	6	8	6	7
Hind limb	124	118	134	126	115	107	111	118	110	120	98	144	101	109	110	111	117	120	104	113	131	86	99
Tibia	39	35	43	39	36	32	35	35	33	36	31	44	32	33	34	35	37	37	32	34	42	26	33
Foot	42	39	43	41	38	36	38	40	35	37	32	52	34	36	35	36	40	42	35	38	46	30	33
3rd toe	24	20	22	22	20	19	21	21	18	21	18	29	18	20	20	20	22	22	20	20	23	16	20
4th "	36	33	38	37	33	30	33	33	28	33	29	46	29	30	31	32	35	36	30	32	38	25	29
5th "	25	24	27	25	23	20	23	23	20	24	21	32	20	21	22	22	25	25	22	22	28	17	22

1–3. Lucknow, Ontario.— 4–6. Cambridge, Mass.— 7. Nashawana, Elizabeth Is., Mass.— 8.–9. New York.— 10–14. Long Is., New York.— 15–16. Shokan, N. Y.— 17–18. Pike Co., Penn.— 19. Philadelphia, Penn.— 20. Marshall, N. Carolina.— 21. Providence, R. I.— 22. Georgia.— 23. Bloomington, Indiana.

Habitat.— North America, east of the Rocky Mountains, from Canada (Quebec and Ontario) to Florida and Louisiana.

5. **Rana onca.**

Rana onca Cope, in Yarrow, Rep. Explor. Surv. w. of 100th Mer., Batr. Rept., p. 528, Pl. xxv, figs. 1–3 (1875); Dickerson, Frog Book, p. 196, Pl. ii, fig. 6 (1906); Bouleng., Ann. and Mag. N. H. (9), III, 1919, p. 409.

Rana draytoni onca Cope, Batr. N. Am., p. 443 (1889).

Rana fischeri Stejneg., N. Am. Faun., No. 7, p. 227, Pl. iii, fig. 5 (1893).

Vomerine teeth in small groups close together between the choanæ or extending a little beyond the level of their posterior borders.

Head as long as broad or broader than long, rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as the eye; canthus rostralis indistinct; loreal region oblique, concave; nostril nearer the eye than the tip of the snout; distance between the nostrils greater than the interorbital width, which is less than that of the upper eyelid; tympanum very distinct, $\frac{3}{4}$ to once the diameter of the eye and close to it.

Fingers moderate, obtusely pointed, first a little longer than the second, third longer than the snout; subarticular tubercles moderately large, prominent.

Hind limb moderately long or rather short, the tibio-tarsal articulation reaching the tympanum or the eye; heels meeting or overlapping when the limbs are folded at right angles to the body; tibia 3 to $3\frac{1}{2}$ times as long as broad, $2\frac{1}{4}$ to $2\frac{1}{3}$ times in length from snout to vent, a little shorter than the fore limb or than the foot. Toes obtuse, $\frac{3}{4}$ webbed, 2 phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles moderate; tarsal fold absent or merely indicated; inner metatarsal tubercle oval, feebly prominent, $\frac{1}{3}$ the length of the inner toe; no outer tubercle.

Skin smooth or with small warts above; a narrow, moderately prominent glandular dorso-lateral fold, from above the tympanum to beyond the sacral region, but not reaching the tip, the distance between the folds 4 to $4\frac{1}{2}$ times in the length from snout to vent.

Pale greenish above, with oval or round dark spots, which may be light-edged; spots on the limbs not forming regular cross-bars. Lower parts white, sometimes profusely sprinkled with brown. Male without external vocal sacs.

Nasal bones large, in contact with each other and with the frontoparietalis, the ethmoid nearly entirely covered.

Measurements of Females, in Millimeters.

From snout to vent	70	68
Head	22	23
Width of head	27	23
Snout	7	7
Eye	7	7
Interorbital width	3	3
Tympanum	7	6
Fore limb	38	34
1st finger	8	8
2nd “	7	7
3rd “	9	9
4th “	6	6
Hind limb	94	100
Tibia	30	31
Foot	31	33
3rd toe	18	21
4th “	27	30
5th “	19	21

Habitat.—Utah and Nevada.

This species is here described from three specimens, two females and one young, from Las Vegas, Nevada, for which I am indebted to the kindness of Dr. Stejneger and Miss Dickerson.

The affinity is very close to *R. clamitans*, the only character which may serve to distinguish *R. onca* being the somewhat greater posterior extension of the dorso-lateral fold. It is difficult to conceive the reasons which have induced Cope to place this frog as a subspecies of *R. draytoni*.

Miss Dickerson says the tympanum is larger in males than in females, and that the body is often bright green in front and brown behind, characters in which it approaches *R. clamitans*, whilst the spotting reminds one of *R. halecina*.

6. **Rana virgatipes.**

Rana virgatipes Cope, Amer. Nat. XXV, 1891, p. 1017; W. J. Davis, Amer. Nat. XXXVIII, 1904, p. 893, and XXXIX, 1905, p. 795; Fowler, Proc. Ac. Philad., LVII, 1905, p. 662, Pl. xl; Dickerson, Frog Book, p. 222, Pl. xiii, and Pl. lxxxiv, figs. 4, 5 (1906); Fowler, Rep. N. Jers. State Mus. 1907, p. 194; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 409.

Vomerine teeth in small groups close together, on a level with or just behind the posterior borders of the choanæ.

Head as long as broad, much depressed; snout rounded, projecting beyond the mouth, as long as or a little longer than the eye, which is large and extremely prominent; canthus rostralis very indistinct; loreal region oblique, concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils much greater than the interorbital width, which is $\frac{1}{3}$ to $\frac{1}{2}$ that of the upper eyelid; tympanum very distinct, $\frac{3}{5}$ to $\frac{5}{6}$ the diameter of the eye in females, as large as the eye in males, 3 to 5 times its distance from the eye.

Fingers rather long and slender, pointed, first as long as or slightly longer than the second, third longer than the snout; subarticular tubercles small, feebly prominent.

Hind limb rather short, the tibio-tarsal articulation reaching the shoulder or the tympanum, the heels meeting or not when the limbs are folded at right angles to the body; tibia 3 to 4 times as long as broad, $2\frac{2}{5}$ to $2\frac{2}{3}$ times in length from snout to vent, much shorter than the fore limb, shorter than the foot. Toes rather long, obtusely pointed, $\frac{3}{4}$ webbed, 2 phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles small, feebly prominent; no tarsal fold; inner metatarsal tubercles elliptic, feebly prominent, $\frac{1}{3}$ to $\frac{2}{5}$ the length of the inner toe; no outer tubercle.

Skin smooth, or with small pearl-like white horny tubercles on the back and limbs; a strong fold from the eye to the shoulder.

Chocolate brown or brownish olive above, with or without small darker spots; a yellowish brown streak from behind the eyes to the sacrum or a little beyond; a yellowish streak from below the eye to the groin; no cross-bands on the limbs. Lower parts yellowish white, or yellow on the throat and hind limbs; throat and belly uniform or more or less spotted with brown; hind limbs spotted or marbled with blackish brown, back of thighs with two blackish brown stripes enclosing a yellow one; web between the toes black.

Male with a large grayish or blackish external vocal sac on each side between the mouth and the shoulder; a feeble pad on the inner side of the first finger.

Nasal bones moderately large, narrowly separated from each other and from the frontoparietals.

Habitat.—New Jersey (Atlantic City and Lakehurst) and North Carolina (Lake Ellis).

Measurements in Millimeters

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
	♂	♂	♂	♀	♀	♀	♀	♀	♀	♀	♀
From snout to vent	56	50	50	52	51	43	36	32	35	33	36
Head	20	18	18	17	17	13	12	11	12	11	11
Width of head	20	18	18	17	17	13	12	11	12	11	11
Snout	6	6	6	6	5.5	4	3.5	3.5	3	3	3
Eye	6	6	6	6	5.5	5	4	3.5	4	4	4
Interorbital width	2.5	1.5	1.5	2	2	1.5	1	1.5	1	1.5	1
Tympanum	6	6	6	5	4	3.5	3	3	3	2.5	2.5
Fore limb	30	26	26	27	27	25	21	18	20	16	17
1st finger	6	5.5	6	6	5.5	4	3.5	3.5	3.5	3.5	3.5
2nd "	6	5	5.5	6	5.5	4	3.5	3.5	3.5	3	3
3rd "	8	7	8	8	7	5.5	5	5	5	5	5
4th "	6	4.5	5	5	5	3.5	3.5	3	3	3	3
Hind limb	80	74	70	75	73	64	55	44	51	46	46
Tibia	23	21	20	21	21	18	15	12	14	13	13
Foot	29	27	26	27	26	21	19	15	17	16	16
3rd toe	17	15	15	16	?	12	11	9	10	10	10
4th "	24	23	22	24	?	19	17	13	15	14	14
5th "	19	16	16	17	?	12	11	9	10	10	10

1–8. Lakehurst, New Jersey.— 9–11. Lake Ellis, N. Carolina.

The species does not exceed a length of 60 millimeters from snout to vent, and may be regarded as a dwarfed form derived from the *R. catesbiana* type.

7. **Rana montezumæ.**

Rana montezumæ Baird, Proc. Ac. Philad., 1855, p. 61; Girard, Rep. U. S. Mex. Bound. Surv., Rept. p. 27, Pl. xxxvi, figs. 1–6 (1859); Brocchi, Miss. Sc. Mex., Batr. p. 14, Pl. iv, fig. 2 (1881); Bouleng. Cat. Batr. Ecaud., p. 35 (1882); Dugès, Naturaleza (2) I, 1888, p. 137; Cope, Batr. N. Am. p. 428, figs. (1889); Günth., Biol. C.-Am., Rept. p. 197 (1900); Bouleng., Ann. and Mag. N. H. (9), III, 1919, p. 409.

Rana adtrita Troschal, in Müller's Reis. Ver. Stat. III, Wirbelte, p. 82 (1865).

Rana montezumæ concolor Cope, Bull. U. S. Nat. Mus., No. 32, 1887, p. 20.

Vomerine teeth in small transverse or oblique groups between or just behind the choanæ.

Head broader than long, much depressed; snout rounded, scarcely projecting beyond the mouth, as long as the eye or a little longer;

canthus rostralis indistinct; loreal region very oblique, scarcely concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the former; distance between the nostrils equal to or a little greater than the interorbital width, which is less than that of the upper eyelid; tympanum moderately or very distinct, ⅔ to once the diameter of the eye, 1½ to 3 times its distance from the latter.

Fingers long and slender, pointed, first usually as long as the second sometimes slightly longer or slightly shorter, third much longer than the snout; a narrow dermal fold sometimes present along the sides of the fingers; subarticular tubercles small, feebly prominent.

Hind limb rather short or moderately long, the tibio-tarsal articulation reaching the tympanum or the eye, the heels meeting when the limbs are folded at right angles to the body; tibia 2½ to 3½ times as long as broad, 2 to 2⅓ times in length from snout to vent, shorter than the fore limb or than the foot. Toes rather long, pointed, webbed to the tips or last phalanx of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles small, feebly prominent; a feeble tarsal fold; inner metatarsal tubercle elliptic or subtriangular and prominent, ¼ to ⅓ the length of the inner toe; no outer tubercle.

Skin smooth above or granulate and with small warts; a narrow or moderately broad, feebly prominent, glandular dorso-lateral fold from above the tympanum to the sacral region or a little beyond, sometimes broken up, sometimes absent; the distance between the folds, on the back, ⅕ to ¼ the length from snout to vent; lower parts smooth, belly and thighs sometimes granulate.

Purplish brown or olive brown above, uniform or speckled, vermiculated, or mottled with lighter or darker, with or without large roundish dark brown spots which may be grass-green in life, and are often light-edged; limbs with or without dark cross-bands. Lower parts dirty white, uniform on throat and breast vermiculated with brown, or brown speckled with white.

Male with a large blackish external vocal sac on each side, forming loose folds from behind the angle of the mouth to the shoulder.

Nasal bones rather large, in contact with each other and narrowly separated from the frontoparietals, which are narrow, grooved along the sagittal suture, and leave but a small part of the ethmoid uncovered; zygomatic branch of squamosal long. Omosternum not forked at the base. Terminal phalanges obtusely pointed.

Larva unknown. Eggs small, measuring 1½ millimeters in diameter in female, 100 millimeters long from snout to vent.

Habitat.— Jalisco, Guanajuato, Plateau of Mexico, Tabasco, Tehuantepec, and (*fide* Cope) Guatemala.— The types are from the City of Mexico, where the frog is common on the market.

The records of *R. montezumæ* from Shoalwater Bay, Washington Territory (3345, Col. J. D. Vaughan) and Upper Mississippi (3349, Dr. J. G. Cooper), in Yarrow's Check-list of 1882, are due to confusion with some other species.

Measurements in Millimeters

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
	♀	♀	♀	♀	♀	♂	♀	♀	♀	♀
From snout to vent	145	140	136	100	100	67	92	80	67	126
Head	40	39	38	32	31	23	29	24	22	42
Width of head	55	51	50	40	35	25	32	26	23	51
Snout	15	13	13	12	10	7	9	8	7	13
Eye	12	10	11	9	10	7	9	8	7	11
Interorbital width	6	5	5	4	3.5	2	3	2.5	2	5
Tympanum	9	9	9	9	9	6	7	6	5	8
Fore limb	88	84	80	56	54	40	52	42	36	81
1st finger	20	20	20	12	14	8	11	11	8	20
2nd "	20	20	20	12	13	8	11	11	8	20
3rd "	25	26	22	15	15	11	15	14	11	23
4th "	22	20	20	11	12	7	10	9	7	20
Hind limb	195	183	188	140	146	106	137	127	102	200
Tibia	69	65	62	46	46	33	42	37	32	63
Foot	72	69	68	50	51	40	50	42	38	68
3rd toe	43	42	40	28	29	23	28	24	23	41
4th "	58	56	58	42	45	35	42	37	33	59
5th "	44	46	46	30	34	25	31	28	25	48

1–3. Lazuna del Castillo, near Guadalajara (Paris Museum).— 4. Guanajuato (Paris Museum).— 5. City of Mexico.— 6–9. L. Texcoco.— 10. L. Zapotlan.

8. Rana halecina.

Rana halecina (Kalm), Linn. Syst. Nat. I, p. 356[4] (1766); Daud., Hist. Rain. Gren. Crap., p. 63 (1803), and Hist. Rept., VIII, p. 122 (1803); Harlan, Journ. Ac. Philad., V, 1826, p. 337; Holbr., N. Am. Herp., I, p. 89, Pl. xiii (1836); Dum. and Bibr., Esp. Gén., VIII, p. 352 (1841); Holbr., op. cit., IV, p. 91, Pl. xxii (1842); Dekay, N. Y. Faun., Rept. p. 62, Pl. xx, fig. 49 (1842);

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[4] This name, latinized by Linnaeus from Kalm's "Sillhoppetosser," appears in the synonymy of *Rana ocellata*.

GÜNTH., Cat. Batr. Sal., p. 13 (1858); COOPER, Rep. U. S. Expl. Surv., XII, ii, p. 304, Pl. XXIX, fig. 1 (1860); BROCCHI, Miss. Sc. Mex., Batr., p. 10 (1881); BOULENG., Cat. Batr. Ecaud., p. 41 (1882); COPE, Proc. Am. Philos. Soc., XXIII, 1886, p. 517; DUGÈS, La Naturaleza (2) I, 1888, p. 136; GÜNTH., Biol. C.-Am., Rept., p. 198 (1900); BOULENG., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana pipiens SCHREBER, Der Naturforscher, XVIII, 1782, p. 182, Pl. IV; GMEL., Syst. Nat. III, p. 1052 (1788); SHAW, Zool., III, p. 105, Pl. XXXII (1802); LECONTE, Proc. Ac. Philad., 1855, p. 424; GARMAN, Bull. Essex Inst., XX, 1888, p. 95; H. GARMAN, Bull. Illin. Lab., III, 1892, p. 322; RHOADS, Proc. Ac. Philad., 1895, p. 394; DICKERSON, Frog Book, p. 171, Pls. XI, LXIII, LXIV (1906); STRECKER, Proc. Biol. Soc. Washingt., XXI, 1908, pp. 60, 83; WRIGHT, Publ. Carnegie Inst. No. 197, 1914, p. 52, Pl. XV.

Rana utricularia HARLAN, Amer. Journ., X, 1825, p. 59, and Journ. Ac. Philad., V, 1826, p. 337; BOULENG., Cat. Batr. Ecaud., p. 49; H. GARMAN, t. c. p. 321.

Rana palustris (non Leconte), GUÉRIN, Icon. R. Anim., Rept., Pl. XXVI, fig. 1 (1844).

Rana lecontii (non Baird and Gir.), GÜNTH., Cat., p. 42; BROCCHI, op. cit. p. 14, Pl. IV, fig. 1; BOULENG., Cat., p. 49; MOCQUARD, Bull. Soc. Philom. (9), I, 1899, p. 158.

Rana oxyrhynchus (non A. Smith), HALLOW., Proc. Ac. Philad., 1856, p. 142.

Rana berlandieri BAIRD, U. S. Mex. Bound. Surv., Rept. p. 27, Pl. XXXVI, figs. 7–10 (1859).

Rana halecina berlandieri COPE, Check-List N. Am. Batr. Rept., p. 32 (1875).

Rana macroglossa BROCCHI, Bull. Soc. Philom. (7) I, 1877, p. 177, and op. cit., p. 12, Pl. III, fig. 1.

Rana maculata BROCCHI, t. c., p. 178, and op. cit., p. 13, Pl. III, fig. 2.

Rana nigricans part., BROCCHI, op. cit., p. 15, Pl. IV, fig. 3.

Rana forreri BOULENG., Ann. and Mag. N. H. (5) XI, 1883, p. 343; GÜNTH. Biol. C.-Am., Rept., p. 199, Pl. LX, fig. A.

Rana virescens (Kalm), GARMAN, Bull. Essex Inst., XVI, 1884, p. 41; COPE, N. Am. Batr., p. 397 (1889); WERNER, Jahresb. Nat. Ver. Magdeb. 1894, p. 128; ATKINSON, Proc. Indiana Ac., 1895, p. 258; DITMARS, Am. Mus. Journ., V, 1905, p. 198, fig.

Rana virescens sphenocephala, brachycephala, austricola, COPE, N. Am. Batr., l. c.

Rana virescens, var. *austricola* IVES, Proc. Ac. Philad., 1891, p. 461.

Rana pipiens brachycephala TEST, Bull. U. S. Fish Comm., XI, 1892, p. 57; STEJNEG., N. Am. Faun. No. 7, p. 228 (1893).

Rana trilobata MOCQUARD, Bull. Soc. Philom. (9), I, 1899, p. 158, Pl. I, fig. 1.

Rana omiltemana, GÜNTH., Biol. C.-Am., Rept., p. 200, Pl. LXI, fig. A (1900).

Rana sphenocephala DICKERSON, op. cit., p. 186, Pls. XII, LXIX, LXX; STRECKER, Proc. Biol. Soc. Washingt., XXI, 1908, p. 83.

Rana austricola NOBLE, Bull. Amer. Mus. N. H. XXXVIII, 1918, p. 315, pl. XV, fig. 1.

Vomerine teeth in short transverse or oblique series or groups between the choanæ or extending a little beyond the level of their posterior borders, close together or at least nearer to each other than to the choanæ.[5]

Head as long as broad, or a little broader than long, or a little longer than broad, moderately depressed; snout rounded or obtusely or acutely pointed, projecting more or less beyond the mouth, as long as the eye or longer; canthus rostralis obtuse; loreal region oblique, feebly concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the former; distance between the nostrils greater than the interorbital width, which equals $\frac{1}{3}$ to $\frac{2}{3}$ that of the upper eyelid; tympanum very distinct, $\frac{1}{2}$ to once the diameter of the eye, close to the latter or separated from it by a space not more than $\frac{1}{2}$ its diameter.

Fingers obtusely pointed, sometimes bordered by a feeble lateral fold, first longer than second, or first and second rarely equal, third as long as or longer than the mouth; subarticular tubercles moderately large or rather small, moderately prominent.

Hind limb moderate or long, the tibio-tarsal articulation reaching the eye or the tip of the snout, or a little beyond, usually between the eye and the tip of the snout, tubercle strongly overlapping; tibia $3\frac{1}{2}$ to $5\frac{1}{2}$ times as long as broad, $1\frac{2}{3}$ to 2 times in length from snout to vent, as long as or shorter or longer than the fore limb or the foot. Toes obtusely pointed, $\frac{2}{3}$ to nearly entirely webbed, usually 2 phalanges of fourth free, sometimes 1 or 3; outer metatarsals separated nearly to the base; subarticular tubercles rather small, moderately prominent; tarsal fold feeble or absent; inner metatarsal tubercle narrow, more or less prominent or compressed, $\frac{1}{3}$ to $\frac{1}{2}$ the length of the inner toe; outer tubercle absent or very small and indistinct.

Skin of upper parts smooth or granular, with more or less prominent longitudinal warts or glandular longitudinal folds on the back; a prominent, narrow or moderately broad glandular dorso-lateral fold, extending from above the tympanum to the hip or not quite so far; the distance between the dorso-lateral folds, on the back, $4\frac{1}{2}$ to 6 times in length from snout to vent; a glandular fold from below the eye to the vocal sac or to above the arm; glandular ridges sometimes present along the tibia. Lower parts smooth, belly exceptionally feebly granulate, hinder half of thighs granulate.

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[5] In the type of *R. trilobata*, a young specimen which I have examined in the Paris Museum, there is a short furrow between the horns of the tongue, such as I have observed in some *R. esculenta*.

Brown, gray, olive, or bright green above, with dark brown or black spots usually disposed with great symmetry[6] rarely without spots; a dark central streak and a dark temporal blotch sometimes present; tympanum often reddish or bronzy, sometimes with a central and white spot; the dorso-lateral folds usually golden or bronzy; limbs with large dark spots or more or less regular cross-bands; hinder side of thighs yellow, spotted or marbled with black, or black with reddish yellow spots. Lower parts white, throat and breast sometimes spotted, marbled, or mottled with gray or brown.

Male with the vocal sacs internal or more or less developed externally, but not retractile, forming loose folds behind the angle of the mouth, above the arm; arms rather strongly thickened; a strong pad on the inner side of the first finger, covered, during the breeding season, with a wart-like gray or blackish horny layer.

Nasal bones moderately large, separated from each other and from the narrow frontoparietals; ethmoid exposed above, squarely truncate in front, not produced between the nasals; zygomatic branch of the squamosal not or but slightly longer than the posterior. Pectoral arch as in *R. esculenta*. Terminal phalanges slightly expanded at the end.

Tadpole very similar to that of *R. esculenta*, the horny teeth disposed in $\frac{\overset{1}{1-1}}{\underset{2}{1-1}}$ series, the back broadly edged with black.

Eggs very small, only 1½ millim. in diameter. Cf. Miss Dickerson's book for a detailed and copiously illustrated account of the development.

Habitat.—North America, as far north as 52°, and not extending west of the Sierra Nevada; Mexico and Central America as far south as Costa Rica. Reaches 8000 ft. altitude in Colorado, 8500 ft. in Mexico, 5000 ft. in Costa Rica.

As is to be expected in a species of so wide a distribution, there are many variations of structure and coloration, which may be regarded as geographical; they cannot, however, be properly defined in the present state of knowledge. Cope has attempted to put order into the matter by dividing the species into four subspecies with the following definition, but he was careful to remark that they "pass into each other by occasional intermediate specimens": —

a. *sphenocephala*, Cope (*oxyrhynchus*, Leconte). Head entering length of head and body 2½ or less than 3 times; males with external vesicles; muzzle more acuminate; no cross-bars on tibia, spots smaller. — Minnesota, Indiana, Georgia, Florida, Louisiana, the types being from Georgia and Florida.

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[6] Sometimes it is quite the reverse, as shown in a female from Cuttyhunk, Mass., which has 3 large spots on one side of the upper surface (1 palpebral, 2 dorsal) and only one (dorsal) on the other.

b. *austricola*, Cope. Head entering length of head and body 2½ to nearly 3 times; no external vocal vesicles; muzzle more or less acuminate; spots less distinct; tibia generally cross-barred; no longitudinal band in front of femur.— The common Mexican form.

c. *virescens*, Kalm (*utricularia*, Harl.). Head acuminate but shorter, entering the length 3 times; males with external vocal vesicles; spots smaller, not so distinctly yellow bordered; cross-bars of tibia generally interrupted; a longitudinal band on the front of the thigh.— Eastern and southern coasts from Maine to the mouth of the Rio Grande, and up the Mississippi to southern Illinois, and the intermediate country.

d. *brachycephala*, Cope. Head shorter and more obtuse, entering the length 3½ times; males without or with rudimental external vocal vesicles; dorsal spots larger, widely yellow bordered; tibial cross-bands complete; no longitudinal band on the front of the thigh.— The common and only form found between the eastern part of the Great Plains and the Sierra Nevada Mountains, from Lake Superior and the state of Washington to Arizona, Texas, and the Plateau of Mexico; also Quebec, Canada.

Much as I should have wished to fall in with the arrangement proposed by so experienced a herpetologist, who had access to a very large material, although I am afraid he did not make full use of it, I have been unable to follow, and Miss Dickerson observes that living material from New England, New York, Michigan, Minnesota, Wisconsin, Colorado, Texas, and Arizona has not enabled her to make a distinction between the forms *virescens* and *brachycephala*, although she raises *R. sphenocephala* to specific rank.

In the specimens examined by me, the proportion of the head to the body does not give very satisfactory results, and the results, such as they are, do not agree with Cope's geographical scheme. In specimens from Ontario and the New England states, the head is 30 to 35 per cent. of the length from snout to vent; in those from Western Canada, Illinois, Michigan, Idaho, Indiana, Colorado, and Arizona, it is 30 to 36 per cent.; and in those from Florida, Mississippi, and Texas, 32 to 37 per cent.[7]; while it is 30 to 40 per cent., usually 33 to 38 per cent. in those from Mexico and Central America. I should have concluded, from the material studied, that the average length of the head increases from north to south, which does not answer to Cope's definitions, the specimens from Texas and some parts of Mexico being referred by him to the *brachycephala* form in which the head is stated to be 3½ times in the length from snout to vent.

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[7] 29 in one specimen from Texas (No. 48 of table).

The shape of the snout varies much individually, exactly as in *R. esculenta.* The most acutely pointed snouts are in specimens from Florida (var. *sphenocephala*) Mississippi, Texas, and Minnesota, and broadly rounded snouts are to be found among specimens from Lucknow, Ont., Cuttyhunk, Mass., Ithaca, N. Y., Texas, Presidio, near Mazatlan, Jalapa, Nogales, Omilteme, La Cumbre de los Arrestados, Guatemala, and Costa Rica.

The development of external vocal vesicles does not seem to be correlated with the distribution, although the fullest development is shown in specimens from Long Island, Florida (var. *sphenocephala*) and Texas; they are large in males from Lucknow, Texas, and Nogales near Orizaba, small or absent in the others.

I cannot reconcile the presence or absence of cross-bars on the tibia with the habitat; these bars are present in two of the specimens from Lucknow and absent in the two others, and they are present or absent in the specimens from the northwestern parts of America which should, according to Cope, be referred to the subspecies *brachycephala.* The dark longitudinal band on the front of the thigh is absent in most of the specimens from Lucknow which, for the length of the head and the large external vocal sacs, answer to Cope's subspecies *virescens;* I find it more or less well marked only in single specimens from Lucknow, Cambridge, New Jersey, Eau Gallie, and Pensacola, Florida, Brownsville and Duval Co., Texas, Bloomington, Ind., Wet Mountain Valley, Colorado (8000 feet), and various localities in Mexico and Costa Rica.

In his description of the subspecies *sphenocephala*, Cope ascribes to it 3 phalanges of fourth toe free, and I find it so in specimens from Eau Gallie and Moon; but in the two male specimens from Pensacola, and in a female from Canaveral, which are surely referable to this form, only 2 phalanges are free, as in most specimens from other localities.

Miss Dickerson, who allows specific rank to *R. sphenocephala*, mentions that the dark spot on the upper surface of the snout is constantly absent whilst it is usually present in the true *R. halecina* (*pipiens*); but I note its very frequent absence in specimens from all parts of the habitat.

As a rule the yellow or pale green border to the dark spots on the body and limbs is well marked in specimens from North America, but I find it absent in those from Florida, Texas, Arizona, Mexico, and Central America.

There is one character, not previously pointed out as geographically distinctive, which deserves attention. In nearly all specimens from Canada and the United States a well defined light streak extends on each side of the head above the dark or mottled upper lip, from near the end of the snout to the end of the glandular fold behind the mouth; this streak is absent in Arizonian, Texan, Mexican and Central American specimens, or only distinct from below the eye, as in *R. palustris*.

The tibio-tarsal articulation as a rule reaches the tip of the snout or between the eye and the tip of the snout, but sometimes only the eye (Moose Jaw and La Palma), sometimes beyond the tip of the snout (Pensacola, Moon, Hitchcock, La Cumbre de los Arrestados, Coban, Bebedero).

Miss Dickerson says *R. sphenocephala* is "peculiar in possessing a circular white spot at the center of the tympanum." I find this spot quite as sharply defined in specimens from New Jersey, Colorado, and Texas which, having an obtuse snout, do not answer to the definition of *R. sphenocephala*.

I am therefore unable to divide the species in minor groups with any precision, and must leave the matter in abeyance for the present. Probably some day it will be possible to draw up satisfactory definitions of the varieties, the principal of which would be *sphenocephala*, Cope, *forreri*, Blgr., and *austricola*, Cope (*lecontii*, Ster., Brocchi, *nigricans*, Brocchi). Var. *austricola* appears to differ from var. *forreri* by its smoother upper parts and its less sharply defined, often much effaced markings. A precise diagnosis of the var. *sphenocephala* is still a desideratum, as Miss Dickerson's definition, head long and pointed, with the eyes set far back, hind legs unusually long, fingers and toes long and slender, web between the toes deeply indented, and a circular, clear-cut white spot in the center of the tympanum, seems to me insufficient, in view of the variation in specimens referable to the typical form.

Rana halecina.

Measurements in Millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.
	♂	♂	♂	♀	♀	♂	♂	♀	♂	♂	♀	♀	♀	♂	♀	♀	♀	♀	♀	♂	♂
From snout to vent	70	66	57	89	50	76	76	77	74	70	83	78	86	69	75	77	60	69	68	64	76
Head	22	23	19	27	17	25	24	25	25	22	27	25	26	24	25	25	21	24	23	21	25
Width of head	23	24	20	29	17	26	26	27	27	24	28	25	28	24	25	24	21	24	24	21	28
Snout	8	8	7	10	7	10	10	9	9	8	9	8	9	9	10	10	7	9	8	7	8
Eye	8	8	7	9	6	8	8	8	8	8	9	8	8	8	8	8	7	8	8	7	8
Interorbital width	3	3	2.5	3.5	2.5	2	2.5	3	2.5	2	3	3	2.5	2.5	2.5	2	2	2.5	3	2	2.5
Tympanum	6	6	5	7	4	5	5	5	6	5.5	6	6	5	5	5	5	4.5	5.5	5	5	5
Fore limb	46	41	35	47	28	42	45	42	44	40	50	43	45	42	43	40	37	40	41	38	47
1st finger	9	7	6	11	6	9	10	9	9	8	10	9	11	8	8	10	8	8	8	8	9
2nd "	8	6	5	9	5	8	8	8	8	7	9	8	9	7	7	9	7	7	7	7	7
3rd "	11	9	9	12	8	10	11	10	11	9	11	11	11	11	11	11	9	10	11	10	11
4th "	6	5	5	6	4	6	6	6	6	5	7	6	6	5	6	7	6	6	5	5	6
Hind limb	128	107	95	147	79	124	127	125	120	115	142	120	135	132	138	136	117	121	119	105	135
Tibia	40	33	30	46	25	38	40	40	38	37	46	38	44	41	45	44	38	40	36	34	44
Foot	40	35	30	47	26	40	42	41	39	37	46	41	44	44	46	45	39	41	39	35	46
3rd toe	24	20	17	27	15	23	25	23	22	21	26	23	25	25	24	23	18	22	23	21	24
4th "	34	29	26	40	22	35	36	36	33	32	38	35	36	35	38	38	32	34	33	32	39
5th "	25	20	17	27	15	25	25	24	23	22	26	24	26	25	26	25	20	23	23	22	27

1–4. Lucknow, Ontario (Lataste Coll.).— 5. Westbourne, Manitoba.— 6–8. Moose Jaw, Assiniboia.— 9–12. Swift Current, Assiniboia.— 13. Calgary, Alberta.— 14–15. Cambridge, Mass. (Lataste Coll.).— 16–17. Cuttyhunk, Elizabeth Is., Mass.— 18–19. New Jersey.— 20. Long Is., N. Y.— 21. Ithaca, N. Y.

Rana halecina.

Measurements in Millimeters.

	22.	23.	24.	25.	26.	27.	28.	29.	30.	31.	32.	33.	34.	35.	36.	37.	38.	39.	40.	41.	42.
	♂	♀	♂	♂	♀	♂	♀	♂	♀	♂	♀	♂	♂	♂	♂	♀	♀	♂	♀	♀	♂
From snout to vent	67	67	78	62	62	59	64	76	90	70	83	57	57	54	74	76	60	64	53	61	72
Head	23	23	26	20	21	19	20	24	28	25	27	20	20	20	27	28	23	25	19	21	26
Width of head	24	23	29	20	21	19	20	25	31	25	26	20	19	19	25	25	20	22	17	20	26
Snout	8	8	11	8	8	7	7	9	10	9	19	7	8	8	12	12	10	11	7.5	9	9
Eye	8	8	8	7	7	6	7	8	9	8	9	7	6.5	6.5	8	8	7	8	6	7	8
Interorbital width	3	2.5	2.5	2.5	3	2	2	2.5	3	3	3	2	2	2	2.5	3	2.5	2.5	2.5	2.5	3
Tympanum	5	6	7	4.5	4.5	4	5	7	7	6	6	5	4.5	4	6	6	4.5	6	4	5	6
Fore limb	39	40	44	34	35	34	34	44	50	41	47	34	32	32	42	43	32	37	28	37	43
1st finger	8	9	10	8	7	7	7	10	12	9	10	6	6	6	9	10	7	7	6	8	10
2nd "	7	8	9	7	6	6	6	9	10	8	9	5	5	5	7	8	6	6	5	6	9
3rd "	10	11	12	9	9	9	9	12	13	11	13	.8	8	8	10	11	9	10	7	?	13
4th "	6	6	8	5	5	5	5	8	8	6	8	4	5	5	6	6	5	6	4	5	8
Hind limb	120	117	134	101	116	106	110	134	152	125	145	107	99	99	136	144	113	123	87	116	135
Tibia	38	37	45	32	36	33	35	45	50	39	48	35	31	31	44	49	36	39	29	38	45
Foot	39	39	43	33	37	34	37	43	50	40	47	35	33	32	45	50	37	40	29	36	43
3rd toe	23	21	23	20	20	20	20	23	25	22	26	17	16	16	24	27	18	21	14	19	23
4th "	34	33	36	27	30	28	31	36	40	34	42	30	29	27	29	43	32	35	24	30	35
5th "	25	23	26	20	20	20	22	26	28	23	29	20	19	17	26	29	20	23	16	?	24

22. Bay View, Michigan.— 23. Oshkosh, Wisconsin.— 24. Port Snelling, Minnesota.— 25. Caldwell, Idaho.— 26. Bloomington, Indiana.— 27–28. Chicago, Illinois.— 29. Douglas Co., Kansas.— 30. Wray, Colorado.— 31. E. of Boulder, Colorado.— 32. Phoenix, Arizona.— 33. Raleigh, N. Carolina.— 34–38. Pensacola, Florida.— 36–37. Eau Gallie, Florida.— 38. Canaveral, Florida.— 39. Ozona, Florida.— 40. Moon, Mississippi.— 41. Hitchcock, Texas.— 42. Brownsville, Texas.

Rana halecina.

Measurements in Millimeters.

	43.	44.	45.	46.	47.	48.	49.	50.	51.	52.	53.	54.	55.	56.	57.	58.	59.	60.	61.	62.	63.
	♂	♂	♀	♀	♂	♀	♀	♀	♀	♀	♀	♀	♀	♀	♂	♂	♂	♂	♀	♀	♂
From snout to vent	70	67	90	82	70	95	81	82	72	61	58	78	75	81	57	52	75	68	80	66	60
Head	24	25	29	27	24	28	26	28	26	22	21	28	26	26	20	19	26	25	28	23	22
Width of head	24	24	29	27	23	28	26	30	28	22	21	29	27	26	20	19	26	25	28	23	22
Snout	9	10	11	11	10	11	10	9	9	8	8	10	10	10	7	6.5	10	10	10	8	7
Eye	8	7	9	9	8	9	9	9	9	8	7	9	9	8	7	6.5	9	9	10	8	7
Interorbital width	3	3	3.5	3	3	3.5	3.5	2.5	2	2.5	2.5	2	3	3	2	2	3	3	3.5	3	2.5
Tympanum	6	6	7	6	5	7	7	7	7.5	5	5	6	6	7	5	5	6	5.5	6	5	7
Fore limb	45	39	49	47	43	47	45	50	43	35	33	44	42	45	34	32	43	42	51	37	38
1st finger	10	8	10	11	8	10	10	12	9	7	7	9	10	10	7	6	8	8	10	8	7.5
2nd "	9	7	9	9	7	9	9	11	9	6	6	8	8	8	6	5	7	7	8	7	6.5
3rd "	12	10	12	12	10	12	12	13	11	8	8	11	11	11	8	8	11	10	12	10	9
4th "	8	5	7	7	6	8	7	8	6	5	5	7	6	6	5	4	6	6	8	6	6
Hind limb	128	115	146	143	130	153	140	141	129	103	101	133	128	141	95	99	130	120	145	118	109
Tibia	42	39	48	47	42	54	46	43	40	32	31	42	40	45	30	31	40	39	47	37	35
Foot	41	38	46	45	40	50	45	48	42	33	31	44	41	45	30	31	44	39	47	39	37
3rd toe	22	19	24	23	21	28	24	29	24	17	16	26	25	25	17	18	26	23	27	22	21
4th "	35	31	40	38	34	42	40	41	37	28	26	39	34	37	24	26	39	33	39	32	31
5th "	25	22	28	27	25	30	27	29	27	19	18	27	25	22	18	20	27	23	30	24	23

43. Brownsville, Texas.— 44–46. Duval Co., Texas.— 47–49. Texas.— 50. Bocoina, Sierra Tarahumari, N. W. Mexico.— 51. Presidio, near Mazatlan (type of *R. forreri*).— 52–53. Presidio, near Mazatlan.— 54–55. Ciudad, Durango.— 56. Hacienda de Abequiza, Guadalajara.— 57–58. La Cumbre de los Arrestados, Jalisco.— 59–62. Jalapa, Vera Cruz.— 63. Omilteme, Guerrero (types of *R. omiltemana*).

Rana halecina.

Measurements in Millimeters

	64.	65.	66.	67.	68.	69.	70.	71.	72.	73.	74.	75.	76.	77.	78.	79.	80.	81.	82.	83.	84.	85.
	♂	♂	♂	♂	♀	♀	♂	♂	♀	♀	♀	♀	♂	♂	♂	♀	♂	♂	♀	♂	♀	♀
From snout to vent	58	58	57	79	92	85	80	72	90	102	100	77	55	61	69	90	57	50	69	59	75	73
Head	23	22	22	30	32	31	29	27	30	35	35	26	20	22	23	27	20	17	25	21	26	26
Width of head	23	22	22	28	32	31	29	25	30	35	35	26	22	24	23	27	20	17	25	21	26	26
Snout	7	7	7	12	13	12	11	10	11	12	12	11	6	8	9	12	7	6	9	7	10	10
Eye	7	7	7	10	10	10	9	8	9	11	11	9	6	7	8	10	7	6	8	7	9	9
Interorbital width	3	2.5	2.5	3	4	3	3.5	3	4	4	4	3.5	3	3	3	3.5	2.5	2	2.5	2	3	3
Tympanum	5.5	6	5.5	6.5	7	5	6	6	6	9	9	6	4	5	5	6	5	4	6	4	5	5
Fore limb	35	36	32	47	50	52	46	45	46	62	60	48	39	36	44	47	35	32	41	35	44	43
1st finger	7.5	7.5	7	10	11	11	9	8	10	15	14	11	6	8	9	10	6	6	9	6	8	8
2nd "	6.5	6.5	6	8	9	9	8	7	9	11	11	9	6	7	8	8	5	5	7	5	7	7
3rd "	8	9	8	12	13	12	11	11	12	16	16	12	8	11	11	12	8	7	10	7	10	10
4th "	6	6	5	7	8	7	7	7	7	9	10	9	6	6	7	7	5	4	6	4	6	6
Hind limb	116	110	101	142	170	146	149	124	153	182	186	143	108	110	124	140	105	91	126	100	136	136
Tibia	36	33	32	47	53	47	47	41	48	62	61	47	34	35	38	46	32	27	41	32	43	42
Foot	40	36	33	46	52	48	47	42	48	57	59	46	34	37	42	46	36	30	41	34	45	44
3rd toe	21	21	19	26	28	25	26	24	26	32	33	26	18	20	22	25	19	16	24	17	26	25
4th "	32	31	28	40	45	38	41	36	41	48	50	40	29	32	35	40	31	25	35	27	38	37
5th "	23	21	20	28	31	26	31	24	29	34	35	27	20	24	25	27	21	17	26	18	26	26

64–66. Omilteme, Guerrero (types of *R. omiltemana*).— 67–68. San Luis Allando.— 69. Tequasixtlan, Tehuantepec.— 70–72. Nogales near Orizaba.— 73–74. N. of Coban, Guatemala.— 75. Duenas, Guatemala.— 76. Totonicapan, Guatemala (type of *R. maculata*).— 77. Guatemala (type of *R. macroglossa*).— 78–79. Guatemala.— 80–82. Bebedero, Costa Rica.— 83–85. La Palma, Costa Rica.

9. Rana palustris.

Rana palustris Leconte, Ann. Lyc. N. Y., I, 1825, p. 282; Harlan, Amer. Journ., X, 1825, p. 59, and Journ. Ac. Philad., V, 1826, p. 339; Holbr., N. Amer. Herp., I, p. 93, Pl. xiv (1836); Dum. & Bibr., Erp. Gén., VIII, p. 356 (1841); Holbr., op. cit. IV, p. 95, Pl. xxiii (1842); Dekay, N. Y. Faun., Rept., p. 62, Pl. xxii, fig. 60 (1842); Leconte, Proc. Ac. Philad., 1855, p. 424; Günth., Cat. Batr. Sal., p. 14 (1858); Weid, N. Acta Ac. Leop.-Carol., XXII, 1865, p. 114; Hinckley, Proc. Bost. Soc. N. H., XXI, 1881, p. 311, Pl. v, fig. 9; Bouleng., Cat. Batr. Ecaud., p. 42 (1882); Cope, Batr. N. Amer., p. 406 (1889); H. Garm., Bull. Illin. Labor., III, 1892, p. 225; Werner, Jahresb. Nat. Ver. Magdeb., 1894, p. 133; Ditmars, Amer. Mus. Journ., V, 1905, p. 200, fig.; Dickerson, Frog Book, p. 188, Pls. xiii and lxxii (1906); Wright, Publ. Carnegie Instit. No. 1917, 1914, p. 61, Pl. xvi; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana pardalis Harlan, Amer. Journ. X, 1825, p. 59.

Vomerine teeth in oblique groups or short series between the choanæ or extending a little beyond the level of their posterior borders, close together or at least nearer to each other than to the choanæ.

Head as long as broad, moderately depressed; snout rounded or obtusely pointed, projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region moderately oblique, feebly concave; nostril equidistant from the eye and from the tip of the snout or a little nearer the former; distance between the nostrils greater than the interorbital width, which is ⅓ to ½ that of the upper eyelid; tympanum very distinct, ⅗ to ⅚ the diameter of the eye, close to the latter or separated from it by a space not more than ½ its diameter.

Fingers obtuse, first longer than second, third as long as or longer than the snout; subarticular tubercles rather large, prominent.

Tibio-tarsal articulation reaching the eye or the tip of the snout, or between these two points; heels strongly overlapping when the limbs are folded at right angle to the body; tibia 4 to 5 times as long as broad, 1⅔ to 2 times in length from snout to vent, as long as or a little longer or a little shorter than the fore limb, or the foot. Toes obtuse, about ¾ webbed, 2 phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles moderately large and prominent; no tarsal fold; inner metatarsal tubercle oval, ⅓ to ⅖ the length of the inner toe; a very small outer tubercle usually present.

Back with four broad and flat glandular longitudinal folds, the

Rana palustris

Measurements in Millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.
	♂	♀	♂	♂	♀	♀	♀	♀	♀	♂	♂	♂	♀	♀	♀	♂	♂	♀
From snout to vent	54	76	56	52	79	63	62	60	57	52	52	50	54	63	59	51	46	75
Head	19	24	20	20	23	20	20	19	19	17	18	18	19	22	21	19	16	25
Width of head	19	24	20	20	23	20	20	19	19	17	18	18	19	22	21	19	16	25
Snout	6	8	7	6	9	6	8	6	6	7	7	6	7	8	7	7	5.5	10
Eye	6	8	6	6	8	6	6	6	6	6	6	6	6	7	7	6	5.5	8
Interorbital space	2	3	1.5	2	3	2.5	2	2	2	2	2	2	2	2.5	2.5	2	2	3
Tympanum	4	6	4	4	6	5	4	4	4	4	4	3.5	4	4	4	4	3.5	6
Fore limb	35	44	34	32	41	34	37	33	34	30	30	29	31	36	36	30	28	43
1st finger	6	9	7	6	8	7	8	7	7	6	6	6	6	7	9	6	5	9
2nd "	5	7	6	5	7	6	6	6	6	5	5	5	5	6	8	5	4	8
3rd "	8	10	9	7.5	10	9	9	8	9	7	8	7	7	9	10	7	7	10
4th "	5	6	6	5	6.5	6	6	5	6	4	5	5	5	6	7	5	4	7
Hind limb	95	120	97	91	116	100	98	94	97	86	90	87	94	108	113	89	80	121
Tibia	30	39	32	30	43	32	33	30	31	28	29	29	30	36	36	28	25	38
Foot	31	40	32	28	37	31	32	31	31	28	29	29	30	36	37	30	26	39
3rd toe	16	20	17	17	21	18	17	18	18	14	16	15	16	20	20	16	13	21
4th "	26	33	27	25	30	27	27	27	27	23	25	25	25	31	32	25	20	33
5th "	18	24	19	18	22	19	21	19	19	17	18	16	18	21	22	19	15	24

1. Lucknow, Ontario (Lataste Coll).— 2. Philadelphia.— 3–9. Long Island, New York.— 10–12. Shokan, New York.— 13. Pike Co., Pennsylvania.— 14. Brevard, N. Carolina.— 15. Pineola, N. Carolina.— 16–18. N. America.

dorso-lateral extending from above the tympanum to the hip, the median pair extending from behind the eyes to the sacral region, where they are replaced by another pair closer together; the distance between the dorso-lateral folds 4 to 6 times in length from snout to vent; a few flat glands sometimes present, in addition to the folds on the back; a glandular fold from below the eye to above the arm. Lower parts smooth.

Brown above, the dorso-lateral folds usually not much lighter, bronzy; a dark brown or blackish spot on the snout and another on each upper eyelid; two series of very large, often squarish, dark brown or black spots along the back, some of them sometimes longitudinally or transversely confluent; sides with one or two series of black spots, or with black vertical base; a dark ventral streak; a light streak on the glandular fold below the eye, rarely extending to the tip of the snout; tympanum reddish or bronzy; limbs with very regular dark brown or black cross-bands; hinder side of thighs yellow or orange, with black spots or marblings. Lower parts white, rarely with pale brown spots, posterior part of belly and hind limbs yellow or orange.

Male with internal vocal sacs; fore limbs very strong, with a large pad on the inner side of the first finger.

Osteological characters as in *R. halecina.*

Tadpole also not essentially different.

Eggs very small, little over 1 millimeter in diameter.

Habitat.— North America east of the Mississippi.

R. palustris is very closely allied to *R. halecina,* and although its specific rank is unquestionable, it is by no means easy to point out constant characters for its distinction.

10. **Rana draytonii.**

Rana draytonii Baird and Gir., Proc. Ac. Philad., 1852, p. 174, and U. S. Explor. Exped., Herp., p. 23, Pl. ii, figs. 19–24 (1858); Cope, Proc. Am. Philos. Soc., XXIII, 1886, p. 521, and Batr. N. Am., p. 441, figs. (1889); Stejneg., N. Am. Faun., No. 7, Pt. ii, p. 225 (1893); Van Denburgh, Proc. Calif. Ac. (2) V, 1896, p. 1008; Dickerson, Frog Book, p. 213, Pls. xv, lxxx, lxxxi (1906); Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana lecontei Baird and Gir., Proc. Ac. Philad., 1853, p. 301.

Rana nigricans (non Ag.), Hallow., Proc. Ac. Philad., 1854, p. 96; Bouleng., Bull. Soc. Zool. France, 1880, p. 207; Brocchi, Miss. Sc. Mex., Batr., p. 15, Pl. iv, fig. 3 (1881); Bouleng., Cat. Batr. Ecaud., p. 43 (1882).

Rana longipes Hallow., Rep. U. S. Explor. Surv., X, iv, Zool., p. 20, Pl. x, fig. 1 (1859).

Rana nigricans, part., Brocchi, op. cit., p. 15.

Rana aurora draytonii Camp, Univ. Calif. Publ. Zool., XVII, 1917, p. 123.

Vomerine teeth in small oblique groups between or just behind the choanæ, close together or at least nearer to each other than to the latter.

Head broader than long, rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region oblique, feebly concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils equal to or a little greater than the interorbital width, which is usually equal to or a little less than that of the upper eyelid, but sometimes only ⅔ that width; tympanum very distinct, smooth, or, exceptionally, with a few granular asperities, ⅔ to once the diameter of the eye, 1⅓ to 2 times its distance from the latter.

Fingers rather long, obtuse, with a more or less distinct dermal margin, first longer than second, third much longer than the snout; subarticular tubercles moderate.

Hind limb moderate or long, the tibio-tarsal articulation reaching the eye, the tip of the snout, or between these two points, the heels overlapping when the limbs are folded at right angles to the body; tibia 3½ to 5 times as long as broad, 1⅔ to 2⅒ times in length from snout to vent, shorter than the fore limb, slightly or considerably shorter than the foot. Toes obtuse, ¾ to nearly entirely webbed, one or two phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles small or rather small, moderately prominent; tarsal fold more or less distinct; inner metatarsal tubercles oval or elliptical, ⅓ to ⅔ the length of the inner toe; outer tubercle indistinct or absent.

Upper parts smooth or with small warts and pearl-like excrescences; a more or less prominent, moderately broad glandular dorso-lateral fold from above the tympanum to the hip or not quite so far, parallel with its fellow, from which it is separated by a space 3⅔ to 5 times in the length from snout to vent; a glandular fold from below the eye to above the arm, sometimes followed by a glandule; lower parts smooth, hinder half of thighs coarsely granulate.

Olive, brown, or reddish brown above, with moderately large roundish dark spots, which may be lighter in the center; a dark central streak and sometimes a more or less distinct dark temporal spot; a light streak from below the eye to the end of the glandular fold behind the mouth; limbs with more or less regular dark cross-bands; lumbar region and sides of thighs yellow, marbled with black, or black with round yellow spots. Lower parts uniform white or spotted or marbled with blackish. In life the sides of the body and the lower surface of the hind limbs may be bright red.

Male without vocal sacs, with a strong pad on the inner side of the first finger.

Nasal bones rather small, widely separated from each other and from the frontoparietals; ethmoid exposed and obtusely pointed in front.

Habitat.—Western North America, from British Columbia to the mountains of Lower California, up to an altitude of 4000 feet.

Measurements in Millimeters

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
	♀	♀	♀	♀	♀	♂	♀	♂	♀	♀	♀	♀
From snout to vent	110	106	84	72	72	70	85	63	75	68	118	115
Head	34	35	27	23	22	25	29	21	23	22	37	34
Width of head	41	40	30	26	25	27	35	23	28	24	45	42
Snout	13	12	10	8	8	9	11	7	7.5	7	12	12
Eye	10	11	8	7	6.5	8	9	7	7.5	7	12	11
Interorbital width	6	6	3.5	3	3	3	5	3.5	4	4	7	7
Tympanum	10	8	5	4	5	6	6	4	5	5	10	9
Fore limb	70	69	55	47	45	45	56	37	46	42	74	68
1st finger	17	17	13	11	11	9	13	8	10	9	18	17
2nd "	15	14	11	10	10	8	11	7	8	8	15	14
3rd "	18	18	16	15	14	13	15	11	12	11	22	20
4th "	13	12	10	9	8	8	9	7	7	7	14	13
Hind limb	171	174	145	127	125	120	159	105	124	113	190	182
Tibia	52	52	45	41	40	38	52	34	41	37	60	58
Foot	59	58	49	43	42	42	52	36	42	39	61	61
3rd toe	34	32	20	23	22	23	29	20	23	22	33	34
4th "	50	50	42	36	36	35	46	30	35	33	53	53
5th "	36	37	30	28	26	25	32	22	25	23	40	39

1–5. San Francisco.—6. San Diego.—7. Tia Juana, San Diego Co.—8–10. Riverside, Cal.—10–12. Pacific Region of N. America.

11. **Rana aurora.**

Rana aurora BAIRD AND GIR., Proc. Ac. Philad., 1852, p. 174, and U. S. Explor. Exped., Herp., p. 18, Pl. II, figs. 1–6 (1858); STEJNEG., N. Am. Faun. No. 7, pt. II, p. 225 (1893); DICKERSON, Frog Book, p. 216, Pls. XIV and LXXXII

(1906); Camp, Univ. Calif. Publ. Zool., XVII, 1917, p. 123; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana temporaria aurora Cope, Proc. Ac. Philad., 1883, pp. 27, 28.

Rana agilis aurora Cope, Proc. Am. Philos. Soc. XXIII, 1886, p. 521, and Batr. N. Am., p. 441, figs. (1889); Meek, Field Col. Mus., Zool. I, 1899, p. 332.

Vomerine teeth in oblique groups close together just behind the level of the choanæ.

Head as long as broad or slightly broader than long, much depressed; snout rounded, feebly projecting beyond the mouth, as long as or slightly longer than the eye; canthus rostralis obtuse; loreal region very oblique, scarcely concave; nostril a little nearer the eye than the tip of the snout; distance between the nostrils greater than the interorbital width, which is ½ to ⅔ that of the upper eyelid; tympanum very distinct, about ⅗ the diameter of the eye, 1⅓ to 2 times its distance from the latter.

Fingers rather long, obtuse, first as long as or a little longer than the second, third much longer than the snout; subarticular tubercles moderate.

Hind limb long, the tibio-tarsal articulation reaching the nostril, the tip of the snout, or a little beyond, the heels overlapping where the limbs are folded at right angles to the body; tibia 4 to 4½ times as long as broad, 1¾ to 1⅞ times in length from snout to vent, shorter than the fore limb, a little shorter than the foot. Toes obtuse, ⅔ webbed, two phalanges of fourth free, outer metatarsals separated nearly to the base; subarticular tubercles small or rather small, moderately prominent; no distinct tarsal fold; inner metatarsal tubercle oval, prominent, ¼ to ⅓ the length of the inner toe; a more or less distinct outer tubercle.

Upper parts smooth; a rather broad, more or less prominent glandular dorso-lateral fold from above the tympanum to the hip or not quite so far, parallel with its fellow, from which it is separated by a space 6 times in the length from snout to vent; a glandular fold from below the eye to above the arm, followed by a glandule. Lower parts smooth, hinder half of thighs, and sometimes posterior part of belly, granulate.

The specimens in the British Museum, for which I am indebted to the kindness of Miss Dickerson, are brown above with small dark spots, a dark temporal blotch, a light streak from below the eye to the shoulder, and dark cross-bands on the limbs, in fact very similar to ordinary specimens of *R. temporaria* except for the absence of the Λ-shaped marking between the shoulders. According to Miss

Dickerson, the upper parts are brown, yellowish, or olive; there may be small spots or specks of dark brown or black on head, back, and sides; much red on the parts of the legs and feet which are concealed when folded, or the sides of the body, and at the base of the arms; reticulations of yellow and black on the body and thighs where they lie against each other. Throat and belly white, mottled with dark.

Male without vocal sacs; fore limbs strong; a thick pad on the inner side of the first finger.

Nasal bones small, transverse, with concave posterior border, separated from each other and from the ethmoid, which is obtusely pointed in front.

The tadpoles of *R. draytoni* and *R. aurora* have not been described.

Habitat.— Washington, Oregon, and California.

Measurements in Millimeters

	♂	♀	♀
From snout to vent	54	76	73
Head	18	26	25
Width of head	19	26	25
Snout	6	9	8
Eye	6	8	8
Interorbital width	2.5	3	3
Tympanum	3.5	5	5
Fore limb	35	48	47
1st finger	7	9	10
2nd “	7	9	9
3rd “	11	13	14
4th “	7	10	10
Hind limb	101	136	128
Tibia	31	42	39
Foot	34	43	41
3rd toe	16	22	21
4th “	28	35	34
5th “	18	26	26

Specimens from Seattle, Washington.

This frog is very closely related to *R. draytonii*, with which I had previously united it (Ann. & Mag. N. H. [6] VIII, 1891, p. 453), but it appears to be distinguishable, in addition to minor points, by the smaller tympanum, as pointed out by Baird & Girard in the original description. However, according to Camp, specimens from Mendocino City, Coast of California, "are intergrades between *aurora* and *draytonii*" and he accordingly regards the latter as a subspecies of the former; but he does not state in what respects they intergrade. I therefore provisionally maintain the two as species.

I regard *R. aurora* as intermediate between *R. draytonii* and *R. pretiosa*, but nearer the former. Until I was able to examine authentic specimens of this frog, which has unaccountably been referred as a subspecies to the European *R. agilis*, I felt somewhat doubtful as to the validity of this species.

The description of Baird and Girard and of Cope do not afford convincing characters for the separation from *R. draytoni*, and Miss Dickerson makes the following statement:—

"*Rana aurora* is easily confused with *R. draytonii*, because of the likeness, both in coloring and proportions. The former frog can be distinguished by the following characteristics: It has a longer leg, the greater length especially noticeable in the tibia.[8] Its skin is very smooth, while that of *R. draytonii*, especially in the full-grown frogs is always tubercular,[9] and may be extremely so (even on the ear).[10] *R. aurora* has a longer foot with a broader web, but with a much smaller inner sole tubercle."[11] It is said not to exceed 90 millimeters from snout to vent.

Cope says the two last phalanges of the fourth toe are free from the web in *R. aurora*, instead of only the last in *R. draytonii*, but the figures in his book show absolute identity between the two species in this respect. He adds that the tongue is "small and narrow" in the former and "not large" in the latter, but the tongue is decidedly small and narrow in our specimens from Riverside, remarkably so in the male,[12] which I cannot otherwise distinguish from *R. draytonii*.

According to Cope's measurements (in percentages) the differences in the proportions between the two species are:—

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[8] Tibio-tarsal articulation reaching beyond end of snout; tibia much longer than femur.—Meek, probably guided by Cope's description, refers to this species a specimen in which the heel only reaches the center of the eye.

[9] Some of my largest *R. draytonii* have the upper parts smooth.

[10] This statement is interesting in connection with Stejneger's comments on my suggestion that *R. boylii* might be a variety of *R. draytoni*.

[11] One third the length of the inner toe, according to Miss Dickerson's figure.

[12] I also find a remarkably small tongue in some males of *R. japonica* from Satsuma, and of *R. temporaria* from the Alps.

Measurements in Millimeters.

	R. draytonii	*R. aurora.*
Snout to vent	100	100
Head	34	36
Width of head	36	35
Eye	12	11
Tympanum	8	7
Fore limb	75	66
Hind limb	173	170
Femur	50	48
Tibia	51	52
Foot	56	53

There is not much light to be derived from these measurements[13] considering the individual variation shown by the specimens tabulated by me under *R. draytonii,* all of which, I am convinced, belong to one species only.

12. Rana pretiosa.

Rana pretiosa Baird and Gir., Proc. Ac. Philad., 1853, p. 378; Baird, Proc. Ac. Philad., 1854, p. 62; Girard, U. S. Explor. Exped., Herp., p. 20, Pl. ii, figs. 13–18 (1858); Cooper, Rep. U. S. Explor. Surv., XII, part 2, p. 304 (1860); Cope, in Hayden's Rep. U. S. Geol. Surv. Montana, p. 469 (1872); Bouleng., Bull. Soc. Zool. France, 1880, p. 208, and Cat. Batr. Ecaud., p. 43 (1882); Cope, Proc. Ac. Philad., 1883, pp. 16, 20; Test, Bull. U. S. Fish Comm. XI, 1892, p. 58; Stejneg., N. Am. Faun. No. 7, part 2, p. 225 (1893); Dickerson, Frog Book, p. 218, Pls. xvi, lxxxiii (1906); Camp, Univ. Calif. Publ. Zool., XVII, 1917, p. 123; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana temporaria pretiosa Cope, Batr. N. Am., p. 432, figs. (1889); Meek, Field Col. Mus., Zool. I, 1899, p. 232.

Rana pretiosa luteiventris H. B. Thompson, Proc. Biol. Soc. Washingt., XXVI, 1913, p. 53; Camp, t. c., p. 123.

Vomerine teeth in oblique groups close together just behind the level of the choanæ.

Head a little broader than long, rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region oblique, feebly concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils usually greater than the interorbital width, which equals ⅔ to ¾ that of the upper eyelid; tympanum more or less distinct, ⅖ to ⅔ the diameter of the eye, 1½ to 2 times its distance from the latter.

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[13] Which, contrary to Miss Dickerson's definition, shows a slightly shorter hind limb in *R. aurora* as compared with *R. draytonii.*

Fingers moderate, obtuse or rather pointed, first longer than second, third usually longer than the snout; subarticular tubercles moderate.

Hind limb moderately long, the tibio-tarsal articulation reaching the tympanum or the eye, the heels meeting or slightly overlapping where the limbs are folded at right angles to the body; tibia 3 to 4½ times as long as broad, 2 to 2⅕ times in length from snout to vent, shorter than the fore limb, shorter than the foot. Toes obtuse, ¾ to nearly entirely webbed, one or two phalanges of fourth free, the web thicker and with convex border in the breeding male; the web extending about half way between the outer metatarsals; subarticular tubercles small, moderately prominent; no tarsal fold; inner metatarsal tubercle oval, ¼ to ⅓ the length of the inner toe; a small, round, or conical outer tubercle usually present.

Upper parts smooth or with small warts; pearl-like horny tubercles often present on the body and hind limbs; a moderately broad and moderately prominent glandular dorso-lateral fold from above the tympanum to the hip; the dorso-lateral folds parallel or slightly converging towards each other on the scapular region, where they are separated from each other by a space contained 4½ to 6 times in the length from snout to vent. Lower parts smooth.

Brown, yellowish, or reddish above, with small darker spots which may have light centers and often with more or less profusely scattered ink-black spots on the back; flanks with large dark spots or marblings; a dark streak from the tip of the snout to the eye and a more or less marked dark temporal spot; a light streak from below the nostril to above the arm; limbs usually with more or less regular dark cross-bands. Lower parts white, uniform or mottled or marbled with gray; in life salmon-red on hind limbs and in a more or less conspicuous U-shaped marking on the belly.

Male without vocal sacs, with very strong fore limbs and a large pad on the inner side of the first finger.

Skeleton similar to that of *R. temporaria.*

The tadpole has not been described, but the eggs are known to be very similar to those of *R. temporaria*, nearly entirely black and nearing 2 to 2½ millimeters in diameter.

Habitat.—North America, from the Rocky Mountains westwards, from British Columbia to California.

Rana pretiosa.

Measurements in Millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.
	♂	♀	♀	♀	♀	♀	♀	♂	♀	♀	♂	♀	♀	♀	♀	♀	♀	♂	♀	♂	♀	♀	♀
From snout to vent..	55	78	78	75	72	63	62	51	66	65	53	69	64	70	73	66	70	56	60	53	65	62	67
Head...............	20	26	25	23	24	22	20	18	20	20	18	20	20	21	21	20	21	19	20	17	21	20	22
Width of head.......	21	30	27	24	26	25	21	19	22	22	19	22	22	23	22	21	23	21	21	20	23	21	24
Snout..............	7	8	8	8	8	7	7	6	7	7	6.5	7	8	7	7	7	7	6.5	7	6	7	7	7
Eye...............	6	7	7	7	7	6	6	6	7	7	6.5	7	7	7	6	7	7	6.5	7	5.5	6	6	7
Interorbital width....	2.5	3	3	3	3	3	3	3	3	3	2	2.5	3	3	2.5	3	3	2.5	3	2	3.5	3	3
Tympanum.........	3	4	4	4	4	3.5	3	3	4	4	3	4	4	4	4	4	4	3.5	4	3	4	4	4
Fore limb...........	35	46	44	43	43	40	37	31	41	42	32	40	37	39	41	39	40	38	39	33	38	35	42
1st finger...........	6	10	10	9	9	8	7.5	6	8	8	6	8	8	8	8	8	8	7	8	6	8	8	10
2nd "	5	8	8	8	8	7	7	5	7	7	5	7	7	7	7	7	7	6	7	5	7	7	9
3rd "	7	11	11	10	10	9	8	7	10	10	7	10	10	10	10	10	10	9	9	8	10	10	11
4th "	5	7	7	7	7	5	5	4	7	7	4.5	6	6	6	6	6	6	6	6	5	7	6	8
Hind limb...........	87	124	119	115	111	108	102	85	107	108	78	106	98	103	108	104	100	91	97	86	103	98	110
Tibia..............	26	38	36	35	33	32	31	26	32	32	24	32	30	33	33	31	32	27	28	26	31	30	34
Foot...............	30	42	40	38	38	37	35	30	37	36	26	36	33	35	37	36	34	32	33	30	36	34	39
3rd toe.............	18	21	21	20	21	19	18	17	20	20	15	18	17	18	20	19	18	17	19	16	19	18	23
4th "	24	32	32	30	30	28	27	25	29	28	21	29	26	27	29	29	27	25	27	23	28	27	31
5th "	18	23	23	22	21	20	19	17	20	20	15	20	18	20	21	21	19	18	20	17	22	30	23

1-7. Banff, Alberta.— 8. L. Louise, Alberta.— 9-10. Rocky Mts., B. N. A.— 11-13. Field, Brit. Columbia.— 14. Kicking Horse R., Mt. Field.— 15-16. Revalstoke, Brit. Columbia.— 17. Ice R. valley, Brit. Columbia.— 18-19. Suman Prairies, Brit. Columbia.— 20-22. Le Grande, Oregon.— 23. Seattle, Washington.

Although this frog bears a general resemblance to *R. temporaria*, it is difficult to understand how a herpetologist of Cope's standing should have regarded it as only subspecifically distinct from that species. It is true the subordination of *R. aurora* to *R. agilis* is an even more inconceivable example of laxity in his treatment of the subject. *R. pretiosa* differs from *R. temporaria* in the more oblique loreal region, the usually narrower interorbital space, the frequent presence of a very distinct outer metatarsal tubercle, and, above all, in the absence of vocal sacs in the male. It is further to be observed that the Λ-shaped glandular ridge, accompanied by a dark marking, which is frequent in *R. temporaria* and in the European and Asiatic species that cluster round it, is constantly absent in *R. pretiosa* as well as in the other American species.

For measurements of *Rana pretiosa* see p. 454.

13. **Rana cantabrigensis.**

Rana cantabrigensis Baird, Proc. Ac. Philad., 1854, p. 62; Bouleng., Bull. Soc. Zool. France, 1880, p. 209, and Cat. Batr. Ecaud., p. 45 (1882); Cope, Proc. Amer. Philos. Soc., XXIII, 1886, p. 519, and Batr. N. Am., p. 435 (1889); Bouleng., Ann. and Mag. N. H. (6) VIII, 1891, p. 453; Howe, Proc. Bost. Soc. XXVIII, 1899, p. 369; Dickerson, Frog Book, p. 211 (1906); Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana temporaria, var. *silvatica*, part., Günth., Cat. Batr. Sal., p. 17 (1858).

Rana sylvatica Bouleng., Bull. Soc. Zool. France, 1879, p. 174.

Rana cantabrigensis latiremis Cope, t. c., p. 520, and op. c., p. 435; Howe, t. c. p. 373.

Rana cantabrigensis evittata Cope, op. c., p. 435.

Vomerine teeth in small oblique groups just behind the level of the choanæ, close together or at least nearer to each other than to the latter.

Head a little broader than long, rather strongly depressed; snout rounded or obtusely pointed, more or less projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region oblique, concave; nostril equidistant from the eye and from the tip of the snout, or nearer the former; distance between the nostrils greater than the interorbital width, which is $\frac{2}{3}$ to $\frac{3}{4}$ that of the upper eyelid; tympanum more or less distinct, about $\frac{1}{2}$ the diameter of the eye and 2 to 3 times its distance from the latter.

Fingers moderate, obtuse, first longer than the second, third a little longer than the snout; subarticular tubercles moderate.

Hind limb moderately long, the tibio-tarsal articulation reaching the tympanum or the eye, the heels meeting or slightly overlapping when the limbs are folded at right angles to the body; tibia 3 to 4 times as long as broad, 2 to 2½ times in length from snout to vent, shorter than the fore limb and than the foot. Toes obtuse, ½ to ¾ webbed, 2 or 3 phalanges of fourth free, the web thicker and with convex border in the breeding male; the web extending about half way down between the outer metatarsals; subarticular tubercles rather small, moderately or feebly prominent; tarsal fold feeble or absent; inner metatarsal strongly projecting, hard, feebly compressed, semicircular in outline, ½ to ⅔ the length of the inner toe; a small, round outer tubercle present or absent.

Skin of upper parts smooth or with small flat glands; a moderately broad, flat or moderately prominent glandular dorso-lateral fold, from above the tympanum to the hip; these folds parallel or converging on the anterior part of the back, where they are separated from each other by a space contained 4½ to 5½ times in length from snout to vent; a second pair of more or less wavy glandular folds along the middle of the back; a glandular fold from below the eye to above the arm. Lower parts smooth, posterior half of thighs with large flat granules.

Yellowish or reddish brown above, usually with dark brown or blackish spots or bands bordering the glandular dorsal folds; a dark streak from the end of the snout to the eye and a large dark temporal spot; a light streak on the upper lip, from the end of the snout to the extremity of the glandular fold behind the mouth; limbs usually with more or less regular dark cross-bands; frequently a yellow line from the tip of the snout to the vent and another along the middle of the thigh and the inner side of the tibia. Lower parts white, uniform, or throat and breast spotted with blackish.

Male with internal vocal sacs, strong fore limbs, a large pad on the inner side of the first finger, and the web between the toes more developed than in the female.

Nasal bones small, oblique, widely separated from each other; ethmoid largely exposed above, angular in front and penetrated between the nasals; frontoparietals separated from each other in front, embracing a fontanelle.

Measurements in millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
	♂	♀	♂	♂	♀	♀	♀	♀	♀	♂	♀	♀
From snout to vent	40	40	44	44	43	56	50	37	37	46	43	51
Head	14	14	15	15	14	17	16	13	13	16	14	16
Width of head	15	14	16	15	15	19	17	14	13	17	15	17
Snout	4.5	4.5	5	5	5.5	6	5.5	4.5	4.5	6	5	6
Eye	4.5	4.5	5	5	5	6	5.5	4.5	4.5	6	5	6
Interorbital width	2.5	2.5	3	2.5	3	3	2.5	2	2	2.5	2.5	2.5
Tympanum	3	3	2.5	3	2.5	3	3	2	2	3	2	3
Fore limb	22	20	25	25	24	30	28	20	21	26	25	27
1st finger	4	4	4	4	4	7	5	3.5	4	4.5	4.5	5
2nd "	3.5	3.5	3.5	3.5	3.5	5.5	4	3	3	3.5	3.5	4
3rd "	6	6	6.5	6	6	8	6.5	5	5	6.5	6.5	7
4th "	3	3	3	3	3	5	3.5	2.5	2.5	3	3	3
Hind limb	58	60	67	67	66	83	76	52	55	75	72	70
Tibia	17	18	20	20	19	25	22	15	16	23	21	20
Foot	19	20	23	23	22	27	24	17	18	25	25	23
3rd toe	10	11	13	13	12	14	13	9	10	13	12	11
4th "	15	15	18	18	17	22	20	14	15	20	20	18
5th "	10	10	11	11	11	13	12	9	10	13	12	11

1–2. Port Smith, Great Slave R., Canada.— 3–5. Banff, Alberta.— 6–7. Rocky Mts., B. N. A.— 8–9. Great Bear Lake.— 10–11. Stanley, Wisc.— 12. N. America.

The larva is unknown.

Habitat.— Western North America, from Alaska and Great Bear Lake to British Columbia, Alberta, Assiniboia, Manitoba, Minnesota, and Illinois.

As pointed out by Howe it is almost certain that the type specimen was not preserved in Massachusetts by Agassiz, but was sent to Baird from Cambridge, whence the confusion.

R. cantabrigensis is the American representative of *R. arvalis*, from which it differs in but very trivial characters. Had I not attached importance to the absence of the Λ-shaped interscapular marking in all the American frogs, I might have been tempted to degrade *R. cantabrigensis* to the rank of a variety of *R. arvalis*, as I am much struck by the resemblance in the cranial characters. It is also noteworthy that the light vertebral line is found in no other North American species, and the light line along the hind limb is a character met with, irrespective of affinities, in many forms from the African and Indo-Malayan regions but not from elsewhere, with the single exception of *R. esculenta*, var. *chinensis*, in which it is very rarely present.

14. **Rana silvatica.**

Rana sylvatica Leconte, Ann. Lyc. N. Y., I, 1825, p. 282; Harl., Amer. Journ. Sc., X, 1825, p. 58, and Journ. Ac. Philad., V, 1825, p. 338; Holbr., N. Am. Herp. I, p. 95, Pl. xv (1836); Dum. and Bibr., Erp. Gén., VIII, p. 362 (1841); Dekay. N. Y. Faun., III, p. 64, Pl. xx, fig. 50, and xxi, fig. 54 (1842); Holbr., N. Am. Herp., Ed. 2, IV, p. 99, Pl. xxiv (1842); Wied, N. Acta Ac. Leop. Carol., XXXII, 1865, p. 114; De l'Isle, Ann. Sc. Nat. (5), XVII, 1872, No. 3, p. 4; Hinckley, Proc. Bost. Soc. N. H., XXI, 1881, p. 310, Pl. v, fig. 6; Bouleng., Cat. Batr. Ecaud., p. 47 (1882); Cope, Batr. N. Am., p. 447, fig. (1889); H. Garm., Bull. Illin. Lab., III, 1892, p. 330; Howe, Proc. Bost. Soc., XXIX, 1899, p. 369; Dickerson, Frog Book, p. 205, Pl. xiv, fig. 1, and Pl. lxxviii and lxxix (1906); Wright, Publ. Carnegie Instit., No. 197, 1914, p. 87, Pl. xxi; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 410.

Rana pennsylvanica Harl., Amer. Journ. Sc., X, 1825, p. 58; Bouleng., Bull. Soc. Zool. France, 1879, p. 188.

Rana temporaria var. *silvatica*, part., Günth., Cat. Batr. Sal., p. 47 (1858).

Vomerine teeth in small oblique groups on a level with or just behind the posterior borders of the choanæ, close together or nearer to each other than to the latter.

Head as long as broad or a little broader than long, rather strongly depressed; snout rounded or obtusely pointed, more or less projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region oblique, concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils equal to or greater than the interorbital width, which is equal to or less than (about ⅔) that of the upper eyelid; tympanum very distinct, ½ to ⅔ the diameter of the eye and 2 to 4 times its distance from the latter.

Fingers moderate, obtuse, first longer than second, third as long as or longer than the snout; subarticular tubercles moderate or large, very prominent.

Hind limbs long, the tibio-tarsal articulation reaching the nostril, the tip of the snout, or a little beyond, the heels strongly overlapping when the limbs are folded at right angles to the body; tibia 4 to 5 times as long as broad, 1⅗ to 2 times in length from snout to vent, as long as or slightly shorter than the fore limb and as long as or slightly longer or shorter than the foot. Toes obtuse, ⅔ to ¾ webbed, 2 phalanges of fourth free, the web thicker and with convex border in the breeding male; the web extending about halfway between the outer metatarsals; subarticular tubercles moderate or rather small and feebly prominent; no tarsal fold; inner metatarsal tubercle very prominent, rather hard, sometimes feebly compressed, ⅖ to ⅗ the length of the inner toe; a small, round outer tubercle present or absent.

Skin of upper parts smooth or with small flat glands; a narrow or moderately broad, prominent glandular dorso-lateral fold, from above the tympanum to the hip; these folds parallel or slightly converging on the anterior part of the back, where they are separated from each other by a space contained 4½ to 5½ times in length from snout to vent; a glandular fold from below the eye to above the arm. Lower parts smooth, posterior half of thighs with large flat granules; the posterior part of the belly sometimes also granular.

Grayish or pinkish brown above, without or with small dark brown or black spots; a dark brown or black streak from the end of the snout to the eye and a large blotch of the same color on the temple, or loreal region and lower half of eye dark like the temple; a light streak on the upper lip, from the end of the snout to the extremity of the glandular fold behind the mouth; a V-shaped dark marking sometimes present between the eyes; dorso-lateral fold paler than the ground color, often edged with blackish or with a series of small blackish spots, on the outer side; limbs with more or less regular dark cross-bands. Lower parts white, posterior part of belly and hind limbs yellow, throat and breast sometimes with small brown or gray spots.

Male with internal vocal sacs, strong fore limbs, and a large pad on the inner side of the first finger; the web between the toes more developed during the breeding season.

Nasal bones small, oblique, widely separated from each other; ethmoid exposed in front of the parietals and forming an obtuse angle penetrating between the nasals.

Tadpole not unlike that of *R. agilis*, with 4 upper and 4 lower series of labial teeth, viz. a long upper marginal and 3 short laterals; 3 continuous lower and one (the innermost) interrupted.

Eggs as in *R. temporaria*, but smaller.

Habitat.—Eastern North America, from Manitoba, Ontario, and Quebec to South Carolina.

This species is strikingly similar to *R. agilis* in form, color, and habits, but the presence of vocal sacs in the males fully justify the specific separation; it remains much smaller. Some specimens appear to be connectant between it and *R. cantabrigensis*, and I am not fully

Rana silvatica.

Measurements in Millimeters.

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
	♂	♂	♂	♂	♂	♀	♂	♀	♀	♂	♀	♀	♂
From snout to vent	34	42	41	41	37	48	40	35	42	42	45	43	45
Head	11	15	14	14	13	16	14	13	15	14	15	15	16
Width of head	12	16	15	15	13	17	14	13	15	15	15	15	16
Snout	4	5.5	5	5	4.5	6	5	4.5	6	5	5	5.5	6
Eye	4	5.5	5	5	4.5	6	5	4.5	5	5	5	5.5	6
Interorbital width	2	2	2.5	2.5	2	2.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5
Tympanum	2	3	3	3	2.5	3	2.5	2.5	3	3	3.5	3	3
Fore limb	19	25	24	25	23	29	24	21	23	26	28	25	26
1st finger	4	4	4	4.5	4	6	4	5	5	5	6	5.5	5.5
2nd "	3.5	3	3	3.5	3	5	3.5	4	4	4	5	4.5	4
3rd "	5	5	6	6	5.5	7	5	6	6	6	8	7	7
4th "	3	2.5	3	3	2.5	4	3	3	3.5	3	4	4	4
Hind limb	60	73	70	75	68	83	68	58	75	79	89	82	81
Tibia	17	23	22	23	21	27	22	20	25	25	28	26	26
Foot	19	23	23	24	21	27	23	20	23	25	28	25	25
3rd toe	10	13	13	14	12	15	13	11	12	13	15	13	13
4th "	15	18	18	19	16	22	18	15	19	19	21	20	20
5th "	10	12	12	13	11	14	12	10	11	13	14	13	13

1. L. Winnipeg.— 2-6. Lucknow, Ontario.— 7-8. London, Ontario.— 9. L. Louise, Manitoba.— 10-11. Cambridge, Mass.— 12. Brooklyn, N. Y.— 13 Long Id., N. Y.

Rana silvatica.

Measurements in Millimeters.

	14. ♀	15. ♂	16. ♀	17. ♂	18. ♀	19. ♀	20. ♀	21. ♀	22. ♀	23. ♀	24. ♀	25. ♀	26. ♀
From snout to vent	40	45	50	41	52	50	45	60	59	59	58	55	54
Head	14	16	16	14	17	16	15	19	19	19	20	19	18
Width of head	14	16	17	15	18	17	16	21	21	20	21	20	19
Snout	5	5.5	6	5	6	5.5	5.5	6	6.5	6.5	6.5	6.5	6
Eye	5	5.5	6	5	6	5.5	5.5	6	6	6.5	6.5	6.5	6
Interorbital width	2.5	3	3	2.5	2.5	2.5	2.5	3	3.5	3.5	3.5	3	3
Tympanum	2.5	3	3.5	3	3.5	3	3	4	4	4	4	4	4.5
Fore limb	23	25	29	25	29	28	27	33	35	32	34	33	32
1st finger	5	4.5	5.5	5	6	5.5	5	7	8	6	8	6	6
2nd "	4	3.5	4.5	4	5	4.5	4	5.5	6	5	6	5	5
3rd "	6	6	7	6	7	7	6.5	9	9	8	9	8	7
4th "	4	3	3.5	3	4	4	3.5	5	5	4	5	5	4
Hind limb	74	74	86	73	86	84	79	101	102	97	99	95	89
Tibia	25	24	27	23	28	27	25	34	34	31	31	30	28
Foot	23	23	26	23	28	27	25	32	33	31	32	30	28
3rd toe	12	12	13	13	14	14	13	17	17	15	17	15	15
4th "	18	18	21	18	21	20	19	24	25	22	25	23	22
5th "	11	12	13	11	13	12	13	15	16	15	16	15	15

14. Long Id., N. Y.— 15-16. Palisades, N. J.— 17-20. Hackensack, N. J.— 21. Chester Co., Penn.— 22. Sullivan Co., Penn.— 23-26. N. America.

convinced that the line of cleavage between the two species corresponds to that traced by Howe, who has endeavored to improve on the definitions given by Cope, which are entirely inadequate.

R. pretiosa, *R. cantabrigensis*, and *R. silvatica* are evidently closely related to the three widely distributed European species *R. temporaria*, *R. arvalis*, and *R. agilis*, which they represent in America; but I am inclined to regard this as a case of independent, parallel evolution in the two parts of the world from a common ancestor, of which *R. draytonii* is perhaps the surviving representative.

15. **Rana godmani.**

?*Levirana vibicaria* Cope, Proc. Ac. Philad., 1894, p. 197.

Rana godmani Günth., Biol. C.-Am., Rept., p. 204, Pl. lxiii, fig. A (1900); Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 411.

Vomerine teeth in feeble oblique groups between the choanæ, equally distant from each other and from the latter, or nearer each other.

Head slightly broader than long, rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as or a little longer than the eye; canthus rostralis obtuse; loreal region feebly oblique, concave; nostril equidistant from the eye and from the end of the snout; distance between the nostrils equal to or a little greater than the interorbital width, which equals that of the upper eyelid; tympanum very distinct, about $\frac{3}{5}$ the diameter of the eye, $1\frac{1}{2}$ to 2 times its distance from the latter.

Fingers moderate, obtuse or a little swollen at the end, first longer than the second, third as long as or a little longer than the snout; subarticular tubercles moderate.

Hind limb moderately long, the tibio-tarsal articulation reaching the eye, the heels overlapping when the limbs are folded at right angles to the body; $4\frac{1}{2}$ to 5 times as long as broad, $2\frac{1}{7}$ to $2\frac{1}{5}$ times in length from snout to vent, shorter than the fore limb, a little shorter than the foot. Toes slightly swollen at the end, $\frac{2}{3}$ webbed, 2 phalanges of fourth and one of third and fifth free; the web separating the outer metatarsal in third distal half; subarticular tubercles rather small, moderately prominent; no tarsal fold; inner metatarsal tubercles oval, about $\frac{1}{3}$ the length of the inner toe; no outer tubercle.

Skin smooth; a broad glandular dorso-lateral fold, as broad as the upper eyelid, from above the tympanum to the hip; the distance between the dorso-lateral folds, on the back, $\frac{1}{6}$ to $\frac{1}{5}$ the length from snout to vent.

Brown or orange above, sometimes speckled and spotted with black; a black streak on the canthus rostralis and along the outer border of the glandular lateral fold; a more or less distinct whitish streak from below the eye to the shoulder; no cross-bars on the limbs, but a more or less distinct dark streak along the thigh and the tibia. Lower parts white.

Male without secondary sexual characters.

Nasal bones narrow, widely separated from each other and from the frontoparietals; only a small part of the ethmoid exposed; zygomatic branch of squamosal long, extending to below the eye.

Measurements, in Millimeters, of the types.

	♂	♀	♀
From snout to vent	56	80	74
Head	19	25	23
Width of head	20	26	25
Snout	7	10	9
Eye	7	9	8
Interorbital width	3.5	5.5	6
Tympanum	4	5	5.5
Fore limb	35	51	43
1st finger	5.5	10	9
2nd "	5	9	8
3rd "	6.5	12	11
4th "	5	9	8
Hind limb	87	122	113
Tibia	26	38	34
Foot	27	41	37
3rd toe	13	23	21
4th "	21	33	29
5th "	13	24	21

Habitat.—Rio Sucio, Costa Rica.

This frog strikingly resembles some specimens of the African *R. galamensis.*—another example of parallelism between New and Old World species, as close genetic relationship can hardly be postulated to account for this resemblance.

I cannot help thinking that *R. godmani* will some day prove to be identical with *Levirana vibicaria*, Cope. Except for the presence of feebly developed vomerine teeth, the longer inner finger and the more extensive web between the toes, in the former, there is almost complete agreement between the descriptions of the two, which are from the same part of Costa Rica.

Cope's description is here reproduced:—

"Identical with *Ranula* but without vomerine teeth. Form robust, muzzle short, wide; hind limbs rather short, the heel reaching to the front of the eye. Canthus rostralis strongly marked, angulated at the nostril, anterior to which it is deflected to the apex, which does not project beyond the lip-border. Loreal region slightly concave; nostril nearly on canthus, .6 the length of the muzzle anterior to the orbit. Tympanum drum subround, about .7 the diameter of the eye-slit. Choanæ rather small, about equal to ostia pharyngea. Tongue large, obovate, deeply emarginate posteriorly. Digits with slight terminal enlargements, those of the fingers scarcely distinct, those of the toes supported by elongate T-shaped phalanges. Second (first) finger a little shorter than third (second); inferior tubercles inconspicuous. Toes half-webbed, fourth digit with three, the others with two free phalanges. Sole with one tubercle, the flat oval praehallax. Integument smooth everywhere. A wide glandular body connecting rictus oris and humerus; a glandular thickening extending from orbit to end of urostyle, which [14] is unusually wide between the orbit and the sacrum. No dermal folds.

Color above a dark olivaceous gray; side of head and a stripe from orbit to urostyle, black. Two rows of small black spots on each side of the vertebral column. Superior side of limbs colored like the back. Inferior surfaces of the body yellowish, more distinctly yellow on inferior surfaces of hind limbs. Concealed surfaces of femur with a reddish tinge, unspotted. Superior face of femur with a black stripe on the distal half. Lips unspotted, with a dark shaded border.

Length of head and body 65 mm.; length of head to rictus oris, 21 mm.; width of head at rictus oris, 25 mm.; length of fore limb, 40 mm.; of head 15 mm.; length of hind limb 101 mm.; of hind foot, 49 mm.; of astragalo-calcaneum, 16 mm."

Rancho Rodondo, on the divide of the Irazu Range, and Isla Nueva, near the head of the Rio Sucio, on the Atlantic side.

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[14] The glandular thickening.—G. A. B.

16. **Rana areolata.**

Rana areolata Baird and Gir., Proc. Ac. Philad., 1852, p. 173; Baird, Rep. U. S. Mex. Bound. Surv., Rept., Pl. xxxvi, figs. 11, 12 (1859); Bouleng., Cat. Batr., Ecaud., p. 41 (1882); Cope, Proc. Am. Philos. Soc., XXIII, 1886, p. 517, and Batr. N. Am., p. 412, figs. (1889); Dickerson, Frog Book, p. 192, Pl. lxxiii (1906); Hurter, Tr. Ac. St. Louis, XX, 1911, p. 115; Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 411.

Rana capito Leconte, Proc. Ac. Philad., 1855, p. 425, Pl. v; Bouleng., Cat., p. 34, and Ann. and Mag. N. H. t. c.

Rana circulosa Rice and Davis, in Jordan, Man. Vert. E. N. Am., 2d ed., p. 355 (1878).

Rana areolata æsopus Cope, t. c., p. 517, and op. cit., p. 412, figs.

Rana areolata capito Cope, t.c., p. 518, and op. cit., p. 415.

Rana areolata circulosa Cope, t. c., p. 518, and op. cit., p. 413, figs.

Rana æsopus Dickerson, op. cit., p. 193, Pl. xiv, fig. 3, and Pl. lxxiv; Deckert, Copeia, No. 80, 1920, p. 26.

Vomerine teeth in short slightly oblique series close together between the choanæ or extending a little beyond the level of their posterior borders.

Head large, 2⅔ to 3 times in length to vent, as long as broad or a little broader than long, much depressed; snout rounded, not or but feebly projecting beyond the mouth, longer than the eye, which is large and prominent; canthus rostralis indistinct or obtuse; loreal region oblique, concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils greater than the interorbital width, which is equal to or a little less than that of the upper eyelid; tympanum very distinct, ⅗ to ⅘ the diameter of the eye, 2 to 3 times its distance from the latter.

Fingers moderate, obtusely pointed, first longer than the second, third as long as the snout or a little longer or a little shorter; subarticular tubercles small, feebly prominent.

Hind limb rather short, the tibio-tarsal articulation reaching the eye or a little beyond, the heels meeting when the limbs are folded at right angles to the body; tibia 3½ to 4 times as long as broad, 1¾ to slightly over 2 times in the length of head and body, as long as or a little shorter than the fore limb or the foot. Toes rather long, obtusely pointed, ½ webbed, 2½ or 3 phalanges of fourth and 1½ or 2 of third and fifth free, the outer metatarsals bound together in their basal half or third; subarticular tubercles small, moderately prominent; no tarsal fold; inner metatarsal tubercle elliptic, prominent, $\frac{2}{5}$ to $\frac{3}{5}$ the length of the inner toe; outer tubercle very small or absent.

Skin of back smooth or rough with elongate warts, or with narrow glandular folds; a rather broad glandular dorso-lateral fold from above the tympanum to the groin; the distance between the folds, on the back, $\frac{1}{5}$ to $\frac{1}{4}$ the length of head and body; sides somewhat corrugated and pustular, or with elongate glandules. Lower parts smooth, or posterior part of belly and thighs feebly granulate.

Upper parts brownish olive, minutely and obsoletely mottled with lighter, with numerous dark brown round blotches edged with yellowish or with yellowish centers; or pale yellow so thickly covered with large reddish brown spots as to reduce the ground-color to a network; glandular dorso-lateral folds yellowish; a white spot in the center of the tympanum; hind limbs with dark cross-bands, the interspaces between them sometimes very narrow. Lower parts pale yellow or white.

Male with external vocal sacs, forming loose folds above the arm; fore limb moderately thickened; a moderately strong pad on the inner side of the first finger.

The above description of the typical *R. areolata* is taken from a large male from Illinois, received from the Museum of Comparative Zoölogy, and a half-grown individual from Hitchcock, Texas, received from the American Museum; in its compilation I have also made use of the works quoted in the synonymy, the excellent photographs in Miss Dickerson's book having been particularly useful.

Of the frog known as *R. æsopus*, I have had a larger material and also the privilege of seeing it alive. It differs in the rather larger head, $2\frac{1}{2}$ to $2\frac{6}{7}$ times in length to vent, always broader than long, the snout sometimes not longer than the eye; the tibio-tarsal articulation reaches the tympanum or the eye, the tibia is 3 to $3\frac{1}{2}$ times as long as broad and up to $2\frac{1}{4}$ times in length from snout to vent; the dorso-lateral fold is usually very broad.

Pale brown or gray above, the dorso-lateral folds pale yellow or orange, speckled with blackish and with numerous round dark brown or blackish spots on the head and body; tympanum reddish brown, with a white central spot; limbs with dark cross-bands, which are much narrower than the spaces between them. Lower parts white, spotted or vermiculate with brown on the throat and breast.

Nasal bones small and widely separated from each other and from the frontoparietals, which leave the ethmoid uncovered in front; zygomatic branch of squamosal long, extending to below the eye.

I have regarded this *R. æsopus* as specifically distinct from *R. areolata*, under the name of *R. capito*. There can be no question that Leconte's figure agrees with *R. æsopus* as defined by Cope and not with *R. areolata*; however, Dr. Barbour has recently informed me that Dr. Stejneger, having at his request reëxamined the type of *R. capito*, and compared it with the types of *R. areolata* and *R. æsopus*, states it to be unquestionably the same as *R. areolata* and not *R. æsopus*. I submit of course to Dr. Stejneger's verdict, but considering the state of things resulting from it, it seems to me that a strict definition of the species is an impossibility and I have therefore restored *R. æsopus* to the rank of a subspecies or variety of *R. areolata*, assigned to it by its original describer.

The larva of *R. areolata* is still unknown.

Measurements in Millimeters.

	Forma typica	var. *æsopus*.										
	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
	♂	♂	♀	♀	♀	♀	♀	♀	♀	♂	♂	♀
From snout to vent	100	73	95	77	93	85	84	68	82	88	80	91
Head	33	30	34	30	34	31	30	26	30	31	30	33
Width of head	35	33	41	33	41	35	36	29	36	38	36	41
Snout	12	10	12	11	12	13	13	10	12	12	11	12
Eye	9	10	11	9	10	9	9	8	10	9	9	11
Interorbital width	4	4	5	4	5	5	5	4	5		4	4
Tympanum	7	7	8	6	7	7	7	6	7	7	6	8
Fore limb	55	42	47	37	51	47	46	36	45	50	43	50
1st finger	11	8	11	10	12	10	10	7	11	10	9	9
2nd "	10	7	8	8	10	8.5	8.5	6	8	9	8	7
3rd "	12	9	13	10	13	11	11	9	11	13	10	11
4th "	8	5	6	6	6	6	7	5	7	7	6	5
Hind limb	145	100	136	116	134	122	126	102	120	132	110	122
Tibia	47	34	45	37	42	40	40	33	40	40	35	41
Foot	47	34	45	39	45	39	41	33	42	45	37	43
3rd toe	26	15	21	18	20	19	19	16	19	21	17	21
4th "	40	25	33	28	33	30	31	27	32	34	26	33
5th "	25	14	20	17	23	19	20	16	20	21	16	?

1. Cathoun, Richland Co., Illinois.— 2–3. Tarpon Springs, Florida.— 4. Fort Meade, Florida.— 5–8. Orlando, Florida.— 9. Crescent City, Florida.— 10–11. Near West Palm Beach, Florida.— 12. Florida.

Habitat.— The typical form is known from Indiana, Illinois, Georgia and Texas; the var. *æsopus* from Florida. R. F. Deckert (l. c.) records true *æsopus* from Hampton Co., North Carolina, but makes no mention of having compared his specimen with the forma typica (*areolata*), which might be expected to occur in both the Carolinas.

17. **Rana tarahumaræ.**

Rana tarahumaræ BOULENG., Ann. and Mag. N. H. (8) XX, 1917, p. 416, and (9) III, 1919, p. 411.

Vomerine teeth in small groups close together behind the level of the choanæ.

Head broader than long, much depressed; snout rounded, feebly projecting beyond the mouth, as long as the eye; canthus rostralis indistinct; loreal region very oblique, slightly concave; nostril equidistant from the eye and from the tip of the snout; distance between the nostrils equal to the interorbital width, which is equal to or a little less than that of the upper eyelid; tympanum distinct, without or with a few small asperities, ⅖ to ½ the diameter of the eye, once to once and a half its distance from the latter.

Fingers moderate, the tips feebly swollen, first longer than the second, third longer than the snout; subarticular tubercles large, prominent.

Hind limb long, the tibio-tarsal articulation reaching the tip of the snout, the heels meeting when the limbs are folded at right angles to the body; tibia 4 to 4½ times as long as broad, 1⅚ to 2 times in length from snout to vent, shorter than the fore limb, as long as or slightly longer or shorter than the foot. Toes with the tips swollen into small disks, the base of which is involved in the very broad web; outer metatarsals separated nearly to the base; subarticular tubercles rather large and prominent; no tarsal fold; inner metatarsal tubercle elliptic, feebly prominent, ⅓ to ⅖ the length of the inner toe; no outer tubercle.

Skin smooth, or upper parts with small pustules; a feeble, curved glandular fold from the eye to the shoulder.

Brown or olive above, with small blackish spots or numerous dots; limbs with irregular dark cross-bars; lower parts white, uniform on throat, breast, and limbs mottled with grayish brown.

Male without vocal sacs, with a thick blackish pad on the inner side of the first finger.

Nasal bones small and widely separated from each other and from the frontoparietals.

Habitat.—Sierra Tarahumare, N. W. Mexico, about 3000 feet altitude.

Measurements of type specimens in Millimeters

	1.	2.	3.	4.	5.	6
	♂	♀	♀	♀	♂	♀
From snout to vent	73	77	65	58	61	45
Head	25	25	21	20	21	16
Width of head	30	29	24	23	23	17
Snout	9	9	8	7.5	8	6
Eye	9	9	8	7.5	8	6
Interorbital width	4	4	3.5	3.5	3.5	2.5
Tympanum	4.5	4.5	3.5	3	3.5	2.5
Fore limb	47	45	40	36	40	30
1st finger	10	10	9	8	9	7
2nd “	9	9	8	7	8	6
3rd “	11.5	11	10	9	9.5	8
4th “	8	8	7	6	7	5
Hind limb	125	122	104	98	103	74
Tibia	40	39	32	29	31	24
Foot	40	38	32	31	32	25
3rd toe	22	21	18	17	18	13
4th “	33	31	27	26	27	20
5th “	25	25	22	21	22	16

1–4. Ioquiro.— 5–6. Barranca del Cobre.

This species is very closely allied to *R. boylii*, differing in the larger eye, the more oblique loreal region, the more distinct tympanum, the shorter tibia, and the absence of vocal sacs.

18. **Rana boylii.**

Rana boylii Baird, Proc. Ac. Philad., 1854, p. 62, and Rep. U. S. Explor. Surv., XII, ii, Pl. xxix, figs. 2, 3 (1860); Cope, N. Am. Batr., p. 444, figs. (1889); Stejneg., N. Am. Faun. No. 7, p. 226 (1893); Dickerson, Frog Book, p. 221, Pl. ii, fig. 7, and Pl. lxxxiv, figs. 1–3 (1906); Camp, Univ. Calif. Publ. Zool., XVII, 1917, p. 117; Bouleng., Ann and Mag. N. H. (9) III, 1919, p. 411.
Rana pachyderma Cope, Proc. Ac. Philad., 1883, p. 25.
Rana boylii muscosa Camp, t. c., p. 118.
Rana boylii sierræ Camp, t. c., p. 120.

Vomerine teeth in very feeble groups or oblique series close together between or extending behind the level of the choanæ, sometimes very indistinct.

Head broader than long, much depressed; snout rounded, feebly projecting beyond the mouth, as long as the eye or a little shorter; canthus rostralis obtuse; loreal region moderately oblique, concave; eye rather small; nostril equidistant from the eye and from the tip of the snout; distance between the nostril, equal to the interorbital width, which is equal to or slightly less than that of the upper eyelid; tympanum feebly distinct, ill-defined, with small asperities as on the neighboring region, $\frac{2}{5}$ to $\frac{1}{2}$ the diameter of the eye, once to twice its distance from the latter.

Fingers moderate, the tips feebly swollen, first longer than the second (exceptionally first and second equal), third longer than the snout; subarticular tubercles moderate or rather large, prominent.

Hind limb long, the tibio-tarsal articulation reaching the tip of the snout or beyond, the heels overlapping when the limbs are folded at right angles to the body; tibia 4 to 6 times as long as broad, $1\frac{3}{5}$ to $1\frac{4}{5}$ times in length from snout to vent, shorter than the fore limb, as long as or a little longer than the foot. Toes with the tips swollen into small disks, the base of which is involved in the very broad web; outer metatarsals separated nearly to the base; subarticular tubercles rather large and prominent; no tarsal fold; inner metatarsal tubercle oval or elliptic, $\frac{2}{5}$ to $\frac{3}{5}$ the length of the inner toe; a round outer metatarsal tubercle.

Skin of upper parts more or less granular, pustular, and porous, sprinkled all over with small horny granules, often with large porous warts on the sides; dorso-lateral fold, if distinct, very broad and flat and confined to the anterior half of the body; a fold from the eye to the shoulder; one or two large glands behind the angle of the mouth; lower parts smooth, except the hinder half of the thighs which is covered with flat granules, some of which show a more or less distinct large central pore or pit.

Grayish brown to dark olive brown above (sometimes brick, and according to Miss Dickerson), usually with a dark cross-band between the eyes, numerous small darker spots on the body and more or less distinct dark cross-bands on the limbs; upper surface of snout sometimes paler than the rest of the head; often an oblique light streak below the eye; hinder side of thighs yellow, spotted or marbled with black; web between the toes yellow. Lower parts white in front and yellow behind, throat often mottled or spotted with blackish.

Male with internal vocal sacs, with very strong fore limbs and a thick pad, covered with a brown horny layer, on the inner side of the first finger.

Eggs nearly entirely black, measuring 2 millimeters in diameter.

Nasal bones small and widely separated from each other and from the frontoparietals; ethmoid exposed and obtusely pointed in front; zygomatic process of squamosal very short. Pectoral arch as in *R. temporaria*. Terminal phalanges feebly expanded distally.

Measurements in Millimeters

	1. ♂	2. ♂	3. ♂	4. ♂	5. ♀	6. ♀	7. ♀	8. ♀	9. ♀	10. ♀	11. ♀	12. ♀
From snout to vent	55	53	49	47	63	63	50	58	57	46	42	40
Head	19	18	17	16	20	20	16	20	20	15	15	14
Width of head	22	20	19	18	24	24	19	23	25	18	17	16
Snout	6	6	5.5	5	6.5	7	5.5	6	6	5	5	4.5
Eye	6	6	5.5	6	6.5	7	5.5	6	6	5	5	4.5
Interorbital width	3.5	4	3.5	3.5	4	4.5	3.5	4	4	3.5	3	3
Tympanum	3	3	2.5	2.5	3.5	3.5	2.5	3	3	2	3	3
Fore limb	38	37	35	33	40	41	31	37	40	29	27	25
1st finger	7	7	6	6	9	8	7	7	8	6	6	5
2nd "	6	6	5	5	8	7.5	6	6	7	5	5.5	5
3rd "	10	9	8	8	10	9	8	9	10	8	8	8
4th "	6	6	5	5	7	6	5	6	7	5	5	4.5
Hind limbs	110	101	87	86	111	116	84	101	107	77	78	72
Tibia	36	34	28	27	36	38	29	34	35	25	26	23
Foot	35	32	27	27	35	34	26	32	33	25	24	22
3rd toe	20	18	15	15	20	19	15	17	19	14	13	12
4th "	30	27	23	23	29	29	22	26	28	21	20	18
5th "	23	20	18	18	23	21	16	19	22	15	14	13

1–7. Palo Alto.— 8. Mt. Diablo.— 9. Camp Taylor, Marion Co.— 10. L. Tahoe.— 11–12. Mill Valley.

Habitat.— Oregon and California, up to 11,500 feet altitude.

19. **Rana pustulosa.**

Rana pustulosa Bouleng., Ann. and Mag. N. H. (5) XI, 1883, p. 343; Günth., Biol. C.-Am., Rept. p. 202, Pl. lxi, fig. 2 (1900); Bouleng., Ann. and Mag. N. H. (9) III, 1919, p. 411.

Vomerine teeth in short oblique series extending beyond the level of the posterior borders of the choanæ, nearer to each other than to the latter.

Head broader than long, rather strongly depressed; snout rounded, feebly projecting beyond the mouth, as long as the eye; canthus rostralis distinct; loreal region moderately oblique and deeply concave; nostril a little nearer the tip of the snout than the eye; distance between the nostrils equal to the interorbital width or to the upper eyelid; tympanum very distinct, $\frac{3}{5}$ the diameter of the eye, from which it is separated by an equal distance.

Fingers moderate, the tips swollen, first longer than the second, third longer than the snout; subarticular tubercles large, prominent.

Hind limb long, the tibio-tarsal articulation reaching the tip of the snout, the heels overlapping when the limbs are folded at right angles to the body; tibia 4 times as long as broad, $1\frac{3}{4}$ times in length from snout to vent, a little shorter than the fore limb, a little longer than the foot. Toes with the tips swollen into small disks, webbed to the tips; outer metatarsals separated nearly to the base; subarticular tubercles rather large and prominent; no tarsal fold; inner metatarsal tubercle oval, $\frac{2}{5}$ the length of the inner toe; no outer tubercle.

Upper parts covered with small pustules; a strong glandular fold from the eye to the shoulder, the upper branch continued as a moderately prominent narrow dorso-lateral fold extending to the hip; the distance between the dorso-lateral folds, or the back, $3\frac{1}{2}$ times in length from snout to vent; lower parts smooth. Olive above, with rather indistinct blackish spots; flanks marbled with blackish and whitish; hinder side of thighs blackish, marbled with gray. Lower parts whitish, the throat and breast soiled with gray.

Male unknown.

Measurements of type specimens.

From snout to vent	106 mm.
Head	35
Width of head	41
Snout	12
Eye	12
Interorbital width	8
Tympanum	7
Fore limb	63
1st finger	15
2nd "	13
3rd "	16
4th "	11
Hind limb	188
Tibia	60
Foot	57
3rd toe	32
4th "	49
5th "	38

This species is known from a single female specimen from Ventanas in Durango, Mexico. It is nearly allied to *R. palmipes.*

20. **Rana palmipes.**

Rana palmipes SPIX, Nov. Spec. Test. Ran. Bras., p. 5, Pl. v, fig. 1 (1824); PETERS, Mon. Berl. Ac. 1872, p. 205, and 1873, p. 622; BOULENG., Cat. Batr. Ecaud., p. 48 (1882), Ann. and Mag. N. H. (6) II, 1888, p. 40, and VIII, 1891, p. 453; GÜNTH., Biol. C.-Am., Rept., p. 202 (1900); BOULENG., Proc. Zool. Soc. 1913, p. 1026, and Ann. and Mag. N. H. (9) III, 1919, p. 412; NOBLE, Bull. Amer. Mus. N. H. XXXVIII, 1918, p. 316, Pl. XIV.

Rana juninensis TSCHUDI, Faun. Per., Herp., p. 64 (1845).

Ranula gollmeri PETERS, Mon. Berl. Ac. 1859, p. 402.

Rana affinis PETERS, l. c., and 1871, p. 402.

Rana clamata var. *guianensis* PETERS, Mon. Berl. Ac. 1863, p. 412.

Ranula affinis COPE, Proc. Ac. Philad., 1866, p. 130, and 1868, p. 117; SUMICHRAST, Bull. Soc. Zool. France, 1880, p. 189.

Pohlia palmipes STEIND., Novara, Amph., p. 15, Pl. I, fig. 5 (1867); GÜNTH., Zool. Rec., IV, p. 147 (1867).

Ranula brevipalmata COPE, Proc. Ac. Philad., 1874, p. 131.

Ranula nigritatus COPE, l. c.

Rana vaillanti BROCCHI, Bull. Soc. Philom. (7) I, 1877, p. 175, and Miss. Sc. Mex., Batr., p. 11, Pl. II, fig. 1 (1882).

Hylarana brevipalmata BROCCHI, op. cit., p. 65.

Rana copii BOULENG., Cat. Batr. Ecaud., p. 49.

Rana nigrilatus BOULENG., l. c.

Ranula palmipes COPE, Bull. U. S. Nat. Mus., No. 32, 1887, p. 19.

Rana bonaccana GÜNTH., Biol. C.-Am., Rept., p. 201, Pl. LX, fig. B (1900).

Rana melanosoma GÜNTH., op. cit., p. 203, Pl. LXIII, fig. B.

Rana brevipalmata FOWLER, Proc. Ac. Philad., 1913, p. 166, Pl. IX.

Rana brevipalmata rhoadsi FOWLER, l. c.

Vomerine teeth in small oblique groups or short transverse or oblique series between the choanæ or on a level with the posterior borders of the latter, equally distant from each other and from the choanæ or nearer to each other.

Head as long as broad or a little broader than long, rather strongly depressed; snout rounded or obtusely pointed, projecting beyond the mouth, as long as or longer than the eye, which is large and very prominent; canthus rostralis distinct; loreal region moderately oblique, concave; nostril equally distant from the eye and from the tip of the snout, or a little nearer the latter; distance between the nostrils equal to or a little greater than the interorbital width, which is equal to or a little less than that of the upper eyelid; tympanum very distinct, ½ to ⅚ the diameter of the eye, 1½ to 4 times its distance from the latter.

Fingers moderate, obtuse, or rather pointed, with more or less distinct lateral dermal border, first longer than the second, third as long as or longer than the snout; subarticular tubercle rather small, feebly prominent.

Hind limb moderate or long, the tibio-tarsal articulation reaching the eye, the tip of the snout, or between these two points, the heels overlapping when the limbs are folded at right angles to the body; tibia 3½ to 5 times as long as broad, 1¾ to 2⅛ times in length from snout to vent, as long as or shorter than the fore limbs, as long as or a little longer (rarely a little shorter) than the foot. Toes with the tips swollen into very small disks, webbed to the tips, on two phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles rather small, feebly prominent; no tarsal fold; inner metatarsal tubercle elliptic; feebly prominent, ¼ to ½ the length of the inner toe; no outer tubercle.

Upper parts smooth or granular with small pearl-like tubercles; a strong glandular fold from the eye to the shoulder, the upper branch continued as a narrow or moderately broad dorso-lateral fold extending to the hip or not quite so far; the distance between the dorso-lateral folds, on the back, 4 to 5½ times in length from snout to vent; a more or less distinct glandular fold, often broken up posteriorly, from below the eye to above the arm; tibia often with narrow glandular folds. Lower parts smooth.

Green, olive, gray, or brown above, sometimes green in front and brown behind, uniform or with small blackish specks or spots; the dorso-lateral fold sometimes lighter and edged with dark brown on the outer side; loreal and temporal regions often dark brown, with a light streak below; the tympanum sometimes reddish brown; hind limbs with or without more or less regular dark cross-bands; hinder side of thighs often marbled with black and yellow. Lower parts white, uniform or spotted or marbled with brown, the throat sometimes entirely brown.

Male with internal vocal sacs and a large pad on the inner side of the first finger, covered, during the breeding season, with a brown velvety horny layer.

Nasal bones small, oblique, widely separated from each other and from the frontoparietals; ethmoid largely exposed above, truncate or obtusely pointed in front and extending or not to between the nasals;

zygomatic process of squamosal larger than the posterior; terminal phalanges feebly expanded at the end.

Tadpole large and very similar to that of *R. catesbiana* in form and general appearance, but mouth larger with the series of horny teeth more numerous, viz. 3 short series on each side behind the long marginal upper series and 4 lower series, the innermost narrowly interrupted in the middle; beak narrowly edged with black.

Habitat.— Central and South America, from southern Mexico to Pernambuco and Peru.

This frog is interesting as the only representative of the genus *Rana* in South America. Few species have been more misunderstood and have given rise to more discussion than this *R. palmipes*, originally described from the Amazonian region of Brazil.

It has been made the type of a distinct genus (*Ranula, Pohlia*) by Peters and by Steindachner, and even referred to the Hylidæ by Günther (1867). Peters founded the genus *Ranula* on the feeble dentition: "Die Zähne des Oberkiefers sind so schwach und wenig zahlreich dass man sie erst bei genauer Untersuchung findet und am Gaumen fehlen sie ganz." The explanation is that *Ranula gollmeri* was described from a recently transformed young, from Caracas, measuring 50 mm. from snout to vent; of this I feel sure, having examined young from Pebas of exactly the same size with a short tail and toothless upper jaw. A second, larger specimen, also from Caracas, was described at the same time by Peters as *Rana affinis*, and regarded as so closely related to *R. temporaria* that it should perhaps rank as a local variety only. A little later, however, Peters recognized that the two supposed species were identical and correctly referred them to *R. palmipes*. In 1866, Cope took up the genus *Ranula* and defined it as the American representative of *Hylorana*, differing in the "important particulars of the incompleteness of the ethmoid arch, its superior plate being represented by cartilage." Cope, who maintained the definition up to the close of his labors, can only have examined young specimens, for in the adult the ethmoid is ossified exactly as in *Rana temporaria*.

Schlegel, Tschudi, and Duméril and Bibron referred *R. palmipes* to the synonymy of *R. esculenta;* Peters described another specimen as a variety of *R. clamitans;* Brocchi's *R. vaillanti* was described as allied to *R. mugiens;* whilst Günther (1900), overlooking the small terminal discs of the toes, compared his *R. bonaccana* to *R. clamitans* and *R. draytonii.*

Rana palmipes.

Measurements in Millimeters.

	1. ♂	2. ♀	3. ♀	4. ♀	5. ♀	6. ♀	7. ♀	8. ♀	9. ♀	10. ♂	11. ♀	12. ♂	13. ♂	14. ♂	15. ♀
From snout to vent	56	73	65	57	103	96	78	66	60	52	63	68	67	57	110
Head	20	25	23	20	39	37	30	23	21	18	21	24	24	20	36
Width of head	20	25	23	20	39	38	31	23	21	18	21	24	24	20	42
Snout	7	10	8	7	17	15	10	9	7.5	6.5	7	9	9	7	13
Eye	7	8	8	7	12	12	9	8	7.5	6.5	7	9	9	7	11
Interorbital width	4	4	4	4	8	8	6	4	4	3.5	3.5	5	5	3.5	8
Tympanum	4.5	4.5	4	5	10	9	7	6	4.5	4	4	5	5	4	9
Fore limb	35	40	38	34	59	57	48	43	36	35	36	43	44	38	60
1st finger	8	9	9	9	15	14	10	9	8	7	9	8	8	7	17
2nd "	7	8	8	8	13	12	9	8	7	6	7.5	7	7	6	15
3rd "	9	11	11	10	16	16	13	11	10	9	10	10	11	9	20
4th "	6	7	7	7	14	13	10	8	7	6	7	6	8	7	15
Hind limb	88	123	114	96	172	163	136	107	91	87	100	113	118	100	180
Tibia	29	42	37	30	53	52	42	35	29	27	32	36	38	32	55
Foot	29	37	35	30	53	52	43	36	30	27	32	36	38	32	57
3rd toe	17	23	20	18	31	29	25	21	18	15	18	19	22	18	32
4th "	24	31	30	26	47	45	37	30	26	23	26	29	32	26	51
5th "	19	24	21	20	36	32	27	21	20	16	20	22	23	20	38

1. Cuernavaca, Mexico.— 2–3. Tequasixtlan, Tehuantepec.— 4. Vera Paz.— 5–9. Bonacca Id., Yucatan (types of *R. bonaccana*).— 10. Lanquin, Guatemala.— 11. Duenas, Guatemala (type of *R. melanosoma*).— 12–14. Hacienda Rosa de Jericho, Nicaragua (types of *R. melanosoma*).— 15. Belize, Honduras (type of *R. vaillanti*).

Rana palmipes.

Measurements in Millimeters.

	16.	17.	18.	19.	20.	21.	22.	23.	24.	25.	26.	27.	28.	29.	30.
	♂	♀	♀	♀	♀	♂	♀	♀	♀	♀	♀	♀	♀	♀	♀
From snout to vent	51	62	96	95	70	87	93	92	76	51	52	95	59	53	113
Head	20	23	34	34	25	30	34	26	27	20	20	34	21	20	40
Width of head	20	23	34	34	25	30	34	27	30	20	20	35	21	20	44
Snout	7	10	14	14	10	13	14	13	11	7	7	14	8	7	17
Eye	7	8	11	11	9	11	11	11	10	7	7	11	8	7	12
Interorbital width	3.5	5	7	7	5	6	7	6	6	4	4	7	5	4	11
Tympanum	5	6	8	9	6	8	9	8	7	5	5	9	5	4	10
Fore limb	33	35	55	53	41	53	55	57	47	31	30	55	35	32	64
1st finger	8	9	13	13	10	13	13	14	10	8	7	14	9	8	16
2nd "	7	8	10	10	9	10	11	11	9	7	6	11	8	7	14
3rd "	10	10	15	15	11	14	16	15	14	9	9	15	11	10	19
4th "	7	8	10	10	8	9	10	10	9	6	5	10	8	7	14
Hind limb	84	100	159	155	113	141	163	170	120	82	82	143	89	85	185
Tibia	26	32	49	49	35	44	50	48	39	25	25	45	29	27	58
Foot	27	33	49	49	36	45	50	53	39	27	26	47	30	27	60
3rd toe	15	18	29	26	22	25	30	29	22	16	15	27	18	16	33
4th "	23	28	43	43	32	39	45	44	33	24	23	40	26	24	52
5th "	17	20	31	30	23	29	31	31	26	18	17	30	20	19	38

16. San Ramon, Nicaragua.—17. **Tung District**, Brit. Guiana.—18–20. **R. Condoto, Choco,** S. W. Colombia.—21–22. Salidero, N. W. Ecuador.—23. **Carandelot, N. W. Ecuador.**—24–25. **Canelos, E. Ecuador.**—26. Sarayaca, E. Ecuador.—27–29. Pebas, Peru.—30. **Yakurrayo, Peru.**

21. **Rana cæruleopunctata.**

Rana cæruleopunctata Steind., Verh. Zool.-bot., Ges. Wien, XIV, 1864, p. 264, Pl. xv, fig. 1; Bouleng., Cat. Batr. Ecaud., p. 50 (1882); Günth., Biol. C.-Am., Rept., p. 205 (1900); Bouleng., Ann. and Mag. N. H. (9), III, 1919, p. 412; Noble, Bull. Amer. Mus. N. H. XXXVIII, 1918, p. 318.

Ranula cæruleopunctata Cope, Proc. Ac. Philad., 1866, p. 130.

Ranula chrysoprasina Cope, l. c.

Hylarana cæruleopunctata Steind., Novara, Amph., p. 48 (1867).

Trypheropsis chrysoprasinus Cope, Proc. Ac. Philad., 1868, p. 117, and Journ. Ac. Philad., (2) VIII, 1876, p. 114, Pl. xxiii, fig. 12.

Hylarana chrysoprasina Brocchi, Min. Sc. Mex., Batr. p. 65 (1882).

Rana chrysoprasina Bouleng., op. cit., p. 49.

Vomerine teeth in oblique groups or short series between the choanæ and extending beyond the level of the posterior borders of the latter or entirely behind them, equally distant from each other and from the choanæ or nearer each other; sometimes very indistinct.

Head as long as broad or a little longer than broad, much depressed; snout rounded or obtusely pointed, moderately or feebly projecting beyond the mouth, as long as the eye; canthus rostralis strong; loreal region vertical or nearly so, slightly concave; nostril a little nearer the tip of the snout than the eye; distance between the nostrils greater than the interorbital width which is equal to or a little less than that of the upper eyelid; tympanum very distinct, ½ to ⅔ the diameter of the eye, 1½ to 3 times its distance from the latter.

Fingers moderate, the tips swollen, first longer than the second, third longer than the snout; subarticular tubercles large, prominent.

Hintd limb moderately long, the tibio-tarsal articulation reaching the eye or between the eye and the tip of the snout, the heels strongly overlapping where the limbs are folded at right angles to the body; tibia 5 to 6 times as long as broad, 1⅔ to 2 times in length from snout to vent, shorter than the fore limb, as long as or a little longer than the foot. Toes with the tips divided into small disks, ⅔ to ¾ webbed, the web reaching the disks of the third and fifth but leaving 2 or 3 phalanges of fourth free; outer metatarsals separated nearly to the base; subarticular tubercles moderate; no tarsal fold; inner metatarsal tubercle oval or elliptic, ¼ to ⅖ the length of the inner toe; a round outer tubercle.

Skin smooth or finely granulate above, or with pearl-like tubercles on the posterior part of the back; a narrow glandular dorso-lateral fold from above the tympanum to the groin, the distance between the two folds, on the back, ⅙ to ⅕ the length from snout to vent; lower parts smooth.

Grayish or reddish brown above, back uniform or with numerous small irregular bluish green spots; sides of head and body, below the canthus rostralis and the lateral fold, which may be better defined by a light line, darker brown or blackish brown; a white or whitish streak on the upper lip, continued to above the axil; limbs with dark cross-bands; hinder side of thighs black with a few large golden spots; lower parts white, throat and breast often brown or mottled with brown; belly sometimes spotted or marbled with brown.

Male without vocal sacs, with the fore limbs very robust and with a strong pad on the inner side of the first finger.

Nasal bones small, oblique, widely separated from each other and from the frontoparietals; upper part of ethmoid exposed in front. Terminal phalanges with feeble transverse expansion.

Eggs measuring 1½ millimeters in diameter, in females 57 millimeters in length from snout to vent.

Mr. G. K. Noble, Bull. Amer. Mus. N. H. XXXVIII, 1918, p. 318, has raised doubts as to *Ranula chrysoprasina* Cope, 1866, from Costa Rica, being a synonym of *R. cæruleopunctata*, and suggested the possibility of the type, now apparently lost, being an aberrant specimen of *R. palmipes*. Against this we have Cope's statement that the loreal region is vertical, which cannot apply to *R. palmipes*, and that the back of the thigh bears "a few golden spots on a black ground behind," which is highly characteristic of *R. cæruleopunctata*. The web is too short for a *R. palmipes;* "toes fully, not widely palmate, three distal phalanges of fourth free." The apparent contradiction between *fully webbed toes* and *three phalanges of fourth free* is no doubt the result of a misprint: "feebly" should be used for "fully." The British Museum has received large collections from Costa Rica, including specimens of *R. cæruleopunctata*, but no *R. palmipes* was among them.

Habitat.— Nicaragua and Costa Rica, up to 1600 meters.— The types, with which some of our specimens perfectly agree, are preserved in the Vienna Museum, their place of origin unknown.

Measurements in Millimeters

	1.	2.	3.	4.	5.	6.	7.	8.	9.
	♂	♀	♂	♀	♀	♂	♂	♀	♀
From snout to vent	39	45	39	56	46	44	41	57	57
Head	13	15	13	18	16	15	14	19	19
Width of head	11	13	12	18	16	13	13	19	17
Snout	5	6	5	6.5	6	6	5.5	6	7
Eye	5	6	5	6.5	6	6	5.5	6	7
Interorbital width	3	3.5	3	4	4	3.5	3.5	4	4
Tympanum	3	3.5	3	4	4	3	3.5	4	4
Fore limb	22	27	25	36	30	28	29	36	33
1st finger	4.5	6	4.5	7	6	6	5	7	7
2nd "	4	5	4	6	5	5	4	6	6
3rd "	6	7	6	9	7	7	7	9	8
4th "	4	5	4	6	5	5	4	6	6
Hind limb	67	70	69	99	70	71	73	101	93
Tibia	21	24	22	31	23	22	24	31	29
Foot	21	22	20	30	21	22	23	31	29
3rd toe	12	12	11	16	12	13	13	17	15
4th "	17	18	16	24	17	19	19	26	23
5th "	12	12	11	17	12	13	13	18	16

1–2. Bebedero, Costa Rico.— 3–5. S. Carlos, Costa Rica.— 6–8. La Palma, Costa Rica, 1600 m.— 9. Costa Rica.

This species is not without analogy with the Old-World forms connecting the subgenera *Rana* and *Hylorana;* another case of parallelism, according to me.

Proceedings of the American Academy of Arts and Sciences.

Vol. 55. No. 10. — November, 1920.

RECORDS OF MEETINGS, 1919–20.

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RECORDS OF MEETINGS.

One thousand and eighty-third Meeting.

OCTOBER 8, 1919.— STATED MEETING.

The Academy met at its House at 8.15 P. M.

The PRESIDENT in the Chair.

Ninety-three Fellows and five guests were present.

The Corresponding Secretary presented the following letters:— from A. A. Blanchard, H. N. Brown, F. H. Fay, P. E. Goddard, R. K. Hack, C. D. Hazen, F. G. Keyes, Joseph Lipka, G. A. Miller, J. B. Moore, F. R. Moulton, Edward Mueller, C. L. Nichols, Raymond Pearl, W. T. Schaller, H. D. Sedgwick, Virgil Snyder, Winslow Warren, R. S. Williams, and A. G. Woodman, accepting Fellowship; from Charles Barrois, J. J. C. Joffre, H. L. Le Chatelier, G. H. F. Nuttall, W. H. Perkin, Raymond Poincaré and G. W. Prothero, accepting Foreign Honorary Membership.

The Corresponding Secretary reported for the Council that E. V. Huntington had resigned as member of the Committee of Publication, and that the Council had authorized the President of the Academy to appoint a committee of three to present a name to fill the vacancy.

The President announced the death of the following Fellows and Foreign Honorary Members: William Brewster, Class II., Section 3; Walter Gould Davis, Class II., Section 1; William Gilson Farlow, Class II., Section 2; Francis Barton Gummere, Class III., Section 4; Abraham Jacobi, Class II., Section 4; William Roscoe Livermore, Class I., Section 4; Charles Herbert Williams, Class I., Section 2; Emil Fischer, Class I., Section 3; Magnus Gustav Retzius, Class II., Section 3; John William Strutt, Baron Rayleigh, Class I., Section 2.

A biographical notice of Charles Richard Van Hise by J. E. Wolff was presented.

The following Communications were presented:—

Professor Elihu Thomson. "Progress in Knowledge of Nature's Electricity."

Dr. William Roscoe Thayer. "The 'Union Académique.'"

On motion of C. H. Moore and after discussion it was

Voted, That a Committee be appointed to consider and report on the proposed American Council of Learned Societies devoted to Humanistic Studies.

The President appointed as this Committee: C. H. Haskins (chairman), Arthur Fairbanks, Kirsopp Lake, C. B. Gulick, C. R. Lanman and W. R. Thayer.

The Meeting was then dissolved.

One thousand and eighty-fourth Meeting.

NOVEMBER 12, 1919.— STATED MEETING.

The Academy met at its House at 8.15 P. M.

The PRESIDENT in the Chair.

Forty-four Fellows and five guests were present.

The Transactions of the last Meeting were read and approved.

The Corresponding Secretary presented letters from C. H. Brent, W. S. Franklin, W. H. Holmes, accepting Fellowship; from E. B. Spear, declining Fellowship; from G. M. Allen resigning Fellowship; from Ferdinand Foch, and A. A. Macdonell, accepting Foreign Honorary Membership.

The Corresponding Secretary announced the receipt of letters and other documents from the papers of Chief Justice Lemuel Shaw, the gift of Mr. Shaw's granddaughter, Josephine MacC. Shaw.

A biographical notice of Charles Card Smith by Arthur Lord was presented.

The Corresponding Secretary reported the action of the Council appointing Professor Louis Derr Chairman of the Committee of Publication in place of Professor E. V. Huntington resigned.

The President directed the attention of the Fellows present to Chief Justice Shaw's diploma of membership in the Academy, dated November 12, 1824.

The President made mention of the important services of Professor E. V. Huntington as Chairman and Professor A. A. Howard as Acting Chairman of the Committee of Publication, and it was

Voted, That the thanks of the Academy be sent to Professor Huntington and Professor Howard for their valued and generous labors in connection with this Committee.

The President announced that at a date in the near future an open meeting would be held in accordance with the plan adopted by vote of the Council, March 12, 1919.

The Recording Secretary presented letters received during the summer from the French Academy, inviting to participation in the "Union Académique"; and from the Académie de Metz, expressing happiness at liberation from German control, and inviting to representation at a Solemn Session on June 12, 1919.

The following Communications were presented and discussed:

Professor C. H. Grandgent. "Trills."

Dr. Harvey Cushing. "Some of the Problems of the Neuro-Surgeon arising from Battle Casualties."

The Meeting was then dissolved.

One thousand and eighty-fifth Meeting.

DECEMBER 6, 1919.—OPEN MEETING.

An Open Meeting was held from four to six o'clock in accordance with the vote of March 12, 1919.

The PRESIDENT in the Chair.

There were about 270 Fellows and guests including ladies present.

Dr. Thomas Morison Legge of London spoke on Stained Glass Windows of French and English Cathedrals with illustrations from lantern slides in the natural colors made by the late Lieut. Hugh Arnold.

Tea was served at five o'clock in the Reception Room on the third floor.

One thousand and eighty-sixth Meeting.

DECEMBER 10, 1919.— STATED MEETING.

The Academy met at its House at 8.15 P. M.

The PRESIDENT in the Chair.

Thirty-three Fellows and one guest were present.

The Transactions of the last Meeting were read and approved.

The Corresponding Secretary presented a letter from Alexis Carrel, accepting Fellowship.

The Corresponding Secretary reported the action of the Council appointing G. R. Agassiz to be Chairman and W. T. Sedgwick to be a member of the House Committee.

The President announced the receipt of a letter of acknowledgment from the editor of "Nature" for congratulations sent on the occasion of its jubilee.

The President announced the death of the following Fellows: Henry Lee Higginson, Class III., Section 4; Frederic Pike Stearns, Class I., Section 4; Thomas Franklin Waters, Class III., Section 3.

C. H. Haskins read the following report for the Committee on the proposed American Council of Learned Societies devoted to Humanistic Studies:

As Chairman of the Committee of the American Academy of Arts and Sciences to which the proposed constitution of the American Council of Learned Societies was referred, I beg to report that the Committee held a meeting 12 November, and unanimously voted to recommend to the Academy the ratification of the covenant and constitution as submitted in print by the conference on 19 September 1919.

It was, *Voted*, That the Academy accept and ratify the covenant and constitution proposed and submitted for such a Council, thereby assuming a place as one of the constituent societies.

The Corresponding Secretary reported that the Council of the Academy had authorized the President to appoint two representatives of the Academy to attend the first meeting of the Council of Learned Societies, and had passed a vote expressing the hope that a better term than "Humanistic" might be found for use in the name of the new Council.

It was, *Voted*, That the Academy defray the expenses of the two delegates to this first meeting.

The following Communication was presented:

Professor William E. Ritter. "The Common Ground of Three Domains of Knowledge, frequently called the Exact, the Descriptive, and the Humanistic Sciences."

Dr. Francis H. Williams presented a brief paper on a possible cause of the Aurora through the action of the wind in a magnetic field.

In the course of the discussion which followed Professor Elihu Thomson presented two letters describing a well-observed case of globular lightning, and Professor Ephraim Emerton described two cases actually seen by him.

The following paper was presented by title:—

"The Specific Heat of Ammonia." By H. A. Babcock. Presented by Henry Crew.

The Meeting was then dissolved.

One thousand and eighty-seventh Meeting.

JANUARY 14, 1920.—STATED MEETING.

The Academy met in its House at 8.15 P. M.

VICE-PRESIDENT THOMSON in the Chair.

Forty-one Fellows and one guest were present.

The Transactions of the last Meeting were read and approved.

The Corresponding Secretary presented a biographical notice of Frederick Remsen Hutton by L. S. Marks.

The Chair announced the death of the following Fellows: Louis Valentine Pirsson, Class II., Section 1; Sir William Osler, Bart., Class II., Section 4; Franklin Carter, Class III., Section 2.

The following Communication was presented:

Dr. Louis A. Bauer. "Observations of the Total Solar Eclipse, May 29, 1919, at Cape Palmas, Liberia, and at other Stations."

A discussion followed.

The Meeting was then dissolved.

One thousand and eighty-eighth Meeting.

JANUARY 31, 1920.— OPEN MEETING.

An Open Meeting was held at the House of the Academy from four to six o'clock.

The PRESIDENT in the Chair.

There were about one hundred Fellows and guests, including ladies, present.

Professor Roland B. Dixon of Harvard University spoke on Western Thibet and the Development of Central Asia with illustrations from lantern slides.

Tea was served at five o'clock in the Reception Room on the third floor.

One thousand and eighty-ninth Meeting.

FEBRUARY 11, 1920.— STATED MEETING.

The Academy met at its House at 8.15 P. M.

The PRESIDENT in the Chair.

Thirty-nine Fellows and one guest were present.

The Transactions of the last two Meetings were read and approved.

The Corresponding Secretary presented a letter from Endicott Peabody resigning Fellowship; and a biographical notice of Pasquale Villari by W. R. Thayer.

On the recommendation of the Council, it was,

Voted, That $300 be appropriated from the General Fund for General-and-Meeting Expenses.

The Council recommended the Amendment of the Statutes to provide for the appointment of a Standing Committee on Biographical Notices which shall be responsible for the preparation of such notices and shall have discretion as to their length, referring also to such other notices as may be conveniently accessible.

Voted, That in accordance with the Statute the recommendation be referred to a committee for report.

The President announced the death of the following Fellows:

Richard Cockburn Maclaurin, Class I., Section 2 (Physics); Elmer Ernest Southard, Class II., Section 4 (Medicine and Surgery).

The President announced the appointment of Professor C. B. Gulick for two years and Professor F. N. Robinson for four years to represent the Academy as members of the American Council of Learned Societies Devoted to Humanistic Studies.

The following Communications were presented and discussed:

Professor George H. Parker. "The Growth and Significance of the Nervous System."

Professor Charles R. Lanman. "The Arrangement of the Ancient Hindu Alphabet, and its Scientific Basis in Phonetics and Anatomy."

The following papers were presented by title:—

"A Monograph of the American Frogs of the Genus Rana." By G. A. Boulenger. Presented by J. B. Woodworth.

"Some Geometric Investigations on the General Problems of Dynamics." By Joseph Lipka.

"Contribution to the General Kinetics of Material Transformations." By A. J. Lotke. Presented by Irving Fisher.

"Rotations in Space of Even Dimensions." By H. B. Phillips and C. L. E. Moore.

"Orbits Resulting from Assumed Laws of Motion." By A. Searle.

The Meeting was then dissolved.

One thousand and nintieth Meeting.

March 10, 1920.—Stated Meeting.

The Academy met at its House at 8.15 P. M.

The President in the Chair.

Thirty-five Fellows and three guests were present.

The Transactions of the last Meeting were read and approved.

The President appointed the following Councillors as Nominating Committee:

G. P. Baxter, of Class I.

W. M. Wheeler, of Class II.

A. C. Coolidge, of Class III.

On recommendation of the Council, the following appropriations were made for the ensuing year:—

From the income of the General Fund, $8,100, to be used as follows:—

for General and Meeting expenses	$1,300.00
for Library expenses	2,800.00
for Books, periodicals and binding	1,200.00
for House expenses	2,200.00
for Treasurer's expenses	600.00

From the income of the Publication Fund, $4,655.24, to be used for publication.

From the income of the Rumford Fund, $3,343.26, to be used as follows:—

for Research	$1,000.00
for Publication	600.00
for use at the discretion of the Committee	1,743.26

From the income of the C. M. Warren Fund, $2,500, to be used at the discretion of the Committee.

A Biographical Notice of W. L. Hooper by A. C. Lane was presented.

The President announced that he had appointed as the Committee to report on the amendment of the Statutes proposed at the last Meeting, A. G. Webster, Barrett Wendell and A. C. Lane.

The President stated that the Council intends to bring forward at the next Meeting of the Academy a request for instructions with regard to the election of women as Fellows of the Academy.

Voted, That the Publication Committee be requested to report what may be done toward providing for the continued publication of learned works in series, the future of which is made precarious or impossible by reason of conditions in Europe.

The following Communication was read and discussed:

Professor Reginald A. Daly. "The Deep Sea Islands of the Pacific."

The following papers were presented by title:—

"The Springfield Rifle and the Leduc Formula." By A. G. Webster.

"On Sabine's Method in Architectural Acoustics without the Use of the Ear." By A. G. Webster.

The Meeting was then dissolved.

One thousand and ninety-first Meeting.

APRIL 14, 1920.—STATED MEETING.

The Academy met at its House at 8.15 P. M.

The PRESIDENT in the Chair.

Seventy-one Fellows and three guests were present.

The Transactions of the last Meeting were read and approved.

The Corresponding Secretary presented for the Council an invitation from the Wisconsin Academy of Sciences, Arts, and Letters to the exercises in commemoration of the fiftieth anniversary of its founding to be held on the 23d of April, 1920, at Madison, Wisconsin, and stated that the Council had voted to arrange for the representation of the Academy at this meeting.

The President announced the death of the following Fellow: Andrew McFarland Davis, Class III., Section 3 (Political Economy and History).

The President announced the gift from Boylston A. Beal Esq., of a painting by Tilton, representing the Bay of Naples, and it was

Voted, That the thanks of the Academy be sent to Mr. Beal for his welcome and valued gift.

The President stated that the Council thought it wise to postpone for discussion at a later meeting the question of the election of women to be Fellows of the Academy, and it was

Voted, That the Persident be authorized to appoint a Committee of three to consider the whole question of elections to the Academy.

Professor Webster moved that it is the sense of those members of the Academy here present that no objection lies against the election of women to be Fellows of the Academy.

Voted, To lay this motion on the table.

The following Communications were read and discussed:

Professor George W. Pierce. "Submarine Detection and Acoustical Aids to Navigation."

Professor Frank W. Taussig. "The Economic Revolution now in Process."

The Meeting was then dissolved.

One thousand and ninety-second Meeting.

May 12, 1920.—Annual Meeting.

The Academy met at its House.

The President in the Chair.

Thirty-six Fellows and one guest were present.

The Transactions of the last Meeting were read and approved.

The following report of the Council was presented:—

Since the last report of the Council, there have been reported the deaths of sixteen Fellows: William Brewster, Franklin Carter, A. McF. Davis, W. G. Davis, W. G. Farlow, F. B. Gummere, H. L. Higginson, Abraham Jacobi, W. R. Livermore, R. C. Maclaurin, Sir William Osler, L. V. Pirsson, E. E. Southard, F. P. Stearns, T. F. Waters, C. H. Williams; and three Foreign Honorary Members: Emil Fischer, M. G. Retzius, J. W. Strutt, Baron Rayleigh.

Twenty-five Fellows and nine Foreign Honorary Members were elected by the Council and announced to the Academy in May, 1919, of which number, one has declined Fellowship. Two Fellows have resigned.

The roll now includes 521 Fellows and 68 Foreign Honorary Members (not including those elected in April 1920).

The annual report of the Treasurer, Henry H. Edes, was read, of which the following is an abstract:—

General Fund.

Receipts.

Balance, April 1, 1919	$8,360.34	
Investments	4,037.24	
Assessments	3,520.00	
Admissions	120.00	
Sundries	206.20	$16,243.78

Expenditures.

Expense of Library	$3,508.73	
Expense of House	2,058.17	
Treasurer	302.85	
Assistant Treasurer	250.00	
General Expense of Society	1,011.03	
President's Expenses	101.02	
Interest on Bonds bought	22.00	
Income transferred to principal	320.52	$7,574.32
Balance, April 1, 1920		8,669.46
		$16,243.78

RUMFORD FUND.

Receipts.

Balance, April 1, 1919	$4,064.23	
Investments	3,758.51	
Grant returned	78.56	$7,901.30

Expenditures.

Research	$3,100.00	
Periodicals and binding	51.52	
Sundries	23.90	
Income transferred to principal	167.80	$3,343.22
Balance, April 1, 1920		4,558.08
		$7,901.30

C. M. WARREN FUND.

Receipts.

Balance, April 1, 1919	$4,776.13	
Investments	1,143.35	
Grant returned	111.61	$6,031.09

Expenditures.

Research	$1,050.00	
Vault rent, part	3.00	
Income transferred to principal	53.04	$1,106.04
Balance, April 1, 1920		4,925.05
		$6,031.09

Publication Fund.

Receipts.

Balance, April 1, 1919	$1,461.50	
Appleton Fund investments	1,181.35	
Centennial Fund investments	2,505.00	
Author's Reprints	33.39	
Sale of Publications	345.86	$5,527.10

Expenditures.

Publications	$1,879.42	
Vault rent, part	10.00	
Income transferred to principal	163.07	$2,052.49
Balance, April 1, 1920		3,474.61
		$5,527.10

Francis Amory Fund.

Receipts.

Investments	$1,324.08	$1,324.08

Expenditures.

Publishing statement	64.90	
Interest on bonds bought	28.25	
Income transferred to principal	1,230.93	$1,324.08

May 12, 1920.

The following reports were also presented:—

REPORT OF THE LIBRARY COMMITTEE.

The Librarian begs to report for the year 1919–1920, as follows:

During the year 1919–20, 65 books have been borrowed by 22 persons, including 17 Fellows, and 1 library. As usual many books have been consulted. All books taken out have been satisfactorily accounted for.

The number of books on the shelves at the time of the last report was 36,237. 796 volumes have been added, making the number now on the shelves 37,033. This includes 25 purchased from the income of the General Fund, 9 from that of the Rumford Fund, and 762 received by gift or exchange.

The books selected from the legacy of the late Professor William Watson, 584 in number, have now all been catalogued by Miss Wyman and added to the shelves. Pamphlets have also been placed alphabetically by authors in the pamphlet boxes, uncatalogued. The remainder of the books and pamphlets of the legacy have been sent to the Harvard Library, for their selection, those then remaining to be passed on to the Institute of Technology Library.

The expenses charged to the Library during the financial year are:—

Salaries	$2,797.03
Binding:—	
General Fund	533.58
Rumford Fund	30.75
Purchase of periodicals and books: –	
General Fund	116.37
Rumford Fund	20.77
Miscellaneous	21.75
Total	$3,520.25

A. G. WEBSTER, *Librarian.*

May 12, 1920.

REPORT OF THE RUMFORD COMMITTEE.

The Rumford Committee respectfully reports as follows:—

The Committee organized by the election of Charles R. Cross, Chairman, and Arthur G. Webster, Secretary.

During the past year grants in aid of research have been made as follows:—

October 8, 1919, to Professor Frances G. Wick in aid of her researches on the Phosphorescence of Hexagonite and of Fluorite at Ordinary and Low Temperatures	300
To Professor Robert W. Wood for the continuation of his optical investigations (additional)	350
December 10, 1919, to Professor Frederick G. Keyes in aid of his research on the Heats of Neutralization at different temperatures	300
January 14, 1920, to Professor Frederick A. Saunders, in aid of his research on Spectral Lines (additional)	150
To Professor David L. Webster in aid of his researches on X-Ray Spectra (additional)	350
March 10, 1920, to Professor Julius Stieglitz in aid of the publication of Marie's Tables of Constants (additional)	250
April 14, 1920, to Professor Leonard R. Ingersoll in aid of his research on the Polarizing Effect of Diffraction Gratings	150
To Professor Harrison M. Randall in aid of his research on the Structure of Spectra in the Infra-Red (additional)	500
To Professor Arthur G. Webster in aid of his research on a new Method in Pyrodynamics and Practical Interior Ballistics (additional)	500
May 12, 1920, to Professor Norton A. Kent in aid of his research on Spectral Lines (additional)	200

Reports of progress in their respective researches have been received from the following persons:— C. G. Abbot, W. M. Baldwin, R. T. Birge, P. W. Bridgman, W. W. Campbell, A. L. Clark, D. F. Comstock (research finished), H. Crew, F. Daniels, H. N. Davis (research finished), A. L. Foley (research finished), E. B. Frost, R. C. Gibbs, H. P. Hollnagel, H. L. Howes, L. R. Ingersoll, N. A. Kent, F. E.

Krester, F. G. Keyes, L. V. King (research discontinued; appropriation returned), C. A. Kraus, E. Kremers, R. A. Millikan, R. S. Minor, C. L. Norton, F. Palmer, Jr., J. A. Parkhurst, H. M. Randall, T. W. Richards, F. K. Richtmyer, F. A. Saunders, W. O. Sawtelle, A. W. Smith, B. J. Spence (research finished), J. Stieglitz, L. T. E. Thompson, O. Tugman, F. W. Very, A. G. Webster, D. L. Webster, F. G. Wick, R. W. Wood.

A paper by Henry A. Babcock, the Specific Heat of Ammonia, will be published shortly with aid from the Rumford Fund in the forthcoming volume of the Proceedings of the Academy, Vol. 55.

Toward the expense of publishing this paper the Committee on April 14 voted to appropriate a sum not exceeding four hundred and fifty dollars.

On April 14 it was voted for the first time, unanimously, and on May 12 for the second time, unanimously, to recommend to the Academy the award of the Rumford Premium to Irving Langmuir for his researches in thermionic and allied phenomena.

Charles R. Cross, *Chairman.*

May 12, 1920.

REPORT OF THE C. M. WARREN COMMITTEE.

The C. M. Warren Committee begs to submit the following report:

The Committee had at its disposal at the time of the last report to the Aacdemy $2,821.50. In December, 1919, Dr. J. H. Ellis returned to the Committee the sum of $111.61 representing the balance of a grant made to him in November, 1917, which was not fully utilized. On May 7, 1920, Professor G. H. Burrows returned the sum of $250 from the grant made to him in 1913, as he has found it impracticable to carry on the investigations for which the grant was originally made. In March, 1920, the Academy placed at the disposal of the Committee the sum of $2,500.

During the year two grants have been made, both to Professor Charles A. Kraus of Clark University, Worcester. The first was a grant of $500 made in June, 1919, for the investigation of properties of water at high temperatures. The second was a grant of $500 made in April, 1920, for the study of compounds which are formed in liquid ammonia solution.

In common with the Committees or Trustees in charge of other funds for research your Committee has received comparatively few applications within the past two or three years. There are indications of increasing interest in such funds, and there is expectation that several applications for grants from the Warren Fund will be made in the near future.

The unexpended balance in the hands of the Committee at the present time is $4,683.11.

Professor E. L. Mark reports that the results of the study of sea water, for which an appropriation from the Warren Fund was made, have been published.

Professor R. E. Wilson reports that his investigations of the phenomena accompanying the hydrolysis of certain organic compounds have been completed and that the results will shortly be published. Professor W. D. Harkins reports that the results of his investigations of the surface energy relations of mercury, of adhesion and negative surface energy, and of the entropy principle concerning the change of molecular kinetic into molecular potential energy have appeared in the Proceedings of the National Academy of Sciences. Others to whom grants have been made have sent in satisfactory reports of progress.

H. P. TALBOT, *Chairman.*

May 12, 1920.

REPORT OF THE PUBLICATION COMMITTEE.

The Committee of Publication reports as follows, for the period from April 1, 1919 to March 31, 1920:

During this period there have been issued No. 5 of Vol. 54 of the Proceedings, and Nos. 1, 2, 3, and 4 of Vol. 55. The number of papers by Fellows of the Academy published in its Proceedings has, as last year, been below the average pre-war figure, and it has therefore been possible under the rules to print several papers of value by authors who are not Fellows.

The financial statement follows:

Balance on hand April 1, 1919	$1,171.77	
Appropriations for 1919–20	4,860.08	
Sales of Publications	345.86	
Sales of authors' reprints	33.39	6,411.10
Expenses		1,879.42
Balance on hand March 31, 1920		4,531.68

Respectfully submitted,

LOUIS DERR, *Chairman.*

May 12, 1920.

REPORT OF THE HOUSE COMMITTEE.

The House Committee submits the following report for 1919–20:—

With the balance of $124.35 left from last year, an appropriation of $2,000, and $45 received from other societies for the use of the rooms, the Committee has had at its disposal the sum of $2,169.35. The total expenditure has been $2,103.17, leaving an unexpended balance on April 1, 1920, of $66.18. The expenditure has been as follows:—

Janitor		$831.00
Electricity	A. Light	151.12
	B. Power	69.62
Coal	Furnace	697.50
	Water Heater	61.20
Care of elevator		49.32
Gas		22.80
Water		8.00
Telephone		65.79
Ice		24.00
Janitor's materials		43.37
Upkeep		69.55
Ash tickets		9.90
Total expenditure		$2,103.17

The amount of $45 contributed by other societies for the use of the building leaves the net expense of the House $2,058.17.

Meetings have been held as follows:—

The Academy	
Regular meetings	8
Open meetings	2
The Physics Section	6
American Antiquarian Society	1
Archeological Institute	1
Colonial Dames	1
Colonial Society	4
Geological Society of Boston	4
Harvard Biblical Club	3
Harvard-Technology Chemical Club	4
Society of Landscape Architects	1
	35

The rooms on the first floor have been used for Council and Committee meetings.

The Committee expended about $200 more than last year which is accounted for in the increased cost of existence.

The wife of the late janitor, Mrs. Reardon, who was retained in the service of the Academy, has sought employment elsewhere, and the janitor's rooms are now occupied by the present janitor and his family.

Respectfully submitted,

G. R. Agassiz, *Chairman.*

May 12, 1920.

It was

Voted, That the thanks of the Academy be given to Mr. Agassiz, the retiring Chairman of the House Committee.

On the recommendation of the Treasurer, it was

Voted, That the Annual Assessment be $10.00.

On the recommendation of the Council, it was

Voted, To appropriate from the income of the General Fund the sum of $650.00 for the use of the Treasurer on account of additional insurance.

The Corresponding Secretary reported that the Special Committee on the amendment to the Statutes to provide for a Standing Committee on Biographical Notices having made an informal report, the Council had authorized the President to appoint a Special Committee to arrange for the preparation of Biographical Notices and to advise the Academy at a later date on a permanent plan with regard to this matter.

On the recommendation of the Rumford Committee, it was

Voted, To award the Rumford Premium to Irving Langmuir for his researches in thermionic and allied phenomena.

The annual election resulted in the choice of the following officers and committees:—

THEODORE W. RICHARDS, *President.*
ELIHU THOMSON, *Vice-President for Class I.*
HARVEY CUSHING, *Vice-President for Class II.*
GEORGE F. MOORE, *Vice-President for Class III.*
HARRY W. TYLER, *Corresponding Secretary.*
JAMES H. ROPES, *Recording Secretary.*
HENRY H. EDES, *Treasurer.*
ARTHUR G. WEBSTER, *Librarian.*

Councillors for Four Years.

FORRIS J. MOORE, *of Class I.*
REID HUNT, *of Class II.*
EPHRAIM EMERTON, *of Class III.*

Finance Committee.

HENRY P. WALCOTT, JOHN TROWBRIDGE,
HAROLD MURDOCK.

Rumford Committee.

Charles R. Cross,
Arthur G. Webster,
Elihu Thomson,
Theodore Lyman,
Louis Bell,
Percy W. Bridgman,
Harry M. Goodwin.

C. M. Warren Committee.

Henry P. Talbot,
Charles L. Jackson,
Gregory P. Baxter,
Walter L. Jennings,
William H. Walker,
Arthur D. Little,
James F. Norris.

Publication Committee.

Louis Derr, *of Class I.*
Herbert V. Neal, *of Class II.*
Albert A. Howard, *of Class III.*

Library Committee.

Harry M. Goodwin, *of Class I.*
Thomas Barbour, *of Class II.*
William C. Lane, *of Class III.*

House Committee.

John O. Sumner,
William T. Sedgwick,
Wm. Sturgis Bigelow.

Committee on Meetings.

The President,
The Recording Secretary,
George H. Parker,
Edwin B. Wilson,
George F. Moore.

Auditing Committee.

George R. Agassiz.
John E. Thayer.

The Council reported that the following gentlemen were elected members of the Academy:—

Harlow Shapley, of Pasadena, as Fellow in Class I., Section 1. (Mathematics and Astronomy.)

Jacques Salomon Hadamard, of Paris, as Foreign Honorary Member in Class I., Section 1. (Mathematics and Astronomy.)

Harry Monmouth Smith, of Brookline, as Fellow in Class I., Section 3. (Chemistry.)

Theodore Harwood Dillon, of Boston, as Fellow in Class I., Section 4. (Technology and Engineering.)

Frederick Law Olmsted, of Brookline, as Fellow in Class I., Section 4. (Technology and Engineering.)

William T. Bovie, of Milton, as Fellow in Class II., Section 3. (Zoölogy and Physiology.)

Benjamin Preston Clark, of Boston, as Fellow in Class II., Section 3. (Zoölogy and Physiology.)

Cecil Kent Drinker, of Boston, as Fellow in Class II., Section 3. (Zoölogy and Physiology.)

William Lyman Underwood, of Belmont, as Fellow in Class II., Section 3. (Zoölogy and Physiology.)

Maurice Caullery, of Paris, as Foreign Honorary Member in Class II., Section 3. (Zoölogy and Physiology.)

Walter Eugene Clark, of Chicago, as Fellow in Class III., Section 2. (Philology and Archaeology.)

Franklin Edgerton, of Philadelphia, as Fellow in Class III., Section 2. (Philology and Archaeology.)

Clark Wissler, of New York, as Fellow in Class III., Section 2. (Philology and Archaeology.)

Charles Howard McIlwain, of Cambridge, as Fellow in Class III., Section 3. (Political Economy and History.)

John Ellerton Lodge, of Boston, as Fellow in Class III., Section 4. (Literature and the Fine Arts.)

The following Communication was presented:—

Hon. William C. Wait. "The Part of the People in the Administration of the Law."

The following paper was presented by title:

"Acoustic Impedance and its Measurement," by A. E. Kennelly and K. Kurokawa.

The Meeting was then dissolved.

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WILLIAM LESLIE HOOPER (1855–1918)

Fellow in Class I, Section 2, 1884

The bare facts of Prof. Hooper's life can be easily abstracted from Who's Who 1917–1918:— "William Leslie Hooper, Prof. Electrical Engineering Tufts College 1890–1918; Born Halifax, N. S., Aug. 2, 1855, son Rev. William and Anne Jane (Whytal) Hooper; grad. Tufts College, Phi Beta Kappa, 1877 (A. M. 1878; Ph.D. 1898; Hon. LL. D. 1915). Married July 9, 1879 Mary E. Heard. Instr. math. and sciences 1878–80; prin. 1882, Bromfield Acad. Harvard, Mass.; Asst. Prof. Physics 1883–90, Tufts College. Acting Pres. Tufts College 1912–14. Died October 3, 1918. Author: Electrical Problems." In the Tufts College Graduate September–November 1918, there is a good likeness and a series of tributes, evidently coming from the heart, from men who knew Prof. Hooper much longer than the writer. It remains to add certain points of personal appreciation.

The first characteristic was his whole-hearted and unselfish devotion to Tufts. When he became Acting President he refused any addition to his salary as Professor, and the expense of his first campaign for funds to meet the deficit he bore himself. This was not because he was a man of large means, for although he stood high among electrical engineers, his devotion to the College and to his work was too great to permit him to heap up wealth from his commercial connections. Yet as Consulting Engineer and Director of the Somerville Electrical Company, Consulting Engineer for the West End Street Railway Company, and for Pearson in his Mexican development he had an enviable reputation.

Some of us have had interests divided between different colleges, but this was not so with him. All his degrees, up to the LL. D. which was the just and unsought reward at the end of his successful administration as Acting President during a critical time, were from Tufts.

Another thing that was characteristic of him was his willingness to work. In committee work or otherwise, he never gave any one else the heavier end of the log. And whether it was funds for the College or funds to send an ambulance unit across, he was a man who stood ready to take up any necessary work.

His love of fresh air and the out-of-doors was typical. I shall never forget his vigorous and spicy lecture on "Colds and How to Prevent Them." In fact he was always interested in medicine, enjoyed greatly his association with the Medical School after he became Acting President, and rejoiced that one of his family took up that profession. His summers were spent at Brompton Lake near Sherbrook, Quebec. Dr. Dresser of the Canadian Survey writes, "Prof. Hooper's cottage, on which I found him working with saw and hammer on my first visit, was built on a small island in an attractive lake of seven or eight miles in extent. The surrounding country is rugged and well forested and there are numerous smaller lakes in the vicinity. A couple of miles distant is the long abandoned Orford Nickel Mine. In these surroundings Prof. Hooper seemed particularly happy. In the varied natural history of the locality, in making surveys of the lakes and neighboring hills, sailing and portaging from lake to lake he had all the enthusiasm of the natural teacher. His genial personality and the cordial welcome from all within made his island a popular 'port of call' for visitors to the vicinity." His indefatigable energy even in vacation time was seen in the fact that he constructed with the aid of his sons, thereby giving them training in surveying, a map of the locality of which Prof. Dresser was glad to make use in his report on the region. Not only that, but for his cottage, with his own hands and by manual labor assisted only by his family, he built a twenty-eight foot chimney involving forty tons of stone.

Sometimes those who make a specialty of literature seem to think in spite of numerous instances to the contrary, that those whose chief work is in other fields have no sense in that direction. His love for his summer home and appreciation of its beauties were embodied in several poems, one of which on an Indian tale of that region is called "A Legend of Outunwitti." The opening words of this poem depict the pleasure he derived from the outdoor life.

Prologue.

You will hear no dissertation
On a theme of weight and moment,
No exalted exhortation,

But a simple woodland story
Written in some idle moments.
But if I shall make remembrance
Of some former glad vacation
Spent within our northern forests,
Spent upon the lakes and rivers,
If the scent of spruce and balsam
Shall be wafted to your nostrils,
If the soughing of the pine trees
And the murmur of the waters
Shall make music in your hearing,
If the glories of the sunset,
Seen across a lake at evening,
You shall vision from my story,
All my task will be accomplished,
I will have achieved my purpose.

Anti-militaristic as he was by environment and temperament, as the great war of 1914 went on, he realized more and more fully, and even before many of his friends, the great part which the United States ought to play. He was as indefatigable as usual in reading upon the subject. His work in the Committee of the Faculty on National Service of which he was chairman, was devoted and enthusiastic, and he gave many lectures under the auspices of the New England group for historical service and did a very great deal toward helping Tufts College to play its part well.

Not the least of the service of any man is to leave a family who will tread worthily in his footsteps. Five children grown to active and useful manhood and womanhood continue the traditions of energy and service to their generation which he handed down to them:—Blanche Heard, Librarian of Tufts College, William Ellsworth, Lieut. U. S. N., 1917–1919, Allen Gunnison, General Electric Company, Dr. Leslie Hooper Macmillan (Mrs. A. S. Macmillan), and Gertrude Mellen.

He died October 3d, 1918.

A. C. LANE.

CHARLES CARD SMITH (1827-1918)

Fellow in Class III, Section 3, 1892

Charles Card Smith, the only child of George and Harriet (Card) Smith, was born in Boston on March 27, 1827, and died in that city on March 20, 1918. His youth was passed in the city of Gloucester, where he attended the public and private schools, but he was not graduated from any university. In 1887 Harvard University, in recognition of his distinguished services as a historian and a writer, conferred on him the honorary degree of Master of Arts.

When sixteen years old Mr. Smith entered the office of the Northampton Woolen Manufacturing Company in Boston, and was continuously employed there until 1853 when he was chosen secretary of the Boston Gas Company and discharged the duties of that office for thirty-six years, until 1889. During that period he served as Treasurer of the American Unitarian Association, from 1862 to 1871.

Thirty years prior to his decease he retired from active business employment and devoted himself mainly to his duties as Editor and Treasurer of the Massachusetts Historical Society, and as a contributor to many historical and literary publications. He was Treasurer of the Massachusetts Historical Society from 1887 to 1907. No one of his predecessors surpassed him in length of service in that position, and only one, Richard Frothingham, held that office for as many years as Mr. Smith. He brought to the discharge of its duties long experience, business sagacity, and wise conservatism, and under his management the investments of the Society increased more than twelve fold. He became the Editor of the Society in 1889 and continued until his resignation as Treasurer and Editor in 1907 at the age of seventy-nine.

He was elected a Fellow of the American Academy of Arts and Sciences in 1892, but was not a frequent attendant at its meetings nor a contributor to its proceedings.

His scholarship, learning, and accuracy, and his critical and painstaking researches made his literary and historical studies and contributions of substantial importance and value. He was a welcome contributor to the *Christian Examiner*, making his first contribution to that paper at the age of twenty; and his writings appeared in the

New York Literary World, Boston Atlas, Boston Courier, Boston Traveller, North American Review (old and new), *Boston Daily Advertiser, Unitarian Review, Boston Post, Harvard Graduates' Magazine*, and the *American Historical Review*. He will be remembered as the author of the chapters in the *Memorial History of Boston*, on "Boston" and the "First French Protestants," and by the chapters in the *Narrative and Critical History of America*, on "Arctic and the Northwest Explorations," and on "Acadia" and "Cape Breton."

He was deeply interested as an officer and member in the Federal Street Church, served on its standing committee, was its Treasurer, and published the short account of the Church in its *Manual* of 1875.

In recognition of his literary abilities he held at various times positions on the examining committee of the Boston Public Library and the visiting committee of Harvard College.

He was elected a member of the American Antiquarian Society in 1876 and served as a member of its publishing committee for sixteen years, and for more than thirty years was a member of the American Historical Association and the Bostonian Society.

His principal contributions, however, were those made to the Proceedings of the Massachusetts Historical Society, of which he was a member for fifty years, and included Memoirs of John S. Barry, William S. Shurtleff, William G. Brooks, Deland A. Goddard, Chandler Robbins, George Dexter, John J. Babson, Thomas Aspinwall, John Amory Lowell, and William S. Appleton and two articles entitled "Short Account of the Society."

No compliment which Mr. Smith ever received was more appreciated by him than the generous words of Lord Macaulay in a letter to him "I could not have wished for a kinder or more liberal critic," or, "a reader so intelligent and enlightened." These words were as gratifying to Mr. Smith as they were deserved.

Mr. Smith's wife survived him but a few days, and he left no children, his only child, Walter Allen Smith, having deceased thirty-six years before Mr. Smith.

The chief interest of his long life was his literary and historical studies. The result of his labors is preserved in those publications to which reference has been made. Simple in his tastes, modest in his bearing and manner, a warm friend and generous helper to all those engaged in the pursuits which he loved, he will long be remembered by his intimate associates. He spared no labors to insure complete accuracy; diligent and painstaking in all his investigations, his conclusions on a disputed point of history met with general acceptance among his associates as the final word upon a subject, and although he had little patience with the superficial investigations of others, he was a kindly though severe critic.

ARTHUR LORD.

CHARLES RICHARD VAN HISE (1857–1918)

Fellow in Class II, Section 1, 1911

President Van Hise was born at Fulton, Wisconsin, in 1857, received his scientific training in the University of Wisconsin, and remained always closely connected with that institution, rising through the different grades of instructor and professor to the presidency, which he held from 1903 to his death in 1918.

Dr. Van Hise was from the beginning interested in geology and allied subjects, and was trained under R. D. Irving, then working on the crystalline pre-Cambrian rocks of the Lake Superior district, whose structure, classification, origin and relation to similar rocks throughout the world was then and still is one of the important problems of geology, enhanced by the vast copper and iron deposits which these rocks contain. Van Hise soon became a collaborator of Irving and on the latter's death carried on the work with an ever broadening field. He was for many years in charge of the Division of Archean Geology of the U. S. Geological Survey, and so had unique opportunity to study the pre-Cambrian in all parts of this country and Canada. Of his two most important geologic works the "Principles of North-American Pre-Cambrian Geology" was the result of these years of study. With a broadening view he then studied the chemical and physical forces which have acted in the formation of these rocks and his "Treatise on Metamorphism," a quarto volume of some 1200 pages was the outcome. This is a vast collection of facts gathered from his own experience or collated from the geological literature of the world, with elaborate discussions and original classifications and conclusions; it is perhaps his greatest single contribution to geology.

Many other publications, large and small (over eighty titles) exhibit his great industry and keen interest in his chosen field; this also included studies and classification of ore-deposits.

As President of the University for fifteen years Van Hise found new scope for his great energy and broad scope, conducting the affairs of that great institution with executive capacity and fruitful initiative, but he could no longer find the time for such monumental geological work. The public aspects of geological problems now interested him; he was for ten years a member of the National Conservation Commission, and his book on the "Conservation of the Natural Resources of the United States" is well known; in many other ways he performed public services, both before and during the War.

Van Hise had an impressive personality; he was a clear and forceful speaker, and always ready to discuss questions which interested him, with a great store of field and laboratory facts and a keen logical mind, seasoned by a touch of humor and an attractive straightforwardness. He was the leader of his generation in the field of pre-Cambrian geology.

Fellow of this Academy since 1911 and a member of the National Academy and many other scientific societies.

JOHN E. WOLFF.

FREDERIC REMSEN HUTTON (1853–1918)

Fellow in Class I, Section 4, 1893

Frederic Remsen Hutton, son of the Rev. Dr. Mancius Smedes Hutton, a well-known clergyman, and of Gertrude Holmes Hutton, was born in New York City on May 28, 1853. He prepared in a private school and then entered Columbia College, graduating with the degree of A. B. in 1873. He then entered the School of Mines and in 1876 received the degrees of C.E., E.M., and A.M. The degree of Ph.D. was conferred on him by Columbia in 1882, and the degree of Sc.D in 1904.

His teaching career began in 1877, when he received the appointment of instructor in Mechanical Engineering at Columbia University — the first appointment in that field at that institution; he was made adjunct professor in 1882, professor in 1891, and Dean of the Schools of Applied Science in 1899. He resigned from Columbia in 1907 and was elected professor emeritus.

His literary output consisted of three text-works, a history of the American Society of Mechanical Engineers and some minor papers. In his latter years he served as consulting engineer to the Department of Water, Gas and Electricity of New York City and to the Automobile Club of America.

Hutton's chief work was not as a teacher, a scientist, or an engineer, but as an organizer and executive. The American Society of Mechanical Engineers, which is now one of the most powerful organizations of its kind, was started by a small group of men in 1880. Three years later, while its membership was less than four hundred, Hutton was appointed Secretary. The history of the Society from that time till 1906 is largely a history of his efforts. During those twenty-three years the membership increased tenfold, the rooms expanded from two small rooms to its present splendid quarters, the library grew from a collection of trade catalogs to the best technical library in the country, and the Society became the recognized central authority in its field. These developments were to a considerable degree the product of his untiring energy, tact, good judgment and cheerfulness.

He died on May 14, 1918, in the sixty-fifth year of his age.

LIONEL S. MARKS.

PASQUALE VILLARI (1827–1917)

Foreign Honorary Member in Class III, Section 3, 1904

Pasquale Villari who died in Rome on December 7, 1917, was born in Naples on October 3, 1827. He studied at the University of Naples, intending to become a lawyer, but like many of the alert young men of his time, he was drawn into the patriotic movement of 1848, and when, after a year and a half, the Bourbon régime was restored, young Villari took refuge in Tuscany. There he supported himself by giving lessons in Italian, devoting all his spare time to historical study, especially to researches in the Archives of Florence. As a result he brought out in 1859 the first volume of his "History of

Savonarola and His Time," a work which immediately gave him a reputation. He filled the chair of History at the University of Pisa and then, after the Union of Tuscany to the new Kingdom of Italy, he became a Professor in the Institute of Higher Studies at Florence. Current political topics more and more engrossed his attention, and he published many letters and pamphlets upon them. His *Lettere Meridionali* on affairs in Southern Italy, and Sicily in particular, caused much comment. He served on various Commissions, was Minister of Public Instruction and, in 1884, by appointment of King Humbert, he became a Senator. History remained, however, his chief intellectual interest. In 1877 he published the first of a three volume biography of Machiavelli, in which he completed his study of the Italian Renaissance. At other times he also printed volumes on earlier Italian history, and for many years to the *Nuova Antologia* he contributed literary criticisms of the important books, or his recollections, historical and personal. By editing the dispatches of the Venetian Ambassador Antonio Giustiniani, he called attention to the importance for historical students of the then almost unexplored Venetian sources. In his later years he was President of the Dante Alighieri Society, organized to promote the study of Italian and the diffusion of Italian ideas outside Italy. He remained active to the last. living to see the invasion of northern Italy by the Teutonic and Slavic hordes, and being thus reminded of the invasions many centuries before. His wife Signora Linda Villari was an English woman, who translated into English his principal works and thus helped to give him a reputation among English readers throughout the world. He was elected a Foreign Honorary Member of the American Academy of Arts and Sciences in 1904.

WILLIAM ROSCOE THAYER.

CHARLES H. WILLIAMS (1850–1918)

Fellow in Class I, Section 2, 1914

Charles Herbert Williams was born in Boston April 29, 1850. He was the eldest son of Dr. H. W. Williams, Professor of Ophthalmology in the Harvard Medical School, and for many years one of the most distinguished specialists in ocular diseases. The son was graduated from Harvard College in the Class of 1871, and then entered the Harvard Medical School from which he was graduated three years later. With unusual aptitude and inclination toward his father's specialty, he then went abroad and spent the most of the next two years in studying ophthalmology in Zurich, Vienna, and then at Utrecht under Professor Donders, the great pioneer in the investigation and treatment of astigmatism. On returning in 1876 he entered practice with his father at the old home, 15 Arlington St. and spent the next decade in active work at his chosen profession.

In 1886 he turned his face westward and assumed an administrative position with the C. B. & Q. Railway, an association which eventually turned his talents into the path along which he was to march far. Presently he became head of the medical and health insurance interests of the great railway system with which he was associated, and his attention was drawn sharply to the grave question of defective vision and especially defective color sense among railway employees. At about that time color blindness as a serious danger was just coming to be fully recognized, and it was to the study of this that Dr. Williams devoted his special attention. About half of the employees of an ordinary railway system have occasion in one way or another to use signals, and since mankind shows three to four per cent of serious cases of color blindness the number of railway employees thus dangerously defective might run to many thousands. With this condition complicated by vision otherwise defective the importance of the field of research undertaken by Dr. Williams is vital.

Up to that time the ordinary means of testing color vision was by the Holmgren worsteds, a scheme most ingeniously devised and in careful hands very effective for diagnosis. Dr. Williams was among the first to recognize the fact that while these worsteds told a truthful story of failing color vision they did not tell the whole of it, since there are many cases of defective central vision due to tobacco or alcoholism which do not show with the Holmgren skeins by reason of these being sufficiently large considerably to overlap the affected area. Further, the psychological effect of testing a man's fitness for employment by setting him at sorting colored wools is bad, for he may not unnaturally think that colored lights would be quite a different matter. Color discs had of course been used in testing but Dr. Williams studied the subject deeply and in 1891 introduced into the C. B. & Q. system a complete scheme for testing the vision of railway men; a set of carefully prepared cards on Snellen's system with special precautions against the possibilities of memorizing, the Holmgren worsteds with the addition of a scheme of numbering to facilitate the complete record of each case, colored signal flags, and finally a lantern showing colored lights, which he afterwards greatly improved.

This was at first used only in dubious cases, but more and more reliance became gradually to be placed on it until in its later form it was accepted as the standard test on many important railway systems. The improvement in Dr. Williams' lantern over the colored glass test introduced by his old instructor Professor Donders, and improved by others, lay in the completeness of its correspondence to the working condition of observing railway signals. It includes use of colored spots of varying size and intensity, of those involving contrast of normal and signal lights, the introduction of confusion colors and such as might come from the modifying influence of atmosphere, and the presentation of combinations so diverse as to give absolutely no chance of passing the test by memorizing, or by the judgments of intensities on which the color blind often consciously or unconsciously depend. It has been found in practice that this lantern can detect many cases that escape examination by the worsteds, and therefore much more effectively eliminate a class of employees which is doubly dangerous from its unconsciousness of its own visual failings.

Dr. Williams also did much to improve the general testing of vision by the design of accurate and uniform test cards and the study of their proper illumination for testing purposes. He also introduced a very ingenious modification of the Snellen types for railway work consisting of characters simulating the various semaphore positions which the railway man has to note.

In 1895 he returned to Boston and began again the regular practice of his profession, not, however, leaving but rather further cultivating the special field of his endeavors. The impression that his work up to this time had made on his colleagues is well noted in a letter received by him from Snellen, in which that great ophthalmologist, recalling pleasant days which he had spent with Dr. Williams abroad, said of his return to general practice, "I am glad that you have come back to the old flag."

While industriously following the daily routine of his profession Dr. Williams kept on with his study of vision and of signalling, and devoted no little time to the improvement and testing of railway signal lights, in which he was most expert and resourceful. He served on many committees of the American Ophthalmological Society and had a large share in the preparation of their test cards and recommendations for improved methods, and wrote many papers before professional societies dealing with the general problem of the critical diagnosis of imperfect vision.

As an incident to his work upon signal lights he devised an extremely simple and useful modification of the photographic wedge photometer, a pocket instrument working on the extinction principle and capable in careful hands of remarkably useful work. This instrument came into considerable use among railway signal experts as well as among those interested in scientific photometry, to whom it has great use as an auxiliary for the quick investigation of photometric conditions. A little later he devised still another photometer of the equality-of-brightness type, employing as before the photographic wedge, which proved particularly useful in comparing the brightness of surfaces and estimating the density of smoke and cloud. He became an early member of the Illuminating Engineering Society and frequently took useful part in its discussions, and was beside a member of many societies and clubs. He was elected to our Academy January 14, 1914, and until very recent years was a frequent attendant at its meetings. The study of color vision he kept up, as far as strength permitted, almost to the end and was for a score of years a frequent lecturer on the problems connected with color blindness at the Harvard Medical School and elsewhere in professional gatherings. In a practical way no American ophthalmologist has done more toward the investigation of this curious and mysterious condition.

In later years a malady of the heart gave his friends much anxiety, and to it, after a gallant struggle, he succumbed on June 9th, 1918. Professionally eminent, he was above all things a charming and courteous gentleman, beloved of his many friends, always ready and eager to lend a helping hand in the advancement of science, and his loss came with a sense of real grief to his colleagues in this Academy.

LOUIS BELL.

American Academy of Arts and Sciences

OFFICERS AND COMMITTEES FOR 1920–21

PRESIDENT.

Theodore W. Richards.

VICE-PRESIDENTS.

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Elihu Thomson,	Harvey Cushing,	George F. Moore.

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COUNCILLORS.

Class I.	Class II.	Class III.
Henry Lefavour,	William T. Sedgwick, *Terms expire* 1921.	Barrett Wendell.
George D. Birkhoff,	Charles H. Warren, *Terms expire* 1922.	Frederic Dodge.
Cecil H. Peabody,	Edward M. East, *Terms expire* 1923.	Charles B. Gulick.
Forris J. Moore,	Reid Hunt, *Terms expire* 1924.	Ephraim Emerton.

COMMITTEE OF FINANCE.

Henry P. Walcott, John Trowbridge, Harold Murdock.

RUMFORD COMMITTEE.

Charles R. Cross, *Chairman*,
Arthur G. Webster, Theodore Lyman, Percy W. Bridgman.
Elihu Thomson, Louis Bell, Harry M. Goodwin.

C. M. WARREN COMMITTEE.

Henry P. Talbot, *Chairman*,
Charles L. Jackson, Walter L. Jennings, Arthur D. Little.
Gregory P. Baxter, William H. Walker, James F. Norris.

COMMITTEE OF PUBLICATION.

Louis Derr, of Class I, *Chairman*,
Herbert V. Neal, of Class II, Albert A. Howard, of Class III.

COMMITTEE ON THE LIBRARY.

Arthur G. Webster, *Chairman*,
Harry M. Goodwin, of Class I, Thomas Barbour, of Class II,
William C. Lane, of Class III.

AUDITING COMMITTEE.

George R. Agassiz, John E. Thayer.

HOUSE COMMITTEE.

John O. Sumner, *Chairman*, Wm. T. Sedgwick, Wm. Sturgis Bigelow.

COMMITTEE ON MEETINGS.

The President,
The Recording Secretary,
George H. Parker, Edwin B. Wilson, George F. Moore.

LIST

OF THE

FELLOWS AND FOREIGN HONORARY MEMBERS.

(Corrected to October 10, 1920.)

FELLOWS.— 529.

(Number limited to six hundred.)

Class I.— *Mathematical and Physical Sciences.*— 190.

Section I.— *Mathematics and Astronomy.*— 42.

George Russell Agassiz	Boston
Raymond Clare Archibald	Providence, R. I.
Solon Irving Bailey	Cambridge
Edward Emerson Barnard	Williams Bay, Wis.
George David Birkhoff	Cambridge
Charles Leonard Bouton	Cambridge
Ernest William Brown	New Haven, Conn.
Sherburne Wesley Burnham	Williams Bay, Wis.
William Elwood Byerly	Cambridge
William Wallace Campbell	Mt. Hamilton, Cal.
Julian Lowell Coolidge	Cambridge
George Cary Comstock	Madison, Wis.
Leonard Eugene Dickson	Chicago, Ill.
Philip Fox	Evanston, Ill.
Fabian Franklin	New York, N.Y.
Edwin Brant Frost	Williams Bay, Wis.
Frank Lauren Hitchcock	Belmont
Edward Vermilye Huntington	Cambridge
Dunham Jackson	Minneapolis, Minn.
Edward Skinner King	Cambridge
Carl Otto Lampland	Flagstaff, Ariz.

Joseph Lipka	Cambridge
Joel Hastings Metcalf	Portland, Me.
George Abram Miller	Urbana, Ill.
Clarence Lemuel Elisha Moore	Watertown
Eliakim Hastings Moore	Chicago, Ill.
Frank Morley	Baltimore, Md.
Forest Ray Moulton	Chicago, Ill.
Henry Bayard Phillips	Boston
William Henry Pickering	Cambridge
Charles Lane Poor	New York, N. Y.
Roland George Dwight Richardson	Middletown, Conn.
Arthur Searle	Cambridge
Harlow Shapley	Pasadena, Cal.
Vesto Melvin Slipher	Flagstaff, Ariz.
Frederick Slocum	Middletown, Conn.
Virgil Snyder	Ithaca, N. Y.
William Edward Story	Worcester
Henry Taber	Worcester
Harry Walter Tyler	Newton
Robert Wheeler Willson	Cambridge
Frederick Shenstone Woods	Newton

CLASS I., SECTION II.— *Physics.*— 55.

Joseph Sweetman Ames	Baltimore, Md.
Carl Barus	Providence, R. I.
Louis Agricola Bauer	Washington, D. C.
Alexander Graham Bell	Washington, D. C.
Louis Bell	Newton
Percy Williams Bridgman	Cambridge
Henry Andrews Bumstead	New Haven, Conn.
George Ashley Campbel	New York, N. Y.
Emory Leon Chaffee	Belmont
Daniel Frost Comstock	Cambridge
William David Coolidge	Schenectady, N. Y.
Henry Crew	Evanston, Ill.
Charles Robert Cross	Brookline

Harvey Nathaniel Davis	Cambridge
Arthur Louis Day	Corning, N. Y.
Louis Derr	Brookline
William Johnson Drisko	Winchester
William Duane	Boston
Alexander Wilmer Duff	Worcester
Arthur Woolsey Ewell	Worcester
William Suddards Franklin	Cambridge
Harry Manley Goodwin	Brookline
George Ellery Hale	Pasadena, Cal.
Edwin Herbert Hall	Cambridge
Hammond Vinton Hayes	Boston
John Charles Hubbard	New York, N. Y.
Gordon Ferrie Hull	Hanover, N. H.
Charles Clifford Hutchins	Brunswick, Me.
James Edmund Ives	Worcester
William White Jacques	Boston
Norton Adams Kent	Cambridge
Frank Arthur Laws	Boston
Henry Lefavour	Boston
Theodore Lyman	Brookline
Thomas Corwin Mendenhall	Ravenna, O.
Ernest George Merritt	Ithaca, N. Y.
Albert Abraham Michelson	Chicago, Ill.
Dayton Clarence Miller	Cleveland, O.
Robert Andrews Millikan	Chicago, Ill.
Harry Wheeler Morse	Los Angeles, Cal.
Edward Leamington Nichols	Ithaca, N. Y.
Ernest Fox Nichols	New Haven, Conn.
Charles Ladd Norton	Boston
George Washington Pierce	Cambridge
Michael Idvorsky Pupin	New York, N. Y.
Frederick Albert Saunders	Cambridge
John Stone Stone	New York, N. Y.
Maurice deKay Thompson	Brookline
Elihu Thomson	Swampscott
John Trowbridge	Cambridge
Arthur Gordon Webster	Worcester

David Locke Webster	Palo Alto, Cal.
Edwin Bidwell Wilson	Brookline
Robert Williams Wood	Baltimore, Md.
John Zeleny	New Haven, Conn.

Class I., Section III.—*Chemistry.*—51.

Wilder Dwight Bancroft	Ithaca, N. Y.
Gregory Paul Baxter	Cambridge
Arthur Alphonzo Blanchard	Cambridge
Marston Taylor Bogert	New York, N. Y.
Bertram Borden Boltwood	New Haven, Conn.
William Crowell Bray	Berkeley, Cal.
Russell Henry Chittenden	New Haven, Conn.
Arthur Messinger Comey	Chester, Pa.
Charles William Eliot	Cambridge
Henry Fay	Boston
George Shannon Forbes	Cambridge
Frank Austin Gooch	New Haven, Conn.
Lawrence Joseph Henderson	Cambridge
Charles Loring Jackson	Boston
Walter Louis Jennings	Worcester
Grinnell Jones	Cambridge
Frederick George Keyes	Cambridge
Elmer Peter Kohler	Cambridge
Charles August Kraus	Worcester
Arthur Becket Lamb	Cambridge
Irving Langmuir	Schenectady, N. Y.
Gilbert Newton Lewis	Berkeley, Cal.
Warren Kendall Lewis	Boston
Arthur Dehon Little	Brookline
Charles Frederic Mabery	Cleveland, O.
Forris Jewett Moore	Boston
George Dunning Moore	Worcester
Edward Williams Morley	West Hartford, Conn.
Harmon Northrop Morse	Baltimore, Md.
Edward Mueller	Cambridge

Samuel Parsons Mulliken	Boston
Charles Edward Munroe	Forest Glen, Md.
James Flack Norris	Boston
Arthur Amos Noyes	Pasadena, Cal.
William Albert Noyes	Urbana, Ill.
Thomas Burr Osborne	New Haven, Conn.
Samuel Cate Prescott	Brookline
Ira Remsen	Baltimore, Md.
Robert Hallowell Richards	Jamaica Plain
Theodore William Richards	Cambridge
Martin André Rosanoff	Pittsburgh, Pa.
Stephen Paschall Sharples	Cambridge
Miles Standish Sherrill	Brookline
Alexander Smith	New York, N. Y.
Harry Monmouth Smith	Brookline
Julius Oscar Stieglitz	Chicago, Ill.
Henry Paul Talbot	Newton
William Hultz Walker	Boston
Willis Rodney Whitney	Schenectady, N. Y.
Robert Seaton Williams	Cambridge
Alpheus Grant Woodman	Watertown

Class I., Section IV.—*Technology and Engineering.*—42.

Henry Larcom Abbot	Cambridge
Comfort Avery Adams	Cambridge
Bernard Arthur Behrend	Boston
William Herbert Bixby	Chicago, Ill.
Francis Tiffany Bowles	Boston
Charles Francis Brush	Cleveland, O.
William Hubert Burr	New Canaan, Conn.
John Joseph Carty	New York, N.Y.
Eliot Channing Clarke	Boston
Harry Ellsworth Clifford	Newton
Theodore Harwood Dillon	Cambridge
Frederic Harold Fay	Boston
Desmond FitzGerald	Brookline
John Ripley Freeman	Providence, R. I.

George Washington Goethals	New York, N. Y.
John Hays Hammond	New York, N. Y.
Rudolph Hering	New York, N. Y.
Ira Nelson Hollis	Worcester
Henry Marion Howe	New York, N. Y.
Hector James Hughes	Cambridge
Alexander Crombie Humphreys	New York, N. Y.
Dugald Caleb Jackson	Cambridge
Lewis Jerome Johnson	Cambridge
Arthur Edwin Kennelly	Cambridge
Gaetano Lanza	Philadelphia, Pa.
Lionel Simeon Marks	Cambridge
Edward Furber Miller	Newton
Hiram Francis Mills	South Hingham
Frederick Law Olmsted	Brookline
Charles Francis Park	Boston
William Barclay Parsons	New York, N.Y.
Cecil Hobart Peabody	Boston
Harold Pender	Philadelphia, Pa.
Albert Sauveur	Cambridge
Peter Schwamb	Arlington
Henry Lloyd Smyth	Cambridge
Charles Milton Spofford	Brookline
Charles Proteus Steinmetz	Schenectady, N. Y.
George Fillmore Swain	Cambridge
George Chandler Whipple	Cambridge
Robert Simpson Woodward	Washington, D. C.
Joseph Ruggles Worcester	Boston

CLASS II.— *Natural and Physiological Sciences.*— 160.

SECTION I.— *Geology, Mineralogy, and Physics of the Globe.*— 47.

Wallace Walter Atwood	Cambridge
George Hunt Barton	Cambridge
Isaiah Bowman	Washington, D. C.
Thomas Chrowder Chamberlin	Chicago, Ill.
John Mason Clarke	Albany, N. Y.

Henry Helm Clayton	Canton
Herdman Fitzgerald Cleland	Williamstown
William Otis Crosby	Jamaica Plain
Reginald Aldworth Daly	Cambridge
Edward Salisbury Dana	New Haven, Conn.
William Morris Davis	Cambridge
Benjamin Kendall Emerson	Amherst
William Ebenezer Ford	New Haven, Conn.
James Walter Goldthwait	Hanover, N. H.
Louis Caryl Graton	Cambridge
Herbert Ernest Gregory	New Haven, Conn.
Ellsworth Huntington	Milton
Oliver Whipple Huntington	Newport, R. I.
Robert Tracy Jackson	Peterborough, N. H.
Thomas Augustus Jaggar	Honolulu, H. I.
Douglas Wilson Johnson	New York, N. Y.
Alfred Church Lane	Cambridge
Andrew Cowper Lawson	Berkeley, Cal.
Charles Kenneth Leith	Madison, Wis.
Waldemar Lindgren	Brookline
Frederic Brewster Loomis	Amherst
Alexander George McAdie	Readville
William John Miller	Northampton
Charles Palache	Cambridge
John Elliott Pillsbury	Washington, D. C.
Raphael Pumpelly	Newport, R. I.
Percy Edward Raymond	Cambridge
William North Rice	Middletown, Conn.
Robert Wilcox Sayles	Cambridge
Waldemar Theodore Schaller	Washington, D. C.
Charles Schuchert	New Haven, Conn.
William Berryman Scott	Princeton, N. J.
Hervey Woodburn Shimer	Watertown
Thomas Wayland Vaughan	Washington, D. C.
Charles Doolittle Walcott	Washington, D. C.
Robert DeCourcy Ward	Cambridge
Charles Hyde Warren	Auburndale
Herbert Percy Whitlock	New York, N. Y.

Bailey Willis	Palo Alto, Cal.
John Eliot Wolff	Cambridge
Jay Backus Woodworth	Cambridge
Frederick Eugene Wright	Washington, D. C.

Class II., Section II.— *Botany*.— 29.

Oakes Ames	North Easton
Irving Widmer Bailey	Cambridge
Liberty Hyde Bailey	Ithaca, N. Y.
Douglas Houghton Campbell	Palo Alto, Cal.
George Perkins Clinton	New Haven, Conn.
John Merle Coulter	Chicago, Ill.
Bradley Moore Davis	Ann Arbor, Mich.
Edward Murray East	Jamaica Plain
Alexander William Evans	New Haven, Conn.
Merritt Lyndon Fernald	Cambridge
George Lincoln Goodale	Cambridge
Robert Almer Harper	New York, N. Y.
John George Jack	East Walpole
Edward Charles Jeffrey	Cambridge
Fred Dayton Lambert	Tufts College
Burton Edward Livingston	Baltimore, Md.
George Richard Lyman	Washington, D. C.
Winthrop John Vanleuven Osterhout	Cambridge
Alfred Rehder	Jamaica Plain
Lincoln Ware Riddle	Cambridge
Benjamin Lincoln Robinson	Cambridge
Charles Sprague Sargent	Brookline
William Albert Setchell	Berkeley, Cal.
Arthur Bliss Seymour	Cambridge
Erwin Frink Smith	Washington, D. C.
John Donnell Smith	Baltimore, Md.
William Codman Sturgis	New York, N. Y.
Roland Thaxter	Cambridge
William Trelease	Urbana, Ill.

Class II., Section III.—*Zoölogy and Physiology*.—54.

Joel Asaph Allen	New York, N. Y.
Thomas Barbour	Cambridge
Francis Gano Benedict	Boston
Henry Bryant Bigelow	Concord
Robert Payne Bigelow	Brookline
William T. Bovie	Milton
John Lewis Bremer	Boston
Charles Thomas Brues	Boston
Hermon Carey Bumpus	Providence, R. I.
Walter Bradford Cannon	Cambridge
William Ernest Castle	Belmont
Charles Value Chapin	Providence, R. I.
Benjamin Preston Clark	Boston
Samuel Fessenden Clarke	Williamstown
Edwin Grant Conklin	Princeton, N. J.
William Thomas Councilman	Boston
Joseph Augustine Cushman	Sharon
William Healey Dall	Washington, D. C.
Charles Benedict Davenport	Cold Spring Harbor, N. Y.
Gilman Arthur Drew	Woods Hole
Cecil Kent Drinker	Boston
Alexander Forbes	Milton
Samuel Henshaw	Cambridge
Leland Ossian Howard	Washington, D. C.
Herbert Spencer Jennings	Baltimore, Md.
Charles Willison Johnson	Brookline
Charles Atwood Kofoid	Berkeley, Cal.
Frederic Thomas Lewis	Waban
Ralph Stayner Lillie	Worcester
Jacques Loeb	New York, N. Y.
Richard Swann Lull	New Haven, Conn.
Edward Laurens Mark	Cambridge
Ernest Gale Martin	Palo Alto, Cal.
Albert Davis Mead	Providence, R. I.
Edward Sylvester Morse	Salem
Herbert Vincent Neal	Tufts College

Henry Fairfield Osborn	New York, N. Y.
George Howard Parker	Cambridge
Raymond Pearl	Baltimore, Md.
John Charles Phillips	Wenham
Herbert Wilbur Rand	Cambridge
William Emerson Ritter	La Jolla, Cal.
William Thompson Sedgwick	Boston
Percy Goldthwait Stiles	Newtonville
John Eliot Thayer	Lancaster
William Lyman Underwood	Belmont
Addison Emory Verrill	Whitneyville, Conn.
John Broadus Watson	Washington, D. C.
Arthur Wisswald Weysse	Boston
William Morton Wheeler	Boston
Harris Hawthorne Wilder	Northampton
Edmund Beecher Wilson	New York, N. Y.
Frederick Adams Woods	Brookline
Robert Mearns Yerkes	Washington, D. C.

Class II., Section IV.— *Medicine and Surgery*.— 30.

Edward Hickling Bradford	Boston
Alexis Carrel	New York, N. Y.
Henry Asbury Christian	Boston
Harvey Cushing	Boston
David Linn Edsall	Cambridge
Harold Clarence Ernst	Jamaica Plain
Simon Flexner	New York, N.Y.
Robert Battey Greenough	Boston
William Stewart Halsted	Baltimore, Md.
Reid Hunt	Brookline
Henry Jackson	Boston
Elliott Proctor Joslin	Boston
William Williams Keen	Philadelphia, Pa.
Frank Burr Mallory	Brookline
Samuel Jason Mixter	Boston
Edward Hall Nichols	Boston
Theophil Mitchell Prudden	New York, N. Y.

William Lambert Richardson	Boston
Milton Joseph Rosenau	Boston
Frederick Cheever Shattuck	Boston
Theobald Smith	Princeton, N. J.
Richard Pearson Strong	Boston
Ernest Edward Tyzzer	Wakefield
Frederick Herman Verhoeff	Boston
Henry Pickering Walcott	Cambridge
John Collins Warren	Boston
William Henry Welch	Baltimore, Md.
Francis Henry Williams	Boston
Simeon Burt Wolbach	Boston
Horatio Curtis Wood	Philadelphia, Pa.

CLASS III.— *Moral and Political Sciences.*— 179.

SECTION I.— *Theology, Philosophy and Jurisprudence.*— 50.

Thomas Willing Balch	Philadelphia, Pa.
Simeon Eben Baldwin	New Haven, Conn.
Willard Bartlett	Brooklyn, N. Y.
Joseph Henry Beale	Cambridge
Melville Madison Bigelow	Cambridge
Charles Henry Brent	Buffalo, N. Y.
Howard Nicholson Brown	Boston
Charles Warren Clifford	New Bedford
Edmund Burke Delabarre	Providence, R. I.
James De Normandie	Roxbury
Frederic Dodge	Belmont
Edward Staples Drown	Cambridge
William Harrison Dunbar	Cambridge
William Herbert Perry Faunce	Providence, R. I.
William Wallace Fenn	Cambridge
Frederick Perry Fish	Brookline
George Angier Gordon	Boston
John Wilkes Hammond	Cambridge
Alfred Hemenway	Boston
Charles Evans Hughes	New York, N. Y.

Frederick John Foakes Jackson	New York, N. Y.
William Lawrence	Boston
Arthur Lord	Plymouth
William Caleb Loring	Boston
Nathan Matthews	Boston
Samuel Walker McCall	Winchester
Edward Caldwell Moore	Cambridge
John Bassett Moore	New York, N. Y.
James Madison Morton	Fall River
George Herbert Palmer	Cambridge
Charles Edwards Park	Boston
Leighton Parks	New York, N. Y.
Francis Greenwood Peabody	Cambridge
George Wharton Pepper	Philadelphia, Pa.
John Winthrop Platner	Cambridge
Roscoe Pound	Belmont
Elihu Root	New York, N. Y.
James Hardy Ropes	Cambridge
Arthur Prentice Rugg	Worcester
Henry Newton Sheldon	Boston
Moorfield Storey	Boston
William Howard Taft	New Haven, Conn.
William Jewett Tucker	Hanover, N. H.
William Cushing Wait	Medford
Williston Walker	New Haven, Conn.
Eugene Wambaugh	Cambridge
Edward Henry Warren	Brookline
Winslow Warren	Dedham
Samuel Williston	Belmont
Woodrow Wilson	Washington, D. C

Class III., Section II.— *Philology and Archæology.*— 53.

Francis Greenleaf Allinson	Providence, R. I
William Rosenzweig Arnold	Cambridge
Maurice Bloomfield	Baltimore, Md.
Franz Boas	New York, N. Y.
Charles Pickering Bowditch	Jamaica Plain

Eugene Watson Burlingame	Albany, N. Y.
Edward Capps	Princeton, N. J.
George Henry Chase	Cambridge
Walter Eugene Clark	Chicago, Ill.
Roland Burrage Dixon	Cambridge
Franklin Edgerton	Philadelphia, Pa.
William Curtis Farabee	Philadelphia, Pa.
Jesse Walter Fewkes	Washington, D. C.
Jeremiah Denis Mathias Ford	Cambridge
Basil Lanneau Gildersleeve	Baltimore, Md.
Pliny Earle Goddard	New York, N. Y.
Charles Hall Grandgent	Cambridge
Louis Herbert Gray	New York, N. Y.
Charles Burton Gulick	Cambridge
Roy Kenneth Hack	Cambridge
William Arthur Heidel	Middletown, Conn.
George Lincoln Hendrickson	New Haven, Conn.
Bert Hodge Hill	Athens, Greece
Elijah Clarence Hills	Bloomington, Ind.
William Henry Holmes	Washington, D. C.
Edward Washburn Hopkins	New Haven, Conn.
Joseph Clark Hoppin	Pomfret, Conn.
Albert Andrew Howard	Cambridge
William Guild Howard	Cambridge
Aleš Hrdlička	Washington, D. C.
Carl Newell Jackson	Cambridge
Hans Carl Gunther von Jagemann	Cambridge
James Richard Jewett	Cambridge
Alfred Louis Kroeber	Berkeley, Cal.
Kirsopp Lake	Cambridge
Henry Roseman Lang	New Haven, Conn.
Charles Rockwell Lanman	Cambridge
David Gordon Lyon	Cambridge
Clifford Herschel Moore	Cambridge
George Foot Moore	Cambridge
Hanns Oertel	New Haven, Conn.
Edward Kennard Rand	Cambridge
George Andrew Reisner	Cambridge

Edward Robinson	New York, N. Y.
Fred Norris Robinson	Cambridge
Rudolph Schevill	Berkeley, Cal.
Edward Stevens Sheldon	Cambridge
Herbert Weir Smyth	Cambridge
Franklin Bache Stephenson	Claremont, Cal.
Charles Cutler Torrey	New Haven, Conn.
Alfred Marston Tozzer	Cambridge
Clark Wissler	New York, N. Y.
James Haughton Woods	Cambridge

Class III., Section III.—*Political Economy and History.*—37.

Brooks Adams	Quincy
George Burton Adams	New Haven, Conn.
Charles McLean Andrews	New Haven, Conn.
Charles Jesse Bullock	Cambridge
Thomas Nixon Carver	Cambridge
John Bates Clark	New York, N. Y.
Archibald Cary Coolidge	Boston
Richard Henry Dana	Cambridge
Davis Rich Dewey	Cambridge
Ephraim Emerton	Cambridge
Henry Walcott Farnam	New Haven, Conn.
Irving Fisher	New Haven, Conn.
Worthington Chauncey Ford	Cambridge
Edwin Francis Gay	New York, N. Y.
Frank Johnson Goodnow	Baltimore, Md.
Evarts Boutell Greene	Champaign, Ill.
Arthur Twining Hadley	New Haven, Conn.
Albert Bushnell Hart	Cambridge
Charles Homer Haskins	Cambridge
Isaac Minis Hays	Philadelphia, Pa.
Charles Downer Hazen	New York, N. Y.
Henry Cabot Lodge	Nahant
Abbott Lawrence Lowell	Cambridge
William MacDonald	New York, N. Y.
Charles Howard McIlwain	Cambridge

Roger Bigelow Merriman	Cambridge
Samuel Eliot Morison	Boston
William Bennett Munro	Cambridge
Charles Lemuel Nichols	Worcester
James Ford Rhodes	Boston
William Milligan Sloane	New York, N. Y.
John Osborne Sumner	Boston
Frank William Taussig	Cambridge
William Roscoe Thayer	Cambridge
Frederick Jackson Turner	Cambridge
George Grafton Wilson	Cambridge
George Parker Winship	Cambridge

Class III., Section IV.— *Literature and the Fine Arts.*— 39.

George Pierce Baker	Cambridge
James Phinney Baxter	Portland, Me.
William Sturgis Bigelow	Boston
Le Baron Russell Briggs	Cambridge
Charles Allerton Coolidge	Boston
Ralph Adams Cram	Boston
Samuel McChord Crothers	Cambridge
Wilberforce Eames	New York, N. Y.
Henry Herbert Edes	Cambridge
Edward Waldo Emerson	Concord
Arthur Fairbanks	Cambridge
Arthur Foote	Brookline
Edward Waldo Forbes	Cambridge
Kuno Francke	Gilbertsville, N. Y.
Daniel Chester French	New York, N. Y.
Horace Howard Furness	Philadelphia, Pa.
Robert Grant	Boston
Morris Gray	Boston
Chester Noyes Greenough	Cambridge
James Kendall Hosmer	Minneapolis, Minn.
Mark Antony DeWolfe Howe	Boston
Archer Milton Huntington	New York, N. Y.
George Lyman Kittredge	Cambridge

William Coolidge Lane	Cambridge
John Ellerton Lodge	Boston
Allan Marquand	Princeton, N. J.
Albert Matthews	Boston
Harold Murdock	Brookline
William Allan Neilson	Northampton
Thomas Nelson Page	Washington, D. C.
Herbert Putnam	Washington, D. C.
Denman Waldo Ross	Cambridge
John Singer Sargent	London, Eng.
Ellery Sedgwick	Boston
Henry Dwight Sedgwick	Cambridge
Richard Clipston Sturgis	Boston
Barrett Wendell	Boston
Owen Wister	Philadelphia, Pa.
George Edward Woodberry	Beverly

FOREIGN HONORARY MEMBERS.—66.

(Number limited to seventy-five.)

CLASS I.—*Mathematical and Physical Sciences.*—22.

SECTION I.—*Mathematics and Astronomy.*—6.

Johann Oskar Backlund	Petrograd
Jacques Salomon Hadamard	Paris
Felix Klein	Göttingen
Tullio Levi-Civita	Rome
Charles Emile Picard	Paris
Charles Jean de la Vallée Poussin	Louvain

CLASS I., SECTION II.—*Physics.*—7.

Svante August Arrhenius	Stockholm
Oliver Heaviside	Torquay
Sir Joseph Larmor	Cambridge
Hendrik Antoon Lorentz	Leyden
Max Planck	Berlin
Sir Ernest Rutherford	Manchester
Sir Joseph John Thomson	Cambridge

CLASS I., SECTION III.—*Chemistry.*—4.

Fritz Haber	Berlin
Henri Louis Le Chatelier	Paris
Wilhelm Ostwald	Leipsic
William Henry Perkin	Oxford

CLASS I.—SECTION IV.—*Technology and Engineering.*—5.

Heinrich Müller Breslau	Berlin
Ferdinand Foch	Paris
Joseph Jacques Césaire Joffre	Paris
Vsevolod Jevgenjevic Timonoff	Petrograd
William Cawthorne Unwin	London

Class II.— *Natural and Physiological Sciences.*— 19.

Section I.— *Geology, Mineralogy, and Physics of the Globe.*— 9.

Frank Dawson Adams	Montreal
Charles Barrois	Lille
Waldemar Christofer Brögger	Christiania
Sir Archibald Geikie	Haslemere, Surrey
Viktor Goldschmidt	Heidelberg
Julius Hann	Vienna
Albert Heim	Zürich
Sir William Napier Shaw	London
Johan Herman Lie Vogt	Trondhjem

Class II, Section II.— *Botany.*— 4.

John Briquet	Geneva
Adolf Engler	Berlin
Ignatz Urban	Berlin
Eugene Warming	Copenhagen

Class II.— Section III.— *Zoölogy and Physiology.*— 3.

Maurice Caullery	Paris
Sir Edwin Ray Lankester	London
George Henry Falkiner Nuttall	Cambridge

Class II., Section IV.— *Medicine and Surgery.*— 3.

Sir Thomas Barlow, Bart	London
Francis John Shepherd	Montreal
Charles Scott Sherrington	Oxford

Class III.— *Moral and Political Sciences.*— 25.

Section I.— *Theology, Philosophy and Jurisprudence.*— 5.

Rt. Hon Arthur James Balfour	Prestonkirk
Heinrich Brunner	Berlin

Albert Venn Dicey	Oxford
Raymond Poincaré	Paris
Rt. Hon. Sir Frederick Pollock, Bart.	London

SECTION II.—*Philology and Archæology.*— 9.

Friedrich Delitzsch	Berlin
Hermann Diels	Berlin
Wilhelm Dörpfeld	Athens
Henry Jackson	Cambridge
Hermann Georg Jacobi	Bonn
Arthur Anthony Macdonell	Oxford
Alfred Percival Maudslay	Hereford
Ramon Menendez Pidal	Madrid
Eduard Seler	Berlin

SECTION III.—*Political Economy and History.*— 6.

Rt. Hon. James Bryce, Viscount Bryce	London
Adolf Harnack	Berlin
Alfred Marshall	Cambridge
Rt. Hon. John Morley, Viscount Morley of Blackburn	London
George Walter Prothero	London
Rt. Hon. Sir George Otto Trevelyan, Bart.	London

SECTION IV.—*Literature and the Fine Arts.*— 5.

Georg Brandes	Copenhagen
Thomas Hardy	Dorchester
Jean Adrien Antoine Jules Jusserand	Paris
Rudyard Kipling	Burwash
Sir Sidney Lee	London

STATUTES AND STANDING VOTES

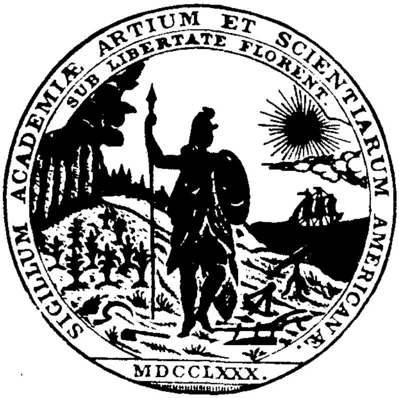
STATUTES

Adopted November 8, 1911: *amended May* 8, 1912, *January* 8, *and May* 14, 1913, *April* 14, 1915, *April* 12, 1916, *April* 10, 1918, *May* 14, 1919.

CHAPTER I

The Corporate Seal

Article 1. The Corporate Seal of the Academy shall be as here depicted:


SIGILLUM ACADEMIÆ ARTIUM ET SCIENTIARUM AMERICANÆ.
SUB LIBERTATE FLORENT.
MDCCLXXX.

Article 2. The Recording Secretary shall have the custody of the Corporate Seal.

See Chap. v. art. 3; chap. vi. art. 2.

CHAPTER II

FELLOWS AND FOREIGN HONORARY MEMBERS AND DUES

ARTICLE 1. The Academy consists of Fellows, who are either citizens or residents of the United States of America, and Foreign Honorary Members. They are arranged in three Classes, according to the Arts and Sciences in which they are severally proficient, and each Class is divided into four Sections, namely:

CLASS I. *The Mathematical and Physical Sciences*
  Section 1. Mathematics and Astronomy
  Section 2. Physics
  Section 3. Chemistry
  Section 4. Technology and Engineering

CLASS II. *The Natural and Physiological Sciences*
  Section 1. Geology, Mineralogy, and Physics of the Globe
  Section 2. Botany
  Section 3. Zoölogy and Physiology
  Section 4. Medicine and Surgery

CLASS III. *The Moral and Political Sciences*
  Section 1. Theology, Philosophy, and Jurisprudence
  Section 2. Philology and Archaeology
  Section 3. Political Economy and History
  Section 4. Literature and the Fine Arts

ARTICLE 2. The number of Fellows shall not exceed Six hundred, of whom not more than Four hundred shall be residents of Massachusetts, nor shall there be more than Two hundred in any one Class.

ARTICLE 3. The number of Foreign Honorary Members shall not exceed Seventy-five. They shall be chosen from among citizens of foreign countries most eminent for their discoveries and attainments in any of the Classes above enumerated. There shall not be more than Twenty-five in any one Class.

ARTICLE 4. If any person, after being notified of his election as Fellow or Resident Associate, shall neglect for six months to accept in writing, or, if a Fellow resident within fifty miles of Boston shall neglect to pay his Admission Fee, his election shall be void; and if any Fellow resident within fifty miles of Boston or any Resident Associate shall neglect to pay his Annual Dues for six months after they are due, provided his attention shall have been called to this Article of the Statutes in the meantime, he shall cease to be a Fellow or Resident Associate respectively; but the Council may suspend the provisions of this Article for a reasonable time.

With the previous consent of the Council, the Treasurer may dispense (*sub silentio*) with the payment of the Admission Fee or of the Annual Dues or both whenever he shall deem it advisable. In the case of officers of the Army or Navy who are out of the Commonwealth on duty, payment of the Annual Dues may be waived during such absence if continued during the whole financial year and if notification of such expected absence be sent to the Treasurer. Upon similar notification to the Treasurer, similar exemption may be accorded to Fellows or Resident Associates subject to Annual Dues, who may temporarily remove their residence for at least two years to a place more than fifty miles from Boston.

If any person elected a Foreign Honorary Member shall neglect for six months after being notified of his election to accept in writing, his election shall be void.

See Chap. vii. art. 2.

ARTICLE 5. Every Fellow resident within fifty miles of Boston hereafter elected shall pay an Admission Fee of Ten dollars.

Every Fellow resident within fifty miles of Boston shall, and others may, pay such Annual Dues, not exceeding Fifteen dollars, as shall be voted by the Academy at each Annual Meeting, when they shall become due; but any Fellow or Resident Associate shall be exempt from the annual payment if, at any time after his admission, he shall pay into the treasury Two hundred dollars in addition to his previous payments.

All Commutations of the Annual Dues shall be and remain permanently funded, the interest only to be used for current expenses.

Any Fellow not previously subject to Annual Dues who takes up his residence within fifty miles of Boston, shall pay to the Treasurer within three months thereafter Annual Dues for the current year, failing which his Fellowship shall cease; but the Council may suspend the provisions of this Article for a reasonable time.

Only Fellows who pay Annual Dues or have commuted them may hold office in the Academy or serve on the Standing Committees or vote at meetings.

ARTICLE 6. Fellows who pay or have commuted the Annual Dues and Foreign Honorary Members shall be entitled to receive gratis one copy of all Publications of the Academy issued after their election.

See Chap. x, art. 2.

ARTICLE 7. Diplomas signed by the President and the Vice-President of the Class to which the member belongs, and countersigned by the Secretaries, shall be given to Foreign Honorary Members and to Fellows on request.

ARTICLE 8. If, in the opinion of a majority of the entire Council, any Fellow or Foreign Honorary Member or Resident Associate shall have rendered himself unworthy of a place in the Academy, the Council shall recommend to the Academy the termination of his membership; and if three fourths of the Fellows present, out of a total attendance of not less than fifty at a Stated Meeting, or at a Special Meeting called for the purpose, shall adopt this recommendation, his name shall be stricken from the Roll.

See Chap. iii.; chap. vi. art. 1; chap. ix, art. 1, 7; chap. x. art. 2.

CHAPTER III

ELECTION OF FELLOWS AND FOREIGN HONORARY MEMBERS

ARTICLE 1. Elections of Fellows and Foreign Honorary Members shall be made by the Council in April of each year, and announced at the Annual Meeting in May.

ARTICLE 2. Nominations to Fellowship or Foreign Honorary Membership in any Section must be signed by two Fellows of that Section or by three voting Fellows of any Sections; but in any one year no Fellow may nominate more than four persons. These nominations, with statements of qualifications and brief biographical data, shall be sent to the Corresponding Secretary.

All nominations thus received prior to February 15 shall be forthwith sent in printed form to every Fellow having the right to vote, with the names of the proposers in each case and a brief account of each nominee, and with the request that the list be returned before March 15, marked to indicate preferences of the voter in such manner as the Council may direct.

All the nominations, with any comments thereon and with the results of the preferential indications of the Fellows, received by March 15, shall be referred at once to the appropriate Class Committees, which shall report their decisions to the Council, which shall thereupon have power to elect.

Persons nominated in any year, but not elected, may be placed on the preferential ballot of the next year at the discretion of the Council, but shall not further be continued on the list of nominees unless renominated.

Notice shall be sent to every Fellow having the right to vote, not later than the fifteenth of January, of each year, calling attention to the fact that the limit of time for sending nominations to the Corresponding Secretary will expire on the fifteenth of February.

See Chap. ii.; chap. vi. art. 1; chap. ix. art. 1.

CHAPTER IV

OFFICERS

ARTICLE 1. The Officers of the Academy shall be a President (who shall be Chairman of the Council), three Vice-Presidents (one from each Class), a Corresponding Secretary (who shall be Secretary of the Council), a Recording Secretary, a Treasurer, and a Librarian, all of whom shall be elected by ballot at the Annual Meeting, and shall hold their respective offices for one year, and until others are duly chosen and installed.

There shall be also twelve Councillors, one from each Section of each Class. At each Annual Meeting three Councillors, one from each Class, shall be elected by ballot to serve for the full term of four years and until others are duly chosen and installed. The same Fellow shall not be eligible for two successive terms.

The Councillors, with the other officers previously named, and the Chairman of the House Committee, *ex officio*, shall constitute the Council.

See Chap. x, art. 1.

ARTICLE 2. If any officer be unable, through death, absence, or disability, to fulfil the duties of his office, or if he shall resign, his place may be filled by the Council in its discretion for any part or the whole of the unexpired term.

ARTICLE 3. At the Stated Meeting in March, the President shall appoint a Nominating Committee of three Fellows having the right to vote, one from each Class. This Committee shall prepare a list of nominees for the several offices to be filled, and for the Standing Committees, and file it with the Recording Secretary not later than four weeks before the Annual Meeting.

See Chap. vi. art. 2.

ARTICLE 4. Independent nominations for any office, if signed by at least twenty Fellows having the right to vote, and received by the Recording Secretary not less than ten days before the Annual Meeting, shall be inserted in the call therefor, and shall be mailed to all the Fellows having the right to vote.

See Chap. vi. art. 2.

ARTICLE 5. The Recording Secretary shall prepare for use in voting at the Annual Meeting a ballot containing the names of all persons duly nominated for office.

CHAPTER V

THE PRESIDENT

ARTICLE 1. The President, or in his absence the senior Vice-President present (seniority to be determined by length of continuous fellowship in the Academy), shall preside at all meetings of the Academy. In the absence of all these officers, a Chairman of the meeting shall be chosen by ballot.

ARTICLE 2. Unless otherwise ordered, all Committees which are not elected by ballot shall be appointed by the presiding officer.

ARTICLE 3. Any deed or writing to which the Corporate Seal is to be affixed, except leases of real estate, shall be executed in the name of the Academy by the President or, in the event of his death, absence, or inability, by one of the Vice-Presidents, when thereto duly authorized.

See Chap. ii. art. 7; chap. iv. art. 1, 3; chap. vi. art. 2; chap. vii. art. 1; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1.

CHAPTER VI

THE SECRETARIES

ARTICLE 1. The Corresponding Secretary shall conduct the correspondence of the Academy and of the Council, recording or making an entry of all letters written in its name, and preserving for the files all official papers which may be received. At each meeting of the Council he shall present the communications addressed to the Academy which have been received since the previous meeting, and at the next meeting of the Academy he shall present such as the Council may determine.

He shall notify all persons who may be elected Fellows or Foreign Honorary Members, or Resident Associates, send to each a copy of the Statutes, and on their acceptance issue the proper Diploma. He shall also notify all meetings of the Council; and in case of the death, absence, or inability of the Recording Secretary he shall notify all meetings of the Academy.

Under the direction of the Council, he shall keep a List of the Fellows, Foreign Honorary Members, and Resident Associates, arranged in their several Classes and Sections. It shall be printed annually and issued as of the first day of July.

See Chap. ii. art. 7; chap. iii. art. 2, 3; chap. iv. art. 1; chap. ix. art. 6; chap. x. art. 1; chap. xi. art. 1.

ARTICLE 2. The Recording Secretary shall have the custody of the Charter, Corporate Seal, Archives, Statute-Book, Journals, and all literary papers belonging to the Academy.

Fellows or Resident Associates borrowing such papers or documents shall receipt for them to their custodian.

The Recording Secretary shall attend the meetings of the Academy and keep a faithful record of the proceedings with the names of the Fellows and Resident Associates present; and after each meeting is duly opened, he shall read the record of the preceding meeting.

He shall notify the meetings of the Academy to each Fellow and Resident Associate by mail at least seven days beforehand, and in his discretion may also cause the meetings to be advertised; he shall apprise Officers and Committees of their election or appointment, and inform the Treasurer of appropriations of money voted by the Academy.

After all elections, he shall insert in the Records the names of the Fellows by whom the successful nominees were proposed.

He shall send the Report of the Nominating Committee in print to every Fellow having the right to vote at least three weeks before the Annual Meeting.

See Chap. iv. art. 3.

In the absence of the President and of the Vice-Presidents he shall, if present, call the meeting to order, and preside until a Chairman is chosen.

See Chap. i.; chap. ii. art. 7; chap. iv. art. 3, 4, 5; chap. ix. art. 6; chap. x. art. 1, 2; chap. xi. art. 1, 3.

ARTICLE 3. The Secretaries, with the Chairman of the Committee of Publication, shall have authority to publish such of the records of the meetings of the Academy as may seem to them likely to promote its interests.

CHAPTER VII

THE TREASURER AND THE TREASURY

ARTICLE 1. The Treasurer shall collect all money due or payable to the Academy, and all gifts and bequests made to it. He shall pay all bills due by the Academy, when approved by the proper officers, except those of the Treasurer's office, which may be paid without such approval; in the name of the Academy he shall sign all leases of real estate; and, with the written consent of a member of the Committee on Finance, he shall make all transfers of stocks, bonds, and other securities belonging to the Academy, all of which shall be in his official custody.

He shall keep a faithful account of all receipts and expenditures, submit his accounts annually to the Auditing Committee, and render them at the expiration of his term of office, or whenever required to do so by the Academy or the Council.

He shall keep separate accounts of the income of the Rumford Fund, and of all other special Funds, and of the appropriation thereof, and render them annually.

His accounts shall always be open to the inspection of the Council.

ARTICLE 2. He shall report annually to the Council at its March meeting on the expected income of the various Funds and from all other sources during the ensuing financial year. He shall also report the names of all Fellows and Resident Associates who may be then delinquent in the payment of their Annual Dues.

ARTICLE 3. He shall give such security for the trust reposed in him as the Academy may require.

ARTICLE 4. With the approval of a majority of the Committee on Finance, he may appoint an Assistant Treasurer to perform his duties, for whose acts, as such assistant, he shall be responsible; or, with like approval and responsibility, he may employ any Trust Company doing business in Boston as his agent for the same purpose, the compensation of such Assistant Treasurer or agent to be fixed by the Committee on Finance and paid from the funds of the Academy.

ARTICLE 5. At the Annual Meeting he shall report in print all his official doings for the preceding year, stating the amount and condition of all the property of the Academy entrusted to him, and the character of the investments.

ARTICLE 6. The Financial Year of the Academy shall begin with the first day of April.

ARTICLE 7. No person or committee shall incur any debt or liability in the name of the Academy, unless in accordance with a previous vote and appropriation therefor by the Academy or the Council, or sell or otherwise dispose of any property of the Academy, except cash or invested funds, without the previous consent and approval of the Council.

See Chap. ii. art. 4, 5; chap. vi. art. 2; chap. ix. art. 6; chap. x. art. 1, 2, 3; chap. xi. art. 1.

CHAPTER VIII

THE LIBRARIAN AND THE LIBRARY.

ARTICLE 1. The Librarian shall have charge of the printed books, keep a correct catalogue thereof, and provide for their delivery from the Library.

At the Annual Meeting, as Chairman of the Committee on the Library, he shall make a Report on its condition.

ARTICLE 2. In conjunction with the Committee on the Library he shall have authority to expend such sums as may be appropriated by the Academy for the purchase of books, periodicals, etc., and for defraying other necessary expenses connected with the Library.

ARTICLE 3. All books procured from the income of the Rumford Fund or of other special Funds shall contain a book-plate expressing the fact.

ARTICLE 4. Books taken from the Library shall be receipted for to the Librarian or his assistant.

ARTICLE 5. Books shall be returned in good order, regard being had to necessary wear with good usage. If any book shall be lost or injured, the Fellow or Resident Associate to whom it stands charged shall replace it by a new volume or by a new set, if it belongs to a set, or pay the current price thereof to the Librarian, whereupon the remainder of the set, if any, shall be delivered to the Fellow or Resident Associate so paying, unless such remainder be valuable by reason of association.

ARTICLE 6. All books shall be returned to the Library for examination at least one week before the Annual Meeting.

ARTICLE 7. The Librarian shall have the custody of the Publications of the Academy. With the advice and consent of the President, he may effect exchanges with other associations.

See Chap. ii. art. 6; chap. x. art. 1, 2.

CHAPTER IX

THE COUNCIL

ARTICLE 1. The Council shall exercise a discreet supervision over all nominations and elections to membership, and in general supervise all the affairs of the Academy not explicitly reserved to the Academy as a whole or entrusted by it or by the Statutes to standing or special committees.

It shall consider all nominations duly sent to it by any Class Committee, and act upon them in accordance with the provisions of Chapter III.

With the consent of the Fellow interested, it shall have power to make transfers between the several Sections of the same Class, reporting its action to the Academy.

See Chap. iii. art. 2, 3; chap. x, art. 1.

ARTICLE 2. Seven members shall constitute a quorum.

ARTICLE 3. It shall establish rules and regulations for the transaction of its business, and provide all printed and engraved blanks and books of record.

ARTICLE 4. It shall act upon all resignations of officers, and all resignations and forfeitures of Fellowship or Resident Associateship; and cause the Statutes to be faithfully executed.

It shall appoint all agents and subordinates not otherwise provided for by the Statutes, prescribe their duties, and fix their compensation. They shall hold their respective positions during the pleasure of the Council.

ARTICLE 5. It may appoint, for terms not exceeding one year, and prescribe the functions of, such committees of its number, or of the Fellows of the Academy, as it may deem expedient, to facilitate the administration of the affairs of the Academy or to promote its interests.

ARTICLE 6. At its March meeting it shall receive reports from the President, the Secretaries, the Treasurer, and the Standing Committees, on the appropriations severally needed for the ensuing financial year. At the same meeting the Treasurer shall report on the expected income of the various Funds and from all other sources during the same year.

A report from the Council shall be submitted to the Academy, for action, at the March meeting, recommending the appropriation which in the opinion of the Council should be made.

On the recommendation of the Council, special appropriations may be made at any Stated Meeting of the Academy, or at a Special Meeting called for the purpose.

See Chap. x. art. 3.

ARTICLE 7. After the death of a Fellow or Foreign Honorary Member, it shall appoint a member of the Academy to prepare a biographical notice for publication in the Proceedings.

ARTICLE 8. It shall report at every meeting of the Academy such business as it may deem advisable to present.

See Chap. ii. art. 4, 5, 8; chap. iv, art. 1, 2; chap. vi. art. 1; chap. vii, art. 1; chap. xi. art. 1, 4.

CHAPTER X.

STANDING COMMITTEES

ARTICLE 1. The Class Committee of each Class shall consist of the Vice-President, who shall be chairman, and the four Councillors of the Class, together with such other officer or officers annually elected as may belong to the Class. It shall consider nominations to Fellowship in its own Class, and report in writing to the Council such as may receive at a Class Committee Meeting a majority of the votes cast, provided at least three shall have been in the affirmative.

See Chap. iii. art. 2.

ARTICLE 2. At the Annual Meeting the following Standing Committees shall be elected by ballot to serve for the ensuing year:

(i) *The Committee on Finance*, to consist of three Fellows, who, through the Treasurer, shall have full control and management of the funds and trusts of the Academy, with the power of investing the funds and of changing the investments thereof in their discretion.

See Chap. iv. art. 3; chap. vii. art. 1, 4; chap. ix. art. 6.

(ii) *The Rumford Committee*, to consist of seven Fellows, who shall report to the Academy on all applications and claims for the

Rumford Premium. It alone shall authorize the purchase of books publications and apparatus at the charge of the income from the Rumford Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iii) *The Cyrus Moors Warren Committee*, to consist of seven Fellows, who shall consider all applications for appropriations from the income of the Cyrus Moors Warren Fund, and generally shall see to the proper execution of the trust.

See Chap. iv. art. 3; chap. ix. art. 6.

(iv) *The Committee of Publication*, to consist of three Fellows, one from each Class, to whom all communications submitted to the Academy for publication shall be referred, and to whom the printing of the Proceedings and the Memoirs shall be entrusted.

It shall fix the price at which the Publications shall be sold; but Fellows may be supplied at half price with volumes which may be needed to complete their sets, but which they are not entitled to receive gratis.

Two hundred extra copies of each paper accepted for publication in the Proceedings or the Memoirs shall be placed at the disposal of the author without charge.

See Chap. iv. art. 3; chap. vi. art. 1, 3; chap. ix. art. 6.

(v) *The Committee on the Library*, to consist of the Librarian, *ex officio*, as Chairman, and three other Fellows, one from each Class, who shall examine the Library and make an annual report on its condition and management.

See Chap. iv. art. 3; chap. viii. art. 1, 2,; chap. ix. art. 6.

(vi) *The House Committee*, to consist of three Fellows, who shall have charge of all expenses connected with the House, including the general expenses of the Academy not specifically assigned to the care of other Committees or Officers.

See Chap. iv. art. 1, 3; chap. ix. art. 6.

(vii) *The Committee on Meetings*, to consist of the President, the Recording Secretary, and three other Fellows, who shall have charge of plans for meetings of the Academy.

See Chap. iv. art. 3; chap. ix. art. 6.

(viii) *The Auditing Committee*, to consist of two Fellows, who shall audit the accounts of the Treasurer, with power to employ an expert and to approve his bill.

See Chap. iv. art. 3; chap. vii. art. 1; chap. ix. art. 6.

ARTICLE 3. The Standing Committees shall report annually to the Council in March on the appropriations severally needed for the ensuing financial year; and all bills incurred on account of these Committees, within the limits of the several appropriations made by the Academy, shall be approved by their respective Chairmen.

In the absence of the Chairman of any Committee, bills may be approved by any member of the Committee whom he shall designate for the purpose.

See Chap. vii. art. 1, 7; chap. ix. art. 6.

CHAPTER XI

MEETINGS, COMMUNICATIONS, AND AMENDMENTS

ARTICLE 1. There shall be annually eight Stated Meetings of the Academy, namely, on the second Wednesday of October, November, December, January, February, March, April and May. Only at these meetings, or at adjournments thereof regularly notified, or at Special Meetings called for the purpose, shall appropriations of money be made or amendments of the Statutes or Standing Votes be effected.

The Stated Meeting in May shall be the Annual Meeting of the Corporation.

Special Meetings shall be called by either of the Secretaries at the request of the President, of a Vice-President, of the Council, or of ten Fellows having the right to vote; and notifications thereof shall state the purpose for which the meeting is called.

A meeting for receiving and discussing literary or scientific communications may be held on the fourth Wednesday of each month, excepting July, August, and September; but no business shall be transacted at said meetings.

ARTICLE 2. Twenty Fellows having the right to vote shall constitute a quorum for the transaction of business at Stated or Special

Meetings. Fifteen Fellows shall be sufficient to constitute a meeting for literary or scientific communications and discussions.

ARTICLE 3. Upon the request of the presiding officer or the Recording Secretary, any motion or resolution offered at any meeting shall be submitted in writing.

ARTICLE 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow or Resident Associate without the consent of the author; and no report shall in any case be published by any Fellow or Resident Associate in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members.

ARTICLE 5. No Fellow or Resident Associate shall introduce a guest at any meeting of the Academy until after the business has been transacted, and especially until after the result of the balloting upon nominations has been declared.

ARTICLE 6. The Academy shall not express its judgment on literary or scientific memoirs or performances submitted to it, or included in its Publications.

ARTICLE 7. All proposed Amendments of the Statutes shall be referred to a committee, and on its report, at a subsequent Stated Meeting or at a Special Meeting called for the purpose, two thirds of the ballot cast, and not less than twenty, must be affirmative to effect enactment.

ARTICLE 8. Standing Votes may be passed, amended, or rescinded at a Stated Meeting, or at a Special Meeting called for the purpose, by a vote of two thirds of the members present. They may be suspended by a unanimous vote.

See Chap. ii. art. 5, 8; chap. iii.; chap. iv. art. 3. 4. 5; chap. v. art. 1; chap. vi. art. 1, 2; chap. ix. art. 8.

STANDING VOTES

1. Communications of which notice has been given to either of the Secretaries shall take precedence of those not so notified.

2. Fellows or Resident Associates may take from the Library six volumes at any one time, and may retain them for three months, and no longer. Upon special application, and for adequate reasons assigned, the Librarian may permit a larger number of volumes, not exceeding twelve, to be drawn from the Library for a limited period.

3. Works published in numbers, when unbound, shall not be taken from the Hall of the Academy without the leave of the Librarian.

4. There may be chosen by the Academy, under such rules as the Council may determine, one hundred Resident Associates. Not more than forty Resident Associates shall be chosen in any one Class.

Resident Associates shall be entitled to the same privileges as Fellows, in the use of the Academy building, may attend meetings and present papers, but they shall not have the right to vote. They shall pay no Admission Fee, and their Annual Dues shall be the same as those of Fellows residing within fifty miles of Boston.

The Council and Committees of the Academy may ask one or more Resident Associates to act with them in an advisory or assistant capacity.

5. Communications offered for publication in the Proceedings or Memoirs of the Academy shall not be accepted for publication before the author shall have informed the Committee on Meetings of his readiness, either himself or through some agent, to use such time as the Committee may assign him at such meeting as may be convenient both to him and to the Committee, for the purpose of presenting to the Academy a general statement of the nature and significance of the results contained in his communication.

RUMFORD PREMIUM

In conformity with the terms of the gift of Sir Benjamin Thompson, Count Rumford, of a certain Fund to the American Academy of Arts and Sciences, and with a decree of the Supreme Judicial Court of Massachusetts for carrying into effect the general charitable intent and purpose of Count Rumford, as expressed in his letter of gift, the Academy is empowered to make from the income of the Rumford Fund, as it now exists, at any Annual Meeting, an award of a gold and a silver medal, being together of the intrinsic value of three hundred dollars, as a Premium to the author of any important discovery or useful improvement in light or heat, which shall have been made and pubished by printing, or in any way made known to the public, in any part of the continent of America, or any of the American Islands; preference always being given to such discoveries as, in the opinion of the Academy, shall tend most to promote the good of mankind; and, if the Academy sees fit, to add to such medals, as a further Premium for such discovery and improvement, a sum of money not exceeding three hundred dollars.

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